FOUR POSITIVE FORMULAE FOR TYPE A QUIVER POLYNOMIALS

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Abstract. We give four positive formulae for the (equioriented type A) quiver polynomials of Buch and Fulton [BF99]. All four formulae are combinatorial, in the sense that they are expressed in terms of combinatorial objects of certain types: Zelevinsky permutations, lacing diagrams, Young tableaux, and pipe dreams (also known as rc-graphs). Three of our formulae are multiplicity-free and geometric, meaning that their summands have coefficient 1 and correspond bijectively to components of a torus-invariant scheme. The remaining (presently non-geometric) formula is a variant of the conjecture of Buch and Fulton in terms of factor sequences of Young tableaux [BF99]; our proof of it proceeds by way of a new characterization of the tableaux counted by quiver constants. All four formulae come naturally in "doubled" versions, two for *double quiver polynomials*, and the other two for their stable limits, the double quiver functions, where setting half the variables equal to the other half specializes to the ordinary case.

Our method begins by identifying quiver polynomials as multidegrees [BB82, Jos84, BB85, Ros89] via equivariant Chow groups [EG98]. Then we make use of Zelevinsky's map from quiver loci to open subvarieties of Schubert varieties in partial flag manifolds [Zel85]. Interpreted in equivariant cohomology, this lets us write double quiver polynomials as ratios of double Schubert polynomials [LS82] associated to Zelevinsky permutations; this is our first formula. In the process, we provide a simple argument that Zelevinsky maps are scheme-theoretic isomorphisms (originally proved in [LM98]). Writing double Schubert polynomials in terms of pipe dreams [FK96] then provides another geometric formula for double quiver polynomials, via [KM03a]. The combinatorics of pipe dreams for Zelevinsky permutations implies an expression for limits of double quiver polynomials in terms of products of Stanley symmetric functions [Sta84]. A degeneration of quiver loci (orbit closures of GL on quiver representations) to unions of products of matrix Schubert varieties [Ful92, KM03a] identifies the summands in our Stanley function formula combinatorially, as lacing diagrams that we construct based on the strands of Abeasis and Del Fra in the representation theory of quivers [AD80]. Finally, we apply the combinatorial theory of key polynomials to pass from our lacing diagram formula to a double Schur function formula in terms of peelable tableaux [RS95a, RS98], and from there to our formula of Buch–Fulton type.

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Introduction

Overview. Universal formulae for cohomology classes appear in a number of different guises. In topology, classes with attached names of Pontrjagin, Chern, and Stiefel–Whitney arise as obstructions to vector bundles having linearly independent sections. Universality of formulae for these and other more general classes can be traced, for our purposes, to the fact that they—the formulae as well as the classes—live canonically on classifying spaces, from which they are pulled back to arbitrary spaces along classifying maps.

In algebra, cohomology classes on certain kinds of varieties (projective space, for example) are called 'degrees', and many universal formulae show up in degree calculations for classical ideals, such as those generated by minors of fixed size in generic matrices [Gia04] (or see [Ful98, Chapter 14]).

In geometry, pioneering work of R. Thom [Tho55] associated cohomology classes to sets of *critical points* of generic maps between manifolds, where the differential drops rank. Using slightly different language, the set of critical points can be characterized as the *degeneracy locus* for the associated morphism of tangent bundles. Subsequently there have appeared numerous extensions of Thom's notion of degeneracy locus, such as to maps between pairs of arbitrary vector bundles.

The cohomology classes Poincaré dual to degeneracy loci for (certain collections of) morphisms of complex vector bundles are expressible as polynomials in the Chern roots of the given vector bundles. It seems to be a general principle that coefficients in universal such expressions as sums of simpler polynomials always seem to be governed by combinatorial rules. This occurs, for example, in [Ful92, Buc01b, BKTY02]. However, even when it is possible to prove an explicit combinatorial formula, it has usually been unclear how the geometry of degeneracy loci reflects the combinatorics directly.

Our original motivation for this work was to close this gap, by bringing the combinatorics of universal formulae for cohomology classes of degeneracy loci into the realm of geometry, continuing the point of view set forth in [KM03a]. To do so, we reduce the degeneracy locus problem in [BF99] to the algebraic perspective mentioned above in terms of degrees of determinantal ideals, by way of the topological perspective mentioned above in terms of classifying spaces, as in [Kaz97, FR02b, KM03a]. This reduction allows us to manufacture formulae that are simultaneously geometric as well as combinatorial, by applying flat degenerations of subvarieties inside torus representations. The cohomology class remains unchanged in the degeneration and is given by summing the classes of all components of the special fiber.

The degenerations we employ result from one-parameter linear torus actions, and hence have Gröbner bases as their natural language. Our theorems on orbit closures and their combinatorics explicitly generalize results from the theory of ideals generated by minors of fixed size in generic matrices, where the prototypical result is Giambelli's degree formula [Gia04]. The ideals that interest us here are generated by minors in *products* of generic matrices. It is intriguing that our combinatorial analysis of these general determinantal ideals proceeds (using the 'Zelevinsky map') via Schubert determinantal ideals [Ful92, KM03a], which are generated by minors of varying sizes in a *single* matrix of variables. Arbitrary Schubert conditions may seem, consequently, to be in some sense more general; but while this may be true, the special form taken by the Schubert rank conditions on single matrices here give rise to a substantially richer combinatorial structure. This richness is fully borne out only by combining methods based on Schubert determinantal ideals with an approach in terms of minors in products of generic matrices, which is rooted more directly in representation theory of quivers.

Our technique of orbit degeneration automatically produces positive geometric formulae that are universal, since they essentially live on classifying spaces. However, there still remains the elucidation of what combinatorics most naturally describes these formulae, or indexes its components. Well over half of our exposition in this paper is dedicated to unearthing rich and often surprising interconnections between a number of combinatorial objects, some known and some new. Combinatorics—of objects including Zelevinsky permutations, lacing diagrams, Young tableaux, and pipe dreams (also known as rc-graphs)—forms the bridge between positive geometric formulae and the explicit algebra of several families of polynomials, including Schur functions, Schubert polynomials, Stanley symmetric functions, and now quiver polynomials. Working in the broader context of double quiver polynomials and their stable versions, the double quiver functions, allows the extra flexibility required for the proof of our variant of the Buch–Fulton conjecture [BF99]. Special cases of this conjecture are proved in [Buc01b, BKTY02].

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The four formulae. In this paper we consider formulae for cohomology classes of degeneracy loci for type A equioriented quivers of vector bundles. This simply means that we start with sequences $E_0 \to E_1 \to \cdots \to E_n$ of vector bundle morphisms over a fixed base. Since we immediately reduce in Theorem 1.20 to considering the equivariant classes of universal degeneracy loci, or quiver loci (to be defined shortly), we present our exposition in that context, referring the reader to [BF99] for an introduction in the language of vector bundles. We work over an arbitrary field $\Bbbk$.

Consider the vector space *Hom* of sequences $V_0 \xrightarrow{\phi_1} V_1 \xrightarrow{\phi_2} \cdots \xrightarrow{\phi_{n-1}} V_{n-1} \xrightarrow{\phi_n} V_n$ of linear transformations between vector spaces of dimensions $r_0, \ldots, r_n$, each thought of as consisting of row vectors. An element $\phi \in \mathit{Hom}$ is called a *quiver representation*, which we view as a sequence of $r_{i-1} \times r_i$ matrices. Each matrix list ϕ determines its *rank array* $\mathbf{r}(\phi) = (r_{ij}(\phi))_{i \le j}$, where $r_{ij}(\phi)$ for $i < j$ equals the rank of the composite map $V_i \to V_j$, and $r_{ii} = \dim(V_i)$. The data of a rank array determines a *quiver locus* $\Omega_{\mathbf{r}}$, defined as the subscheme of matrix lists $\phi \in \mathit{Hom}$ with rank array dominated by $\mathbf{r}$ entrywise: $r_{ij}(\phi) \le r_{ij}$ for all $i \le j$. Thus $\Omega_{\mathbf{r}}$ is the zero scheme of the ideal $I_{\mathbf{r}}$ in the coordinate ring of Hom generated by all minors of size $(1 + r_{ij})$ in the product of the appropriate $j - i$ matrices of variables, for all $i < j$ (Definition 1.1).

We assume throughout that $\mathbf{r} = \mathbf{r}(\phi)$ for some matrix list $\phi \in \Omega_{\mathbf{r}}$. This is a nontrivial condition equivalent to the irreducibility of $\Omega_{\mathbf{r}}$. Alternatively, it means that $\Omega_{\mathbf{r}}$ is an orbit closure for the group $GL = \prod_{i=0}^{n} GL(V_i)$ that acts on Hom by change of basis in each vector space V_i. In particular $\Omega_{\mathbf{r}}$ is stable under the action of any torus in GL. Picking a maximal torus T, we define the *quiver polynomial* as the T-multidegree of $\Omega_{\mathbf{r}}$. We show in Proposition 1.19 that this polynomial coincides with the T-equivariant class of $\Omega_{\mathbf{r}}$ in the Chow ring $A_T^*(\mathit{Hom})$, or alternatively in cohomology when $\Bbbk = \mathbb{C}$. This Chow ring is a polynomial ring $\mathbb{Z}[\mathbf{x}_{\mathbf{r}}]$ over the integers in an alphabet $\mathbf{x}_{\mathbf{r}}$ of size $r_0 + \cdots + r_n$, the union of bases $\mathbf{x}^0, \ldots, \mathbf{x}^n$ for the weight lattices of the maximal tori in $GL(V_i)$. Since $\Omega_{\mathbf{r}}$ is stable under GL and not just the torus T, its equivariant class is symmetric in each of the alphabets $\mathbf{x}^i$. More naturally, this is precisely the statement that the quiver polynomial lies inside $A_{GL}^*(\mathit{Hom}) \subset A_T^*(\mathit{Hom})$, and in this sense does not depend on our choice of T. We express the quiver polynomial determined by the rank array $\mathbf{r}$ as $\mathcal{Q}_{\mathbf{r}}(\mathbf{x} - \mathring{\mathbf{x}})$ for reasons that become clear in the ratio formula and double versions.

Next we present our four positive combinatorial formulae for quiver polynomials, in the same order that we will prove them in the main body of the text: the ratio formula, the pipe formula, the component formula, and the tableau formula. Besides making the overview of their proofs more coherent, this choice will serve to emphasize an important point that is worth bearing in mind before seeing the statements: while each formula stands well enough on its own, this paper is not merely a catalog of four different perspectives on quiver loci. Connections among the formulae lend added insight to each, and transformations between them often form crucial parts of proofs.

Before going into any sort of detail about the formulae, let us put them briefly in perspective (cross-references can be found in the more detailed subsections to come). The ratio formula writes $\mathcal{Q}_{\mathbf{r}}$ as a ratio of double Schubert polynomials. It arises geometrically from a comparison between the quiver locus $\Omega_{\mathbf{r}}$ and a related matrix Schubert variety. The pipe formula expresses $\mathcal{Q}_{\mathbf{r}}$ as a sum over combinatorial gadgets called pipe dreams. It is an easy consequence of the ratio formula, given that double Schubert polynomials expand as sums over the same gadgets. The component formula breaks $\mathcal{Q}_{\mathbf{r}}$ into a sum of products of double Schubert polynomials. It arises geometrically because the quiver locus $\Omega_{\mathbf{r}}$ degenerates to a limit scheme $\Omega_{\mathbf{r}}(0)$ that is usually reducible; each component is a product of matrix Schubert varieties, so its multidegree is a product of double Schubert polynomials. The pipe formula shows up here to provide a combinatorial upper bound on which components can occur in $\Omega_{\mathbf{r}}(0)$. Finally, the tableau formula recovers the decomposition of $\mathcal{Q}_{\mathbf{r}}$ into sums of products of Schur polynomials from [BF99]. The positive combinatorial properties of the coefficients in this decomposition derive principally from the ratio formula, by expanding the numerator Schubert polynomial as a sum of Demazure characters. However, the Schur-product decomposition of $\mathcal{Q}_{\mathbf{r}}$ also results by expressing (certain limits of) the summands in the component formula in terms of Schur functions, and it is this transformation that provides the bridge to the quiver constants from [BF99].

For each formula, we shall give here a precise statement, although many details of definitions will be left for later. Each formula will be presented along with a pointer to its location in the text, and an example using the following $n = 3$ rank array $\mathbf{r} = (r_{ij})$.

$$(*) \qquad \mathbf{r} \;=\; \begin{array}{cccc|c} 3 & 2 & 1 & 0 & i/j \\ \hline & & & 1 & 0 \\ & & 3 & 1 & 1 \\ & 3 & 2 & 1 & 2 \\ 1 & 1 & 1 & 0 & 3 \end{array}$$

We use a priori infinite alphabets $\mathbf{a} = \mathbf{x}^0$, $\mathbf{b} = \mathbf{x}^1$, $\mathbf{c} = \mathbf{x}^2$, and $\mathbf{d} = \mathbf{x}^3$ in the examples, keeping in mind that for the above ranks $\mathbf{r}$, the only variables actually appearing are a_1 from the $\mathbf{a}$ alphabet, b_1, b_2, b_3 from the $\mathbf{b}$ alphabet, c_1, c_2, c_3 from the $\mathbf{c}$ alphabet, and d_1 from the $\mathbf{d}$ alphabet. With these conventions the quiver polynomial is

$$\mathcal{Q}_{\mathbf{r}} = \mathcal{Q}_{\mathbf{r}}(\mathbf{x} - \mathring{\mathbf{x}}) = (b_1 + b_2 + b_3 - c_1 - c_2 - c_3)(a_1 - d_1).$$

In the general setting we distinguish between the symbol $\mathbf{x}$, which denotes a sequence $\mathbf{x}^0, \ldots, \mathbf{x}^n$ of infinite alphabets $\mathbf{x}^i = x^i_1, x^i_2, x^i_3, \ldots$, and $\mathbf{x}_{\mathbf{r}}$, which denotes a sequence $\mathbf{x}^0_{\mathbf{r}}, \ldots, \mathbf{x}^n_{\mathbf{r}}$ of finite alphabets, with $\mathbf{x}^i_{\mathbf{r}}$ having cardinality r_i. In the notation $\mathcal{Q}_{\mathbf{r}}(\mathbf{x} - \mathring{\mathbf{x}})$ for quiver polynomials, only the finitely many variables $\mathbf{x}_{\mathbf{r}} \subset \mathbf{x}$ actually appear, even though we view $\mathcal{Q}_{\mathbf{r}}$ as taking a sequence of infinite alphabets as input.

Ratio formula (Zelevinsky permutation). Associated to each rank array $\mathbf{r}$ is a permutation $v(\mathbf{r})$ in the symmetric group S_d for $d = r_0 + \cdots + r_n$. This *Zelevinsky permutation* $v(\mathbf{r})$ has a block structure determined by $\mathbf{r}$. Proposition 1.6 gives a complete characterization. Given the dimensions $r_0, \ldots, r_n$, there is a unique Zelevinsky permutation $v(\mathit{Hom})$ of minimal length. It is associated to the dense GL-orbit in Hom, that is, the maximal rank array, whose ij entry is the maximum possible rank for a map $V_i \to V_j$ that factors through V_k for $i \le k \le j$. The diagram of $v(\mathit{Hom})$, as defined after Proposition 1.6, is particularly simple.

The last ingredients for our first formula are the *double Schubert polynomials* of Lascoux and Schützenberger [LS82] (Section 2.1). These represent torus-equivariant cohomology classes of Schubert varieties in flag manifolds and are characterized geometrically as the multidegrees of matrix Schubert varieties [FR02a, KM03a]. The double Schubert polynomials $\mathfrak{S}_w(\mathcal{X} - \mathcal{Y})$ are indexed by permutations $w \in S_d$ and take as input two alphabets $\mathcal{X}$ and $\mathcal{Y}$ of size d. View $\mathbf{x}_{\mathbf{r}}$ as an alphabet of size d by concatenating its $n+1$ alphabets, and write $\mathring{\mathbf{x}}_{\mathbf{r}} = \mathbf{x}^n_{\mathbf{r}}, \ldots, \mathbf{x}^0_{\mathbf{r}}$ for the size d alphabet obtained by block reversing $\mathbf{x}_{\mathbf{r}}$.

Theorem (Ratio formula). $\mathcal{Q}_{\mathbf{r}}(\mathbf{x} - \mathring{\mathbf{x}}) \;=\; \dfrac{\mathfrak{S}_{v(\mathbf{r})}(\mathbf{x}_{\mathbf{r}} - \mathring{\mathbf{x}}_{\mathbf{r}})}{\mathfrak{S}_{v(\mathit{Hom})}(\mathbf{x}_{\mathbf{r}} - \mathring{\mathbf{x}}_{\mathbf{r}})}.$

This result is proved in Theorem 2.9. It is geometric in the sense that it interprets in equivariant cohomology the *Zelevinsky map* (Section 1.3), which takes quiver loci $\Omega_{\mathbf{r}}$ isomorphically to open subvarieties of Schubert varieties in partial flag manifolds. The geometry of the Zelevinsky map explains why Schubert polynomials appear.

Example. The rank array $\mathbf{r}$ in $(*)$ has the associated Zelevinsky permutations

$$v(\mathbf{r}) \;=\; \begin{array}{c|c|ccc|ccc|c|} \cline{2-9} 5 & \square & \square & \square & \square & \times & \cdot & \cdot & \cdot \\ \cline{2-9} 2 & \square & \times & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 3 & \square & \cdot & \times & \cdot & \cdot & \cdot & \cdot & \cdot \\ 6 & \square & \cdot & \cdot & \square & \cdot & \times & \cdot & \cdot \\ \cline{2-9} 1 & \times & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 4 & \cdot & \cdot & \cdot & \times & \cdot & \cdot & \cdot & \cdot \\ 8 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \square & \times \\ \cline{2-9} 7 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \times & \cdot \\ \cline{2-9} \end{array} \qquad \text{and} \qquad v(\mathit{Hom}) \;=\; \begin{array}{c|c|ccc|ccc|c|} \cline{2-9} 5 & * & * & * & * & \times & \cdot & \cdot & \cdot \\ \cline{2-9} 2 & * & \times & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 3 & * & \cdot & \times & \cdot & \cdot & \cdot & \cdot & \cdot \\ 4 & * & \cdot & \cdot & \times & \cdot & \cdot & \cdot & \cdot \\ \cline{2-9} 1 & \times & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 6 & \cdot & \cdot & \cdot & \cdot & \cdot & \times & \cdot & \cdot \\ 7 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \times & \cdot \\ \cline{2-9} 8 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \times \\ \cline{2-9} \end{array}$$

The numbers running down the left sides constitute one-line notation for $v(\mathbf{r})$ and $v(Hom)$; $v(\mathbf{r})$ takes $1 \mapsto 5$, $2 \mapsto 2$, $3 \mapsto 3$, $4 \mapsto 6$, and so on. We have replaced all 1 entries in the permutation matrices for $v(\mathbf{r})$ and $v(Hom)$ with $\times$ entries because it is sometimes convenient (as in Section 8.1) to use integers in these arrays for other purposes. The boxes $\square$ denote cells in the diagram of $v(\mathbf{r})$, whereas the $*$ entries denote cells in the diagram of $v(Hom)$. That $v(Hom)$ has such a simple diagram means that the denominator in the formula

$$\mathcal{Q}_{\mathbf{r}} \quad = \quad \frac{\mathfrak{S}_{52361487}(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} - \mathbf{d}, \mathbf{c}, \mathbf{b}, \mathbf{a})}{\mathfrak{S}_{52341678}(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} - \mathbf{d}, \mathbf{c}, \mathbf{b}, \mathbf{a})}$$

equals the product $(a_1 - c_3)(a_1 - c_2)(a_1 - c_1)(a_1 - d_1)(b_1 - d_1)(b_2 - d_1)(b_3 - d_1)$ of linear factors corresponding to the locations of the $*$ entries.

Pipe formula. One of many elementary ways to see that the denominator in the ratio formula is always the corresponding product of linear factors, and divides the numerator, is through our next combinatorial formula. It is in fact little more than an application of the Fomin–Kirillov double version [FK96] of the Billey–Jockusch–Stanley formula for Schubert polynomials [BJS93, FS94].

For a permutation $v \in S_d$, denote by $\mathcal{RP}(v)$ its set of *reduced pipe dreams* (also known as *planar histories* or *rc-graphs*). These are certain fillings the $d \times d$ grid with square tiles of the form $+$ or ╯╭, in which two pipes either cross or avoid each other (Section 5.1). For example, the permutation $v(Hom)$ has only one reduced pipe dream D_{Hom}, with crossing tiles $+$ in the diagram of $v(Hom)$ and elbow tiles ╯╭ elsewhere. For Zelevinsky permutations $v(\mathbf{r})$, we have $D \supseteq D_{Hom}$ whenever $D \in \mathcal{RP}(v(\mathbf{r}))$ is a reduced pipe dream. We identify pipe dreams with their sets of crossing tiles as subsets of the $d \times d$ grid.

Label the rows of the grid from top to bottom with the ordered alphabet $\mathbf{x}_{\mathbf{r}}$, and label the columns from left to right with $\mathring{\mathbf{x}}_{\mathbf{r}}$. Given a pipe dream D, define $(\mathbf{x}_{\mathbf{r}} - \mathring{\mathbf{x}}_{\mathbf{r}})^D$ as the product of linear binomials $(x_{\mathrm{row}(+)} - x_{\mathrm{col}(+)})$, one for each crossing tile $+$ in D.

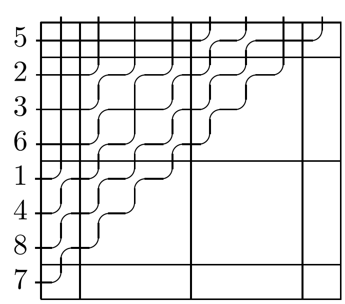
Theorem (Pipe formula). $\mathcal{Q}_{\mathbf{r}}(\mathbf{x} - \mathring{\mathbf{x}}) \; = \sum_{D \in \mathcal{RP}(v(\mathbf{r}))} (\mathbf{x}_{\mathbf{r}} - \mathring{\mathbf{x}}_{\mathbf{r}})^{D \smallsetminus D_{Hom}}.$

This follows from Theorem 5.5. It is geometric in the sense that summands on the right-hand side are equivariant classes of coordinate subspaces in a flat (Gröbner) degeneration [KM03a, Theorem B] of the matrix Schubert variety for $v(\mathbf{r})$ (Section 2.1). This is the largest matrix Schubert variety whose quotient modulo the appropriate parabolic subgroup contains the Zelevinsky image of $\Omega_{\mathbf{r}}$ as an open dense subvariety.

Example. Here is a typical pipe dream for the Zelevinsky permutation $v(\mathbf{r})$ from the previous Example, drawn in two ways.

	d_1	c_1	c_2	c_3	b_1	b_2	b_3	a_1
a_1	+	+	+	+	·	·	+	·
b_1	+	·	·	·	·	·	·	·
b_2	+	·	+	·	·	·	·	·
b_3	+	·	·	·	·	·	·	·
c_1	·	·	·	·	·	·	·	·
c_2	·	·	·	·	·	·	·	·
c_3	·	·	·	·	·	·	·	·
d_1	·	·	·	·	·	·	·	·

$\longleftrightarrow$



$= \quad D$

The right-hand diagram depicts the $+$ and ╯╭ tiles, but it omits those parts of pipes below the main antidiagonal, since the "sea" of ╯╭ tiles there can be confusing to look at.

The elbows have been entirely omitted from the left-hand diagram, but its row and column labels are present. The pipe dream D_{Hom} has $+$ tiles precisely in the diagram of $v(Hom)$, where the $*$ entries are in the ratio formula example. The expression $(\mathbf{x_r} - \mathring{\mathbf{x}}_\mathbf{r})^{D \smallsetminus D_{Hom}}$ for the above pipe dream D is simply $(a_1 - b_3)(b_2 - c_2)$.

For this permutation $v(\mathbf{r})$, moving the '+' in the **bc** block to either of the two remaining available cells on the antidiagonal of the **bc** block produces another reduced pipe dream for $v(\mathbf{r})$. Independently, the other '+' can move freely along the antidiagonal on which it sits. None of the '+' entries in D_{Hom} can move. Therefore the right-hand side of the pipe formula becomes a product of two linear forms, namely the sums

$$\big((a_1-b_3) + (b_1-b_2) + (b_2-b_1) + (b_3-c_3) + (c_1-c_2) + (c_2-c_1) + (c_3-d_1)\big)$$
$$\text{and} \qquad \big((b_1-c_3) + (b_2-c_2) + (b_3-c_1)\big)$$

of the binomials associated to cells on the corresponding antidiagonals. Cancellation occurs in the longer of these two linear forms to give $(a_1 - d_1)$, so the product of these two forms returns $\mathcal{Q}_\mathbf{r} = (b_1 + b_2 + b_3 - c_1 - c_2 - c_3)(a_1 - d_1)$ again.

Component formula (lacing diagrams). Double Schubert polynomials are not symmetric in general. Stanley introduced certain symmetrized versions [Sta84] which are now called *double Stanley symmetric functions* or *stable double Schubert polynomials* and denoted by $F_w(\mathcal{X} - \mathcal{Y})$. They are indexed by permutations w and take as arguments a pair of infinite alphabets $\mathcal{X}$ and $\mathcal{Y}$. When we evaluate such a symmetric function on a finite alphabet, we mean to set all remaining variables to zero. Stanley functions are produced by an algebraic limiting procedure (Proposition 6.5) from Schubert polynomials; hence the term 'stable polynomial'. In general, if a double Schubert polynomial $\mathfrak{S}_w$ and double Stanley function F_w are evaluated on the same pair of finite alphabets, the polynomial F_w, which is symmetric separately in each of the two alphabets, will tend to have many more terms.

Schubert polynomials and Stanley functions can be defined for any *partial permutation* w, by which we mean a rectangular matrix filled with zeros except for at most one 1 in each row and column, by canonically extending w to a permutation (Section 2.1). Partial permutations are matrices with entries in the field $\Bbbk$, so it makes sense to say that a list $\mathbf{w} = (w_1, \ldots, w_n)$ of partial permutations lies in Hom, if each w_i has size $r_{i-1} \times r_i$. Such lists of partial permutation matrices can be identified with nonembedded graphs drawn in the plane, called *lacing diagrams*, that we define in Section 3. The vertex set of $\mathbf{w}$ consists of $n+1$ columns of dots, where column i has r_i dots. An edge of w connects the dot at height α in column $i-1$ to the dot at height β in column i if the $\alpha\beta$ entry of w_i is 1.

Lacing diagrams come with a natural notion of length derived from the Bruhat order. The minimum possible length for a lacing diagram with rank array $\mathbf{r}$ is the codimension of the quiver locus $\Omega_\mathbf{r}$, which is the total degree of $\mathcal{Q}_\mathbf{r}(\mathbf{x} - \mathring{\mathbf{x}})$. Denote by $W(\mathbf{r})$ the set of minimal length lacing diagrams with rank array $\mathbf{r}$ (characterized combinatorially in Theorem 3.8).

Since Schubert polynomials can be indexed by partial permutations, we write

$$\mathfrak{S}_\mathbf{w}(\mathbf{x} - \mathring{\mathbf{x}}) \quad = \quad \mathfrak{S}_{w_1}(\mathbf{x}^0_\mathbf{r} - \mathbf{x}^1_\mathbf{r}) \cdots \mathfrak{S}_{w_n}(\mathbf{x}^{n-1}_\mathbf{r} - \mathbf{x}^n_\mathbf{r})$$

for products of double Schubert polynomials indexed by the partial permutations in $\mathbf{w}$. Similarly, we have the product notation

$$F_\mathbf{w}(\mathbf{x} - \mathring{\mathbf{x}}) \quad = \quad F_{w_1}(\mathbf{x}^0 - \mathbf{x}^1) \cdots F_{w_n}(\mathbf{x}^{n-1} - \mathbf{x}^n)$$

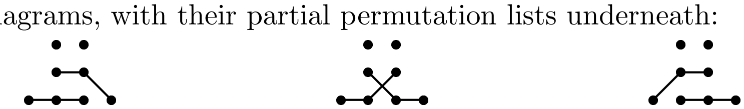
for double Stanley symmetric functions. Again, $\mathfrak{S}_\mathbf{w}$ and $F_\mathbf{w}$ take sequences of infinite alphabets as input, but only finitely many variables appear in $\mathfrak{S}_\mathbf{w}(\mathbf{x} - \mathring{\mathbf{x}})$, and we are allowed to write $F_\mathbf{w}(\mathbf{x_r} - \mathring{\mathbf{x}}_\mathbf{r})$ if we want to evaluate $F_\mathbf{w}$ on sequences of finite alphabets.

Theorem (Component formula).
$$\mathcal{Q}_{\mathbf{r}}(\mathbf{x}-\mathring{\mathbf{x}}) = \sum_{\mathbf{w}\in W(\mathbf{r})} \mathfrak{S}_{\mathbf{w}}(\mathbf{x}-\mathring{\mathbf{x}}) = \sum_{\mathbf{w}\in W(\mathbf{r})} F_{\mathbf{w}}(\mathbf{x}_{\mathbf{r}}-\mathring{\mathbf{x}}_{\mathbf{r}})\,.$$

The two versions are Corollary 6.17 and Corollary 6.23. The theorem implies, in particular, that all the extra terms in the Stanley version cancel.

The first sum in this theorem is combinatorially positive in a manner that most directly reflects the geometry of quiver loci. The basic idea is to flatly degenerate the group action of GL under which the quiver locus $\Omega_{\mathbf{r}}$ is an orbit closure (Section 4). As GL degenerates, so do its orbits, and the flat limits of the orbits are stable under the action of the limiting group. (The general version of this statement is Proposition 4.1; the specific case of interest to us is Proposition 4.5.) The limit need not be irreducible; its components are precisely the closures of orbits (under the limiting group) through minimal length lacing diagrams, which are matrix lists in *Hom*. This, together with the statement that the components in the degenerate limit of $\Omega_{\mathbf{r}}$ are generically reduced, is precisely the content of Theorem 6.16. It is our main geometric theorem concerning quiver loci and it immediately implies the Schubert version of the component formula.

Example. Continuing with $\mathbf{r}$ from $(*)$, the set $W(\mathbf{r})$ consists of the following three minimal length lacing diagrams, with their partial permutation lists underneath:



$$\left(\begin{bmatrix}1&0&0\end{bmatrix}, \begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\right) \qquad \left(\begin{bmatrix}1&0&0\end{bmatrix}, \begin{bmatrix}0&1&0\\1&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}1\\0\\0\end{bmatrix}\right) \qquad \left(\begin{bmatrix}0&1&0\end{bmatrix}, \begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}, \begin{bmatrix}1\\0\\0\end{bmatrix}\right)$$

The two versions (Schubert and Stanley) of the component formula read
$$\begin{aligned}\mathcal{Q}_{\mathbf{r}} &= \mathfrak{S}_{1243}(\mathbf{b}-\mathbf{c})\cdot\mathfrak{S}_{213}(\mathbf{c}-\mathbf{d}) + \mathfrak{S}_{2143}(\mathbf{b}-\mathbf{c}) + \mathfrak{S}_{213}(\mathbf{a}-\mathbf{b})\cdot\mathfrak{S}_{1243}(\mathbf{b}-\mathbf{c})\\ &= F_{1243}(\mathbf{b}_3-\mathbf{c}_3)F_{213}(\mathbf{c}_3-\mathbf{d}_1) + F_{2143}(\mathbf{b}_3-\mathbf{c}_3) + F_{213}(\mathbf{a}_1-\mathbf{b}_3)F_{1243}(\mathbf{b}_3-\mathbf{c}_3)\end{aligned}$$

In the bottom line above, we have written $\mathbf{a}_1 = \{a_1\}$, $\mathbf{b}_3 = \{b_1, b_2, b_3\}$, $\mathbf{c}_3 = \{c_1, c_2, c_3\}$, and $\mathbf{d}_1 = d_1$ for the finite alphabets corresponding to $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, and $\mathbf{d}$. We have indexed the Schubert polynomials and Stanley functions by permutations instead of partial permutations by completing each partial permutation w to a permutation $\widetilde{w}$.

Tableau formula (peelable tableaux and factor sequences). Our previous formulae write quiver polynomials in terms of binomials, double Schubert polynomials (in two completely different ways), and double Stanley symmetric functions. Now we turn to Schur functions. For a list $\underline{\lambda} = (\lambda_1, \ldots, \lambda_n)$ of partitions, write
$$s_{\underline{\lambda}}(\mathbf{x}-\mathring{\mathbf{x}}) = s_{\lambda_1}(\mathbf{x}^0-\mathbf{x}^1)\cdots s_{\lambda_n}(\mathbf{x}^{n-1}-\mathbf{x}^n)$$
to denote the corresponding product of double Schur functions in infinite alphabets (Section 1.4). Using our Theorem 1.20 to identify our quiver polynomials with those in [BF99], the Main Theorem of Buch and Fulton states that there exist unique integers $c_{\underline{\lambda}}(\mathbf{r})$ satisfying
$$Q_{\mathbf{r}}(\mathbf{x}-\mathring{\mathbf{x}}) = \sum_{\underline{\lambda}} c_{\underline{\lambda}}(\mathbf{r}) s_{\underline{\lambda}}(\mathbf{x}_{\mathbf{r}}-\mathring{\mathbf{x}}_{\mathbf{r}}),$$
and exhibits an explicit way to generate them. Although this procedure involves negative integers, Buch and Fulton conjectured the positivity of all the *quiver constants* $c_{\underline{\lambda}}(\mathbf{r})$. In

addition, starting with a rank array $\mathbf{r}$, they produced a concrete recursive method for generating lists of semistandard Young tableaux (Section 7), called *factor sequences* (Section 8.1). Defining $\Phi(\mathbf{r})$ to be the set of factor sequences coming from the rank array $\mathbf{r}$, every list $W \in \Phi(\mathbf{r})$ of tableaux has an associated list $\underline{\lambda}(W)$ of partitions. The combinatorial conjecture of [BF99] says that $c_{\underline{\lambda}}(\mathbf{r})$ counts the number of factor sequences $W \in \Phi(\mathbf{r})$ of shape $\underline{\lambda}$.

The direct connection between factor sequences and the combinatorial geometry of quiver polynomials was from the beginning—and still remains now—a mystery to us. It seems that the question should come down to finding an appropriate geometric explanation for Schur-positivity of Stanley symmetric functions, which is known both combinatorially and algebraically from various points of view. Although we lack a geometric framework, one of these other points of view, namely that of *Demazure characters* (Section 7.1), still allows us to deduce a combinatorial formula for the quiver constants $c_{\underline{\lambda}}(\mathbf{r})$.

The argument is based on the fact that every Stanley function F_w expands as a sum $\sum_\lambda \alpha_w^\lambda s_\lambda$ of Schur functions s_λ with nonnegative integer *Stanley coefficients* α_w^λ (Section 6.1). The main point, stated precisely in Theorem 7.14, is that every quiver constant is a Stanley coefficient for the Zelevinsky permutation: $c_{\underline{\lambda}}(\mathbf{r}) = \alpha_{v(\mathbf{r})}^\lambda$ for a certain partition λ.

It is known that every Stanley coefficient counts a set of *peelable tableaux* (Section 7.4). Our case focuses on the set $\mathrm{Peel}(D_\mathbf{r})$ of peelable tableaux for the diagram $D_\mathbf{r}$ of the Zelevinsky permutation $v(\mathbf{r})$. A semistandard tableau P lies in $\mathrm{Peel}(D_\mathbf{r})$ if it satisfies a readily checked, mildly recursive condition. The shape of every such tableau contains the partition whose shape is the diagram D_{Hom} of $v(Hom)$, so removing D_{Hom} leaves a skew tableau $P - D_{Hom}$. The connected components of this skew tableau form a list $\Psi_\mathbf{r}(P)$ of tableaux, read northeast to southwest (Definition 8.13), whose list of shapes we denote by $\underline{\lambda}(P)$.

Theorem (Tableau formula; Buch–Fulton factor sequence conjecture).

$$\begin{aligned}\mathcal{Q}_\mathbf{r}(\mathbf{x} - \mathring{\mathbf{x}}) &= \sum_{P \in \mathrm{Peel}(\mathbf{r})} s_{\underline{\lambda}(P)}(\mathbf{x}_\mathbf{r} - \mathring{\mathbf{x}}_\mathbf{r}) \\ &= \sum_{W \in \Phi(\mathbf{r})} s_{\underline{\lambda}(W)}(\mathbf{x}_\mathbf{r} - \mathring{\mathbf{x}}_\mathbf{r}).\end{aligned}$$

The two formulae in this theorem come from Theorem 7.21 and Corollary 8.23 (see Theorem 7.10, as well, which says that our $c_{\underline{\lambda}}(\mathbf{r})$ agree with those in [BF99]). In contrast to the Schubert and Stanley versions of the component formula, the peelable and factor sequence versions of the tableau formula are equal summand by summand; this is immediate from Theorem 8.22, which says simply that $P \mapsto \Psi_\mathbf{r}(P)$ is a bijection from peelable tableaux to factor sequences. We shall mention more about the proof of the tableau formula later in the Introduction.

Example. The set $\mathrm{Peel}(D_\mathbf{r})$ of peelable tableaux for $D_\mathbf{r}$ consists of four tableaux:

$$\begin{array}{ccccc}1&1&1&1&7\\2&4&&&\\3&&&&\\4&&&&\end{array}\qquad \begin{array}{cccc}1&1&1&1\\2&4&7&\\3&&&\\4&&&\end{array}\qquad \begin{array}{cccc}1&1&1&1\\2&4&&\\3&7&&\\4&&&\end{array}\qquad \begin{array}{cccc}1&1&1&1\\2&4&&\\3&&&\\4&&&\\7&&&\end{array}$$

Hence, under the bijection $\Psi_\mathbf{r}$ that acts by removing entries whose locations lie in the diagram of $v(Hom)$ (the $*$ entries in the ratio formula example), there are four factor sequences:

$$(\boxed{7}, \boxed{4}, \varnothing) \qquad (\varnothing, \boxed{4\,|\,7}, \varnothing) \qquad \left(\varnothing, \begin{array}{|c|}\hline 4\\ \hline 7\\ \hline\end{array}, \varnothing\right) \qquad (\varnothing, \boxed{4}, \boxed{7})$$

Using finite alphabets as in the component formula example, the tableau formula for $\mathcal{Q}_{\mathbf{r}}$ is

$$\mathcal{Q}_{\mathbf{r}} \;=\; s_{\square}(\mathbf{a}_1 - \mathbf{b}_3)s_{\square}(\mathbf{b}_3 - \mathbf{c}_3) + s_{\square\square}(\mathbf{b}_3 - \mathbf{c}_3) + s_{\substack{\square\\\square}}(\mathbf{c}_3 - \mathbf{d}_1) + s_{\square}(\mathbf{b}_3 - \mathbf{c}_3)s_{\square}(\mathbf{c}_3 - \mathbf{d}_1).$$

Proofs via double versions and limits. The formulae we presented above are all for *ordinary* quiver polynomials. One of the innovations in this paper is the idea of working with certain *double* versions of quiver polynomials, defined via the ratio formula. We clarify their combinatorics by applying limits that combine the way Stanley symmetric functions come from Schubert polynomials with the way Buch's series P_r come from quiver K-classes [Buc02, Section 4]. Our need for these limits arises because we require a component formula with symmetric summands. Our need for double versions arises because at a key stage we must set the $\mathring{\mathbf{x}}$ variables to zero without affecting the $\mathbf{x}$ variables.

The doubling construction simply replaces the sequence $\mathring{\mathbf{x}}$ of alphabets with a new sequence $\mathring{\mathbf{y}}$ consisting of variables independent of the $\mathbf{x}$ variables. All of the definitions go through verbatim after replacing each symbol $\mathring{\mathbf{x}}$ or $\mathbf{x}^j$ that has a minus sign in front of it by the corresponding $\mathbf{y}$ symbol. For example, we define the *double quiver polynomial* $\mathcal{Q}_{\mathbf{r}}(\mathbf{x}-\mathring{\mathbf{y}})$ by making this replacement in the ratio formula (Definition 2.5). Given this definition, the content of the ratio formula for ordinary quiver polynomials is that specializing $\mathring{\mathbf{y}} = \mathring{\mathbf{x}}$ in the ratio $\mathcal{Q}_{\mathbf{r}}(\mathbf{x} - \mathring{\mathbf{y}})$ of double Schubert polynomials yields the multidegree of the quiver locus $\Omega_{\mathbf{r}}$ inside *Hom*. Interpreting the numerator and denominator of $\mathcal{Q}_{\mathbf{r}}(\mathbf{x} - \mathring{\mathbf{y}})$ as multidegrees of matrix Schubert varieties (Section 2.1), this is accomplished in Section 2.3 using the *Zelevinsky map* from Section 1.3, which takes the quiver locus $\Omega_{\mathbf{r}}$ isomorphically to the intersection of the matrix Schubert variety $\overline{X}_{v(\mathbf{r})}$ with an appropriate "opposite big cell".

The Zelevinsky map was introduced in a slightly different form in [Zel85]. The fact that it is a scheme-theoretic isomorphism is equivalent to the main result in [LM98]. However, since we need additionally the explicit combinatorial connection to Zelevinsky permutations, we derive a new proof (Theorem 1.14), which happens to be quite elementary. It does not even rely on the primeness of the ideal $I_{\mathbf{r}}$ defining $\Omega_{\mathbf{r}}$ as a subscheme of *Hom*.

The double version of the pipe formula, Theorem 5.5, is automatic from the formula for double Schubert polynomials in [FK96], as we explain in Section 5.1.

Unlike the ratio and pipe formulae, simply replacing $\mathring{\mathbf{x}}$ by $\mathring{\mathbf{y}}$ in the component formula almost always yields a provably false statement (Remark 6.26). On the other hand, it turns out not to be useful to have a double component formula in which the summands are not symmetric, since (as we shall explain shortly) the point is to decompose the summands in terms of Schur polynomials. To rectify both the symmetry problem and the obstruction to doubling the variables, we resort to algebraic limits (reviewed in Section 6.1) of the sort used to construct Stanley functions from Schubert polynomials. The essential point is that when a rank array $\mathbf{r}$ is replaced by rank array $m + \mathbf{r}$, in which each r_{ij} is increased by m, nothing fundamental changes about the geometry, algebra, or combinatorics of the quiver locus $\Omega_{\mathbf{r}}$. The development of this idea culminates in the following master component formula (Theorem 6.20), which is arguably the single most important result in the paper.

Theorem (Stable double component formula).

$$\mathcal{F}_{\mathbf{r}}(\mathbf{x} - \mathring{\mathbf{y}}) \;:=\; \lim_{m\to\infty} \mathcal{Q}_{m+\mathbf{r}}(\mathbf{x} - \mathring{\mathbf{y}}) \;=\; \sum_{\mathbf{w}\in W(\mathbf{r})} F_{\mathbf{w}}(\mathbf{x} - \mathring{\mathbf{y}}).$$

The most important stability result on the way to this formula is Proposition 4.13: the lacing diagrams indexing the components of the quiver degenerations for the rank arrays $\mathbf{r}$ and $m + \mathbf{r}$ are in canonical bijection, and furthermore the multiplicities of the corresponding components are equal. Taking multidegrees, this immediately implies the positivity part of the Buch–Fulton conjecture for quiver constants geometrically, using none of the four formulae (Remark 4.15). Further expository details concerning the geometric, algebraic, and combinatorial roles of limits and stability in this paper, especially the manner in which they enter into the proof of the above formula, are included in the introduction to Section 6.

Why do we need double and stable versions? At a crucial point in our derivation of the peelable tableau formula from a component formula (see Remark 7.15), we need certain key polynomials to equal the products of Schur polynomials that they a priori only approximate. We check this by applying the theory of Demazure characters via isobaric divided differences (Section 7.2). But in the literature, these have only been developed for ordinary (not double) Schubert polynomials, so we must work with ratios $\mathfrak{S}_{v(\mathbf{r})}(\mathbf{x})/\mathfrak{S}_{v(Hom)}(\mathbf{x}) = \mathcal{Q}_{\mathbf{r}}(\mathbf{x})$ of ordinary Schubert polynomials obtained from double quiver polynomials by setting $\mathring{\mathbf{y}} = \mathbf{0}$.

To be more precise, consider the two sides of the equation

$$\frac{\mathfrak{S}_{v(\mathbf{r})}(\mathbf{x})}{\mathfrak{S}_{v(Hom)}(\mathbf{x})} = \sum_{\mathbf{w}\in W(\mathbf{r})} F_{\mathbf{w}}(\mathbf{x}_{\mathbf{r}}),$$

which we prove in Proposition 7.13 from the stable double component formula. Both sides expand into sums of products $s_{\underline{\lambda}}(\mathbf{x}_{\mathbf{r}})$ of ordinary Schur polynomials, the left side by Demazure characters, and the right side by using the fact that each summand is a product of symmetric polynomials in the sequence $\mathbf{x}_{\mathbf{r}}$. This formula simultaneously explains our need for doubled alphabets as well as for stabilization: Schur-expansion of the left-hand side demands double versions because it requires $\mathring{\mathbf{y}} = \mathbf{0}$, while Schur-expansion of the right-hand side demands stability because it requires the summands to be symmetric functions.

We show in Theorem 7.10 that the coefficients in the Schur-expansion of the right-hand side are the quiver constants from [BF99], while the above Demazure character argument implies that the coefficients on the left-hand side are Stanley coefficients (Theorem 7.14). These count peelable tableaux, proving the peelable tableau formula. The factor sequence conjecture follows from the peelable tableau formula via the bijection $\Psi_{\mathbf{r}}$. The proof of this bijection in Section 8 is a complex but essentially elementary computation. It works by breaking the diagram $D_{\mathbf{r}}$ into pieces small enough so that putting them back together in different ways yields peelable tableaux on the one hand, and factor sequences on the other.

Related notions and extensions. Quiver loci have been studied by a number of authors in their roles as universal degeneracy loci for vector bundle morphisms, as generalizations of Schubert varieties, and in representation theory of quivers; see [AD80, ADK81, LM98, FP98, Ful99, BF99, FR02b], for a sample. The particular rank conditions we consider here appeared first in [BF99] as the culmination of an increasingly general progression of degeneracy loci. Snapshots from this progression include the following; for more on the history, see [FP98].

- The case of two vector bundles and a fixed map required to have rank bounded by a given integer is known as the Giambelli–Thom–Porteous formula; see [Ful98, Chapter 14.4]. In this case both the ratio formula and the component formula reduce directly to the Giambelli–Thom–Porteous formula: the denominator in the ratio formula equals 1, while its numerator is the appropriate Schur polynomial; and there is just one lacing diagram with no crossing laces.
- Given two filtered vector bundles, there are degeneracy loci defined by Schubert conditions bounding the ranks of induced morphisms from subbundles in the filtration of the source to quotients by those in the target. The resulting universal formulae are double Schubert polynomials [Ful92].

- The same Schubert conditions can be placed on a quiver like those considered here, with $2n$ bundles of ranks increasing from 1 up to n and then decreasing from n down to 1. The degeneracy locus formula produces polynomials that Fulton called 'universal Schubert polynomials' [Ful99], because they specialize to quantum and double Schubert polynomials. We propose to call them *Fulton polynomials*. Results similar to the ones we prove here have been obtained independently for the special case of Fulton polynomials in [BKTY02].

One can consider even more general quiver loci and degeneracy loci, for arbitrary quivers and arbitrary rank conditions. We expect that the resulting quiver polynomials should have interesting combinatorial descriptions, at least for finite type quivers. Some indications of this come from [FR02b], where the same general idea of reducing to the equivariant study of quiver loci also appears (but the specifics differ in what kinds of statements are made concerning quiver polynomials), along with Rimányi's Thom polynomial proof of the component formula [BFR03], which he produced in response to seeing the formula.

For nilpotent cyclic quivers the orbit degeneration works just as for the type A_{n+1} quiver considered in this paper, and the Zelevinsky map can be replaced by Lusztig's embedding of a nilpotent cyclic quiver into a partial flag variety [Lus90]. The technique of orbit degeneration might also extend to arbitrary finite type quivers, but Zelevinsky maps do not extend in any straightforward way. One of the advantages of orbit degeneration is that it not only implies existence of a combinatorial formula, but provides strong hints (Proposition 4.11) as to the format of such a formula. Thus, in contrast to [BF99, p. 668], where much of the work lay in "discovering the shape of the formula", the geometry of degeneration provides a blueprint automatically.

Among the cohomological statements in this paper lurks a single result in K-theory: Theorem 2.7 gives a combinatorial formula for quiver K-classes as ratios of double Grothendieck polynomials. This K-theoretic analogue of the ratio formula (which is cohomological) immediately implies the K-theoretic analogue of our pipe formula by [FK94] (see [KM03b, Section 5]). There is also a K-theoretic generalization of our component formula, proved independently in two papers subsequent to this one [Buc03, Mil03]. It implies Buch's conjecture [Buc02] that the K-theoretic analogues of the quiver constants $c_{\underline{\lambda}}(\mathbf{r})$ exhibit a certain sign alternation, generalizing the positivity of $c_{\underline{\lambda}}(\mathbf{r})$.

It is an interesting problem to find a K-theoretic analogue of the factor sequence tableau formula. Substantial work in this direction has already been done by Buch [Buc02]. In analogy with our proof of the cohomological Buch–Fulton factor sequence conjecture, a complete combinatorial description of its K-theoretic analogue will involve a better understanding of how stable Grothendieck polynomials indexed by permutations expand into stable Grothendieck polynomials indexed by partitions. Lascoux has given an algorithm for this expansion based on a transition formula for Grothendieck polynomials [Las01].

A note on the field $\Bbbk$. Although to simplify the exposition we use language at times as if the field $\Bbbk$ were algebraically closed, all of our results hold for an arbitrary field $\Bbbk$. In fact the schemes we consider are all defined over the integers $\mathbb{Z}$, and our geometric statements concerning flat (Gröbner) degenerations work in that context.

Section 1. Geometry of quiver loci

1.1. Quiver loci and ideals

Fix a sequence $V = (V_0, V_1, \ldots, V_n)$ of vector spaces with $\dim(V_i) = r_i$, and denote by *Hom* the variety $\mathrm{Hom}(V_0, V_1) \times \cdots \times \mathrm{Hom}(V_{n-1}, V_n)$ of type A_{n+1} **quiver representations** on V. That is, *Hom* equals the vector space of sequences

$$\phi : \qquad V_0 \xrightarrow{\phi_1} V_1 \xrightarrow{\phi_2} \cdots \xrightarrow{\phi_{n-1}} V_{n-1} \xrightarrow{\phi_n} V_n$$

of linear transformations. By convention, set $V_{-1} = 0 = V_{n+1}$, and $\phi_0 = 0 = \phi_{n+1}$. Once and for all fix a basis[1] for each vector space V_i, and express elements of V_i as row vectors of length r_i. Doing so identifies each map ϕ_i in a quiver representation ϕ with a matrix of size $r_{i-1} \times r_i$. Thus the coordinate ring $\Bbbk[\mathit{Hom}]$ becomes a polynomial ring in variables $(f^1_{\alpha\beta}), \ldots, (f^n_{\alpha\beta})$, where the i^{th} index β and the $(i+1)^{\text{st}}$ index α run from 1 to r_i. Let Φ be the **generic** quiver representation, in which the entries in the matrices $\Phi_i : V_{i-1} \to V_i$ are the variables $f^i_{\alpha\beta}$.

Definition 1.1. Given an array $\mathbf{r} = (r_{ij})_{0 \le i \le j \le n}$ of nonnegative integers, the **quiver locus** $\Omega_{\mathbf{r}}$ is the zero scheme of the **quiver ideal** $I_{\mathbf{r}} \subset \Bbbk[\mathit{Hom}]$ generated by the union over all $i < j$ of the minors of size $(1 + r_{ij})$ in the product $\Phi_{i+1} \cdots \Phi_j$ of matrices:

$$I_{\mathbf{r}} \quad = \quad \langle \text{minors of size } (1 + r_{ij}) \text{ in } \Phi_{i+1} \cdots \Phi_j \text{ for } i < j \rangle.$$

Thus the quiver locus $\Omega_{\mathbf{r}}$ gives a natural scheme structure to the set of quiver representations whose composite maps $V_i \to V_j$ have rank at most r_{ij} for all $i < j$.

Two quiver representations (V, ϕ) and (W, ψ) on sequences V and W of n vector spaces are isomorphic if there are linear isomorphisms $\eta_i : V_i \to W_i$ for $i = 0, \ldots, n$ commuting with ϕ and ψ. Also, we can take the direct sum $V \oplus W$ in the obvious manner. Quiver representations that cannot be expressed nontrivially as direct sums are called **indecomposable**. Whenever $0 \le p \le q \le n$, there is an (obviously) indecomposable representation

$$I_{p,q} : \qquad 0 \to \cdots \to 0 \to \underset{p}{\Bbbk} = \cdots = \underset{q}{\Bbbk} \to 0 \to \cdots \to 0$$

having copies of the field $\Bbbk$ in spots between p and q, with identity maps between them and zeros elsewhere. Here is a standard result, c.f. [LM98, Section 1.1], generalizing the rank–nullity theorem from linear algebra.

Proposition 1.2. *Every indecomposable quiver representation is isomorphic to some $I_{p,q}$, and every quiver representation $\phi \in \mathit{Hom}$ is isomorphic to a direct sum of these.*

The space *Hom* of quiver representations carries an action of the group

$$GL^2 \quad = \quad GL(V_0)^2 \times GL(V_1)^2 \times \cdots \times GL(V_{n-1})^2 \times GL(V_n)^2 \tag{1.1}$$

of linear transformations. Specifically, if we think of elements in each V_i as row vectors and writing $GL(V_i)^2 = \overleftarrow{GL}(V_i) \times \overrightarrow{GL}(V_i)$ for $i = 0, \ldots, n$, the factor $\overleftarrow{GL}(V_i)$ acts by inverse multiplication on the right of $\mathrm{Hom}(V_{i-1}, V_i)$, while the factor $\overrightarrow{GL}(V_i)$ acts by multiplication on the left of $\mathrm{Hom}(V_i, V_{i+1})$. The group

$$GL \quad = \quad GL(V_0) \times \cdots \times GL(V_n)$$

embeds diagonally inside GL^2, so that $\gamma = (\gamma_0, \ldots, \gamma_n) \in GL$ acts by

$$\gamma \cdot \phi \quad = \quad (\ldots, \gamma_{i-1}\phi_i\gamma_i^{-1}, \gamma_i\phi_{i+1}\gamma_{i+1}^{-1}, \ldots).$$

[1] This is not so bad, since we shall in due course require a maximal torus in $GL(V_0) \times \cdots \times GL(V_n)$.

This action of GL preserves ranks, in the sense that $GL \cdot \Omega_{\mathbf{r}} = \Omega_{\mathbf{r}}$ for any ranks $\mathbf{r}$, because γ_i^{-1} cancels with γ_i when the maps in $\gamma \cdot \phi$ are composed.

Lemma 1.3. *The group GL has finitely many orbits on Hom, and every quiver locus $\Omega_{\mathbf{r}}$ is supported on a union of closures of such orbits.*

Proof. Since $GL \cdot \Omega_{\mathbf{r}} = \Omega_{\mathbf{r}}$, quiver loci are unions of orbit closures for GL, so it is enough to prove there are finitely many orbits. Proposition 1.2 implies that given any quiver representation $\phi \in Hom$, we can choose new bases for $V_0, \ldots, V_n$ in which ϕ is expressed as a list of partial permutation matrices (all zero entries, except for at most one 1 in each row and column). Hence ϕ lies in the GL orbit of one of the finitely many lists of partial permutation matrices in Hom. □

Definition 1.1 makes no assumptions about the array $\mathbf{r}$ of nonnegative integers, but only certain arrays can actually occur as ranks of quivers. More precisely, associated to each quiver representation $\phi \in Hom$ is its **rank array** $\mathbf{r}(\phi)$, whose nonnegative integer entries are given by the ranks of composite maps $V_i \to V_j$:

$$r_{ij}(\phi) \quad = \quad \mathrm{rank}(\phi_{i+1}\phi_{i+2}\cdots\phi_j) \quad \text{for} \quad i < j,$$

and $r_{ii}(\phi) = r_i$ for $i = 0, \ldots, n$. It is a consequence of the discussion above that the support of a quiver locus $\Omega_{\mathbf{r}}$ is irreducible if and only if $\mathbf{r} = \mathbf{r}(\phi)$ occurs as the rank array of some quiver representation $\phi \in Hom$, and this happens if and only if $\Omega_{\mathbf{r}}$ equals the closure of the orbit of GL through ϕ.

Convention 1.4. We make the convention here, once and for all, that in this paper we consider only rank arrays $\mathbf{r}$ that can occur, so $\mathbf{r} = \mathbf{r}(\phi)$ for some $\phi \in \Omega_{\mathbf{r}}$.

That being said, we remark that this assumption is unnecessary in the part of Section 1.4 preceding Theorem 1.20, as well as in Section 4.

1.2. Rank, rectangle, and lace arrays

Given a rank array $\mathbf{r} = \mathbf{r}(\phi)$ that can occur, Proposition 1.2 implies that there exists a unique array of nonnegative integers $\mathbf{s} = (s_{ij})_{0 \le i \le j \le n}$ such that $\phi = \bigoplus_{p \le q} I_{p,q}^{\oplus s_{pq}}$. In other words, the ranks $\mathbf{r}$ can occur if and only if there is an associated array $\mathbf{s}$. We call this data the **lace array**; the nomenclature is explained by Lemma 3.2, and the symbol '**s**' recalls that our construction is based on the 'strands' appearing in [AD80]. Taking ranks, we find that

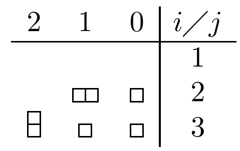
$$r_{ij} \quad = \quad \sum_{\substack{k \le i \\ j \le m}} s_{km} \quad \text{for} \quad i \le j. \tag{1.2}$$

The (transposed[2]) Buch–Fulton **rectangle array** is the array of rectangles (R_{ij}) for $0 \le i < j \le n$, such that R_{ij} is the rectangle of height $r_{i,j-1} - r_{ij}$ and width $r_{i+1,j} - r_{ij}$.

[2]The rectangle R_{ij} of [BF99] is the transpose of ours.

Example 1.5. Consider the rank array $\mathbf{r} = (r_{ij})$, its lace array $\mathbf{s} = (s_{ij})$, and its rectangle array $\mathbf{R} = (R_{ij})$, which we depict as follows.

$$\mathbf{r} \;=\; \begin{array}{cccc|c} 3 & 2 & 1 & 0 & i/j \\ \hline & & & 2 & 0 \\ & & 3 & 2 & 1 \\ & 4 & 2 & 1 & 2 \\ 3 & 2 & 1 & 0 & 3 \end{array} \qquad \mathbf{s} \;=\; \begin{array}{cccc|c} 3 & 2 & 1 & 0 & i/j \\ \hline & & & 0 & 0 \\ & & 0 & 1 & 1 \\ & 1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 0 & 3 \end{array} \qquad \mathbf{R} \;=$$



The relation (1.2) says that an entry of $\mathbf{r}$ is the sum of the entries in $\mathbf{s}$ that are weakly southeast of the corresponding location. The height of R_{ij} is obtained by subtracting the entry r_{ij} from the one above it, while the width of R_{ij} is obtained by subtracting the entry r_{ij} from the one to its left. □

It follows from the definition of R_{ij} that

$$\sum_{k\geq j} \text{height}(R_{ik}) = r_{i,j-1} - r_{i,n} \leq r_{i,j-1} \qquad \text{for all } i \tag{1.3}$$

$$\sum_{\ell\leq i} \text{width}(R_{\ell j}) = r_{i+1,j} - r_{0,j} \leq r_{i+1,j} \qquad \text{for all } j. \tag{1.4}$$

(This will be applied in Proposition 8.12.) The relation (1.2) can be inverted to obtain

$$s_{ij} \quad = \quad r_{ij} - r_{i-1,j} - r_{i,j+1} + r_{i-1,j+1} \tag{1.5}$$

for $i \leq j$, where $r_{ij} = 0$ if i and j do not both lie between 0 and n. Therefore the rank array $\mathbf{r}$ can occur for V if and only if $r_{ii} = r_i$ for $i = 0, \ldots, n$ and

$$r_{ij} - r_{i-1,j} - r_{i,j+1} + r_{i-1,j+1} \quad \geq \quad 0 \tag{1.6}$$

for $i \leq j$, the left-hand side being simply s_{ij}. We shall interchangeably use a lace array $\mathbf{s}$ or its corresponding rank array $\mathbf{r}$ to describe a given irreducible quiver locus.

It follows from (1.6) that

$$\text{height}(R_{ij}) \text{ decreases as } i \text{ decreases.} \tag{1.7}$$

$$\text{width}(R_{ij}) \text{ decreases as } j \text{ increases.} \tag{1.8}$$

Given a rank array $\mathbf{r}$ or equivalently a lace array $\mathbf{s}$, we shall construct a permutation $v(\mathbf{r}) \in S_d$, where $d = r_0 + \cdots + r_n$. In general, any matrix in the space M_d of $d \times d$ matrices comes with a decomposition into block rows of heights $r_0, \ldots, r_n$ (from top to bottom) and block columns of lengths $r_n, \ldots, r_0$ (from left to right). Note that our indexing convention may be unexpected, with the square blocks lying along the main block *anti*diagonal rather than on the diagonal as usual. With these conventions, the i^{th} block column refers to the block column of length r_i, which sits i blocks from the *right*.

We draw the matrix for the permutation w with a symbol $\times$ (instead of a 1) at each position $(q, w(q))$ and zeros elsewhere.

Proposition 1.6. *Given a rank array* $\mathbf{r}$ *for* V*, there exists a unique element* $v(\Omega_{\mathbf{r}}) = v(\mathbf{r}) \in S_d$ *satisfying the following conditions. Consider the block in the* i^{th} *block* column *and* j^{th} *block* row*.*

1. *If* $i \leq j$ *(that is, the block sits on or below the main block antidiagonal) then the number of* $\times$ *entries in that block equals* s_{ij}*.*
2. *If* $i = j+1$ *(that is, the block sits on the main block superantidiagonal) then the number of* $\times$ *entries in that block equals* $r_{j,j+1}$*.*

8	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$\times$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
9	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$\cdot$	$\times$	$\cdot$	$\cdot$	$\cdot$
4	$*$	$*$	$*$	$\times$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
5	$*$	$*$	$*$	$\cdot$	$\times$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
11	$*$	$*$	$*$	$\cdot$	$\cdot$	$\square$	$\square$	$\cdot$	$\cdot$	$\square$	$\times$	$\cdot$
1	$\times$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
2	$\cdot$	$\times$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
6	$\cdot$	$\cdot$	$\square$	$\cdot$	$\cdot$	$\times$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
12	$\cdot$	$\cdot$	$\square$	$\cdot$	$\cdot$	$\cdot$	$\square$	$\cdot$	$\cdot$	$\square$	$\cdot$	$\times$
3	$\cdot$	$\cdot$	$\times$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
7	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\times$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
10	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\times$	$\cdot$	$\cdot$

FIGURE 1. Zelevinsky permutation and its diagram

3. *If $i \geq j+2$ (that is, the block lies strictly above the main block superantidiagonal) then there are no $\times$ entries in that block.*
4. *Within every block row or block column, the $\times$ entries proceed from northwest to southeast, that is, no $\times$ entry is northeast of another $\times$ entry.*

Definition 1.7. $v(\mathbf{r})$ is the **Zelevinsky permutation** for the rank array $\mathbf{r}$.

Proof. We must show that the number of $\times$ entries in any block row, as dictated by conditions 1–3, equals the height of that block row (and transposed for columns), since condition 4 then stipulates uniquely how to arrange the $\times$ entries within each block. In other words the height r_j of the j^{th} block row must equal the number $r_{j,j+1}$ of $\times$ entries in the superantidiagonal block in that block row, plus the sum $\sum_i s_{ij}$ of the number of $\times$ entries in the rest of the blocks in that block row (and a similar statement must hold for block columns). These statements follow from (1.2). $\square$

The **diagram** $D(v)$ of a permutation $v \in S_d$, which by definition consists of those cells in the $d \times d$ grid having no $\times$ in v due north or due west of it, refines the data contained in the Buch–Fulton rectangle array [BF99]. The following Lemma is a straightforward consequence of the definition of the Zelevinsky permutation $v(\mathbf{r})$ and equation (1.5); to simplify notation, we often write $D_{\mathbf{r}} = D(v(\mathbf{r}))$.

Lemma 1.8. *In any block of the diagram $D_{\mathbf{r}} = D(v(\mathbf{r}))$, the cells form a rectangle justified in the southeast corner of the block. If the block is above the superantidiagonal, this rectangle consists of all the cells in the block. If the block is on or below the superantidiagonal, say in the i^{th} block column and j^{th} block row, this rectangle is the Buch–Fulton rectangle $R_{i-1,j+1}$.*

Example 1.9. Let $\mathbf{r}, \mathbf{s}, \mathbf{R}$ be as in Example 1.5. The Zelevinsky permutation is

$$v(\mathbf{r}) \quad = \quad \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 8 & 9 & 4 & 5 & 11 & 1 & 2 & 6 & 12 & 3 & 7 & 10 \end{pmatrix},$$

whose permutation matrix is indicated by $\times$ entries in Fig. 1, and whose diagram $D_{\mathbf{r}} = D(v(\mathbf{r}))$ is indicated by the union of the $*$ and $\square$ entries there. $\square$

Definition 1.10. Consider a fixed dimension vector $(r_0, r_1, \ldots, r_n)$ and a varying rank array $\mathbf{r}$ with that dimension vector.

1. The diagram $D_{Hom} := D(v(Hom))$ for the quiver locus Hom consists of all cells above the block superantidiagonal. Thus $D_{\mathbf{r}} \supset D_{Hom}$ for all occurring rank arrays $\mathbf{r}$. Define the **(skew) diagram** $D_{\mathbf{r}}^{\square}$ for $\mathbf{r}$ as the difference
$$D_{\mathbf{r}}^{\square} \quad = \quad D_{\mathbf{r}} \smallsetminus D_{Hom}.$$
In our depictions of $D_{\mathbf{r}}$, each cell of $D_{\mathbf{r}}^{\square}$ is indicated by a box $\square$, while each cell of D_{Hom} is indicated by an asterisk $*$ (see Fig. 1, for example).
2. For the zero quiver Ω_0, whose rank array $\mathbf{r}_0$ is zero except for the r_{ii} entries, the diagram $D(\Omega_0)$ consists of all locations strictly above the antidiagonal.

1.3. The Zelevinsky map

Let GL_d be the invertible $d \times d$ matrices. Denote by $P \subset GL_d$ the parabolic subgroup of block *lower* triangular matrices, where the diagonal blocks have sizes $r_0, \ldots, r_n$, and let B_+ be the group of *upper* triangular matrices.

The group P acts by multiplication on the left of GL_d, and the quotient $P\backslash GL_d$ is the manifold of partial flags with dimension jumps $r_0, \ldots, r_n$. Schubert varieties X_v in $P\backslash GL_d$ are orbit closures for the (left) action of B_+ by inverse right multiplication. In writing $X_v \subseteq P\backslash GL_d$ we shall always assume that the permutation $v \in S_d$ has minimal length in its right coset $(S_{r_0} \times \cdots \times S_{r_n})v$, which means that v has no descents within a block. Graphically, this condition is reflected in the permutation matrix for v by saying that no $\times$ entry is northeast of another in the same block row. Observe that this condition holds by definition for Zelevinsky permutations $v(\mathbf{r})$.

The preimage in GL_d of a Schubert variety $X_v \subseteq P\backslash GL_d$ is the closure in GL_d of the double coset PvB_+. The closure $\overline{PvB_+}$ of that inside M_d is the 'matrix Schubert variety' $\overline{X}_v$, whose definition we now recall. Since we shall need matrix Schubert varieties in the slightly more general setting of partial permutations later on, we use language here compatible with that level of generality.

Definition 1.11 ([Ful92])**.** Let w be a **partial permutation** of size $k \times \ell$, meaning that w is a matrix with k rows and ℓ columns having all entries equal to 0 except for at most one entry equal to 1 in each row and column. Let $M_{k\ell}$ be the vector space of $k \times \ell$ matrices. The **matrix Schubert variety** $\overline{X}_w$ is the subvariety
$$\overline{X}_w \quad = \quad \{Z \in M_{k\ell} \mid \operatorname{rank}(Z_{q\times p}) \leq \operatorname{rank}(w_{q\times p}) \text{ for all } q \text{ and } p\}$$
inside $M_{k\ell}$, where $Z_{q\times p}$ consists of the top q rows and left p columns of Z.

If $v \in S_d$ is a permutation, we do not require v to be a minimal element in its coset of $S_{r_0} \times \cdots \times S_{r_n}$ when we write $\overline{X}_v$ (as opposed to X_v); however, the matrix Schubert variety will fail to be a $P \times B_+$ orbit closure in M_d unless v is minimal.

Let Y_0 be the variety of all matrices in GL_d whose antidiagonal blocks are all identity matrices, and whose other blocks below the antidiagonal are zero. Thus Y_0 is obtained from the unipotent radical $U(P_+)$ of the block upper-triangular subgroup P_+ by a global left-to-right reflection followed by block left-to-right reflection. This has the net effect of reversing the order of the block columns of $U(P_+)$. If w_0 is the **long element** in S_d, represented as the antidiagonal permutation matrix, and $\mathbf{w}_0$ is the **block long element**, with antidiagonal permutation matrices in each diagonal block, then $Y_0 = U(P_+)w_0\mathbf{w}_0$. The variety Y_0 maps isomorphically (scheme-theoretically) to the **opposite big cell** U_0 in $P\backslash GL_d$ under projection modulo P. In other words, $U_0 \to Y_0$ is a section of the projection $GL_d \to P\backslash GL_d$ over the open set U_0.

Using the isomorphism $U_0 \cong Y_0$, the intersection $X_v \cap U_0$ of a Schubert variety in $P\backslash GL_d$ with the opposite big cell is a closed subvariety Y_v of Y_0 (we assume v is minimal in its coset when we write Y_v). It will be important for Theorem 1.14 to make the (standard) comparison between the equations defining Y_v inside Y_0 and the equations defining the corresponding matrix Schubert variety $\overline{X}_v$ inside M_d.

Lemma 1.12. *The ideal of Y_v as a subscheme of Y_0 is obtained from the ideal of $\overline{X}_v$ as a subscheme of M_d by setting all variables on the diagonal of each antidiagonal block equal to* 1 *and all other variables in or below the block antidiagonal to zero.*

Proof. The statement is equivalent to its geometric version, which says that Y_v equals the scheme-theoretic intersection of $\overline{X}_v$ with Y_0. To prove this version, note that $\overline{X}_v \cap Y_0 = (\overline{X}_v \cap GL_d) \cap Y_0$, and that $\overline{X}_v \cap GL_d$ projects to X_v as a fiber bundle over X_v with fiber P. The intersection of $\overline{X}_v \cap GL_d$ with the image Y_0 of the section $U_0 \to GL_d$ is scheme-theoretically the image Y_v of the section restricted to $X_v \cap U_0$. □

Definition 1.13. The **Zelevinsky map** $\mathcal{Z}$ takes $\phi \in \mathit{Hom}$ to the block matrix

$$(\phi_1, \phi_2, \ldots, \phi_n) \quad \stackrel{\mathcal{Z}}{\longmapsto} \quad \begin{bmatrix} 0 & & 0 & \phi_1 & \mathbf{1} \\ 0 & & \phi_2 & \mathbf{1} & 0 \\ 0 & \cdot^{\cdot^{\cdot}} & \mathbf{1} & 0 & 0 \\ \phi_n & \cdot^{\cdot^{\cdot}} & 0 & 0 & 0 \\ \mathbf{1} & & 0 & 0 & 0 \end{bmatrix}. \tag{1.9}$$

Zelevinsky's original map [Zel85] sent each $\phi \in \mathit{Hom}$ to the matrix $(\mathcal{Z}(\phi) w_0 \mathbf{w}_0)^{-1}$, which is obtained from $\mathcal{Z}(-\phi)$ by reversing the block columns and taking the inverse. Zelevinsky proved that it was bijective. The following theorem is therefore equivalent to the main theorem in [LM98], which proved using the primality of ideals of Plücker relations that Zelevinsky's original map is an isomorphism of schemes. The simpler form $\mathcal{Z}$ of Zelevinsky's map allows us to prove that the relevant ideals are equal without assuming that either is prime; more importantly, it connects quiver loci to the explicit combinatorics of matrix Schubert varieties for Zelevinsky permutations, as we require later.

Theorem 1.14. *The Zelevinsky map $\mathcal{Z}$ induces a scheme isomorphism from each quiver locus $\Omega_{\mathbf{r}}$ to the closed subvariety $Y_{v(\mathbf{r})}$ inside the Schubert subvariety $X_{v(\mathbf{r})}$ of the partial flag manifold $P\backslash GL_d$. In other words, $\mathcal{Z}(\Omega_{\mathbf{r}}) = Y_{v(\mathbf{r})}$ as subschemes of Y_0.*

Proof. Both schemes $\mathcal{Z}(\Omega_{\mathbf{r}})$ and $Y_{v(\mathbf{r})}$ are contained in $\mathcal{Z}(\mathit{Hom})$: for $\mathcal{Z}(\Omega_{\mathbf{r}})$ this is by definition, and for $Y_{v(\mathbf{r})}$ this is because the diagram of $v(\mathbf{r})$ contains the union of all blocks strictly above the block superantidiagonal, whence every coordinate above the block superantidiagonal is zero on $Y_{v(\mathbf{r})}$. Since the Zelevinsky map $\Omega_{\mathbf{r}} \to \mathcal{Z}(\Omega_{\mathbf{r}})$ is obviously an isomorphism of schemes, we must show that $\mathcal{Z}(\Omega_{\mathbf{r}})$ is defined by the same equations in $\Bbbk[\mathcal{Z}(\mathit{Hom})]$ defining $Y_{v(\mathbf{r})}$.

The set-theoretic description of $\overline{X}_v$ in Definition 1.11 implies that the Schubert determinantal ideal $I(\overline{X}_v) \subseteq \Bbbk[M_d]$ contains the union (over $q, p = 1, \ldots, d$) of minors with size $1 + \mathrm{rank}(v_{q \times p})$ in the northwest $q \times p$ submatrix of the $d \times d$ matrix Φ of variables. In fact, using Fulton's 'essential set' [Ful92, Section 3], the ideal $I(\overline{X}_v)$ is generated by those minors arising from cells (q, p) at the southeast corner of some block, along with all the variables strictly above the block superantidiagonal.

Consider a box (q,p) at the southeast corner of $B_{i+1,j-1}$, the intersection of block column $i+1$ and block row $j-1$, so that by definition of Zelevinsky permutation,

$$\mathrm{rank}(v(\mathbf{r})_{q\times p}) \quad = \quad \sum_{\substack{\alpha>i\\ \beta<j}} s_{\alpha\beta} + \sum_{k=i+1}^{j} r_{k-1,k}. \tag{1.10}$$

Lemma 1.15. *The number* $\mathrm{rank}(v(\mathbf{r})_{q\times p})$ *in* (1.10) *equals* $r_{ij} + \sum_{k=i+1}^{j-1} r_k$.

Proof. The coefficient on $s_{\alpha\beta}$ in $r_{ij} + \sum_{k=i+1}^{j-1} r_k$ is the number of elements in $\{r_{ij}\} \cup \{r_{i+1,i+1}, \ldots, r_{j-1,j-1}\}$ that are weakly northwest of $r_{\alpha\beta}$ in the rank array $\mathbf{r}$ (when the array $\mathbf{r}$ is oriented so that its southeast corner is r_{0n}). This number equals the number of elements in $\{r_{i,i+1}, \ldots, r_{j-1,j}\}$ that are weakly northwest of $r_{\alpha\beta}$, unless $r_{\alpha\beta}$ happens to lie strictly north and strictly west of r_{ij}, in which case we get one fewer. This one fewer is exactly made up by the sum of entries from $\mathbf{s}$ in (1.10). $\square$

Resuming the proof of Theorem 1.14, we set the appropriate variables in the generic matrix Φ to 0 or 1 by Lemma 1.12, and consider the equations coming from the northwest $q \times p$ submatrix for (q,p) in the southeast corner of $B_{i+1,j-1}$. Since $Y_{v(\mathbf{r})} \subseteq \mathcal{Z}(\mathit{Hom})$, these equations are minors of (1.9). In particular, using Lemma 1.15, and assuming that $i,j \in \{0,\ldots,n\}$, we find that these equations in $\Bbbk[\mathcal{Z}(\mathit{Hom})]$ are the minors of size $1+u+r_{ij}$ in the generic $(u+r_i)\times(u+r_i)$ block matrix

$$\begin{bmatrix} 0 & 0 & & 0 & \phi_{i+1} \\ 0 & 0 & & \phi_{i+2} & \mathbf{1} \\ 0 & 0 & \iddots & \mathbf{1} & 0 \\ 0 & \phi_{j-1} & \iddots & 0 & 0 \\ \phi_j & \mathbf{1} & & 0 & 0 \end{bmatrix}, \tag{1.11}$$

where $u = \sum_{k=i+1}^{j-1} r_k$ is the sum of the ranks of the subantidiagonal **1** blocks. The ideal generated by these minors of size $1+u+r_{ij}$ is preserved under multiplication of (1.11) by any matrix in $SL_{u+r_i}(\Bbbk[\mathcal{Z}(\mathit{Hom})])$. In particular, multiply (1.11) on the left by

$$\begin{bmatrix} \mathbf{1} & -\phi_{i+1} & \phi_{i+1}\phi_{i+2} & \cdots & \pm\phi_{i+1,j-2} & \mp\phi_{i+1,j-1} \\ & \mathbf{1} & & & & \\ & & \mathbf{1} & & & \\ & & & \ddots & & \\ & & & & \mathbf{1} & \\ & & & & & \mathbf{1} \end{bmatrix},$$

where $\phi_{i+1,k} = \phi_{i+1}\cdots\phi_k$ for $i+1 \leq k$. The result agrees with (1.11) except in its first block row, which has left block $(-1)^{j-1-i}\phi_{i+1}\cdots\phi_j$ and all other blocks zero. Therefore the minors coming from the southeast corner (q,p) of the block $B_{i+1,j-1}$ generate the same ideal in $\Bbbk[\mathcal{Z}(\mathit{Hom})]$ as the size $1+r_{ij}$ minors of $\phi_{i+1}\cdots\phi_j$, for all $i \leq j$. We conclude that the ideals of $\mathcal{Z}(\Omega_{\mathbf{r}})$ and $Y_{v(\mathbf{r})}$ inside $\Bbbk[\mathcal{Z}(\mathit{Hom})]$ coincide. $\square$

The proof of Theorem 1.14 never uses the fact that the minors vanishing on matrix Schubert varieties generate a prime ideal. Nonetheless, they do [Ful92, KM03a], and in fact one can conclude much more by citing (as in [LM98]) other properties of Schubert varieties from [Ram85, RR85], and using Lemma 1.12.

Corollary 1.16 ([LM98, AD80])**.** *Irreducible quiver loci $\Omega_{\mathbf{r}}$ are reduced, normal, and Cohen–Macaulay with rational singularities. The total area*

$$d(\mathbf{r}) \quad = \quad \ell(v(\mathbf{r})) - \ell(v(Hom)) \quad = \sum_{0\le k\le i<j\le m\le n} s_{k,j-1}s_{i+1,m}$$

of the rectangles R_{ij} equals the codimension of $\Omega_{\mathbf{r}}$ in Hom.

Remark 1.17. One can prove even more simply that the Zelevinsky map $\mathcal{Z}$ induces an isomorphism of the reduced variety underlying the scheme $\Omega_{\mathbf{r}}$ with the opposite 'cell' in *some* Schubert variety X_v inside $P\backslash GL_d$, without identifying $v = v(\mathbf{r})$. Indeed, $\mathcal{Z}$ is obviously an isomorphism (in fact, a linear map) from *Hom* to $\mathcal{Z}(Hom)$, which is easily seen to equal $Y_{v(Hom)}$. Then one uses equivariance of $\mathcal{Z}$ under appropriate actions of GL to conclude that it takes orbit closures (the reduced varieties underlying quiver loci) to orbit closures (opposite Schubert 'cells').

1.4. Quiver polynomials as multidegrees

In this section we define quiver polynomials, our title characters, as multidegrees of quiver loci $\Omega_{\mathbf{r}}$, and prove that these agree with the polynomials appearing in [BF99]. For the reader's convenience, we present the definition of multidegree [BB82, Jos84, BB85, Ros89] and introduce related notions here; those wishing more background and explicit examples can consult [KM03a, Sections 1.2 and 1.7] or [MS04, Chapter 8].

Let $\Bbbk[\mathbf{f}]$ be a polynomial ring graded by $\mathbb{Z}^d$. The gradings we consider here are all[3] **positive** in the sense of [KM03a, Section 1.7], which implies that the graded pieces of $\Bbbk[\mathbf{f}]$ have finite dimension as vector spaces. Writing $\mathcal{X} = x_1, \ldots, x_d$ for a basis of $\mathbb{Z}^d$, each variable $f \in \Bbbk[\mathbf{f}]$ has an **ordinary weight** $\deg(f) = \mathbf{u}(f) = \sum_{i=1}^d u_i(f)\cdot x_i \in \mathbb{Z}^d$, and the corresponding **exponential weight** $\mathrm{wt}(f) = \mathcal{X}^{\mathbf{u}(f)} = \prod_{i=1}^d x_i^{u_i(f)}$, which is a Laurent monomial in the ring $\mathbb{Z}[x_1^{\pm1}, \ldots, x_d^{\pm1}]$. Every finitely generated $\mathbb{Z}^d$-graded module $\Gamma = \bigoplus_{\mathbf{u}\in\mathbb{Z}^d}\Gamma_{\mathbf{u}}$ over $\Bbbk[\mathbf{f}]$ has a finite $\mathbb{Z}^d$-graded free resolution

$$\mathcal{E}_{\bullet}:\ 0 \leftarrow \mathcal{E}_0 \leftarrow \mathcal{E}_1 \leftarrow \mathcal{E}_2 \leftarrow \cdots \qquad \text{where} \qquad \mathcal{E}_i = \bigoplus_{j=1}^{\beta_i}\Bbbk[\mathbf{f}](-\mathbf{t}_{ij})$$

is $\mathbb{Z}^d$-graded, with the j^{th} summand of $\mathcal{E}_i$ generated in degree $\mathbf{t}_{ij} \in \mathbb{Z}^d$. In this case, the K**-polynomial** of Γ is

$$\mathcal{K}(\Gamma;\mathcal{X}) \quad = \quad \sum_i(-1)^i\sum_j \mathcal{X}^{\mathbf{t}_{ij}}.$$

Given any Laurent monomial $\mathcal{X}^{\mathbf{u}} = x_1^{u_1}\cdots x_d^{u_d}$, the rational function $\prod_{j=1}^d(1-x_j)^{u_j}$ can be expanded as a well-defined (that is, convergent in the $\mathcal{X}$-adic topology) formal power series $\prod_{j=1}^d(1-u_jx_j+\cdots)$ in $\mathcal{X}$. Doing the same for each monomial in an arbitrary Laurent polynomial $\mathcal{K}(\mathcal{X})$ results in a power series denoted by $\mathcal{K}(\mathbf{1}-\mathcal{X})$. The **multidegree** of a $\mathbb{Z}^d$-graded $\Bbbk[\mathbf{f}]$-module Γ is

$$\mathcal{C}(\Gamma;\mathcal{X}) \quad = \quad \text{the sum of terms of degree } \mathrm{codim}(\Gamma) \text{ in } \mathcal{K}(\Gamma;\mathbf{1}-\mathcal{X}),$$

where $\mathrm{codim}(\Gamma) = \dim(\mathrm{Spec}(\Bbbk[\mathbf{f}])) - \dim(\Gamma)$. If $\Gamma = \Bbbk[\mathbf{f}]/I$ is the coordinate ring of a subscheme $X \subseteq \mathrm{Spec}(\Bbbk[\mathbf{f}])$, then we may also write $[X]_{\mathbb{Z}^d}$ or $\mathcal{C}(X;\mathcal{X})$ to mean $\mathcal{C}(\Gamma;\mathcal{X})$.

[3]Actually, a nonpositive grading appears in the proof of Proposition 2.6, but we apply no theorems to it—the nonpositive grading there is only a transition between two positive gradings.

In the context of quiver loci, $d = r_0 + \cdots + r_n$, and we write $\mathbf{x_r}$ for the alphabet $\mathcal{X}$, which is split into a sequence of $n+1$ alphabets

$$\mathbf{x_r} \;=\; \mathbf{x}^0_{\mathbf{r}}, \ldots, \mathbf{x}^n_{\mathbf{r}} \tag{1.12}$$

of sizes $r_0, \ldots, r_n$. Thus the i^{th} alphabet is

$$\mathbf{x}^i_{\mathbf{r}} \;=\; x^i_1, \ldots, x^i_{r_i}. \tag{1.13}$$

The coordinate ring $\Bbbk[Hom]$ is graded by $\mathbb{Z}^d$, with the variable $f^i_{\alpha\beta} \in \Bbbk[Hom]$ having

$$\begin{aligned} \text{ordinary weight } \deg(f^i_{\alpha\beta}) &= x^{i-1}_\alpha - x^i_\beta \\ \text{and exponential weight } \ \operatorname{wt}(f^i_{\alpha\beta}) &= x^{i-1}_\alpha / x^i_\beta \end{aligned} \tag{1.14}$$

for each $i = 1, \ldots, n$. Under this grading by $\mathbb{Z}^d$, the quiver ideal $I_{\mathbf{r}}$ is homogeneous.

Definition 1.18. The **(ordinary) quiver polynomial** is the multidegree

$$\mathcal{Q}_{\mathbf{r}}(\mathbf{x} - \mathring{\mathbf{x}}) \;=\; \mathcal{C}(\Omega_{\mathbf{r}}; \mathbf{x_r})$$

of the subvariety $\Omega_{\mathbf{r}}$ inside Hom, under the $\mathbb{Z}^d$-grading in which $\deg(f^i_{\alpha\beta}) = x^{i-1}_\alpha - x^i_\beta$.

For the moment, the argument $\mathbf{x} - \mathring{\mathbf{x}}$ of $\mathcal{Q}_{\mathbf{r}}$ can be regarded as a formal symbol, denoting that $n+1$ alphabets $\mathbf{x} = \mathbf{x}^0, \ldots, \mathbf{x}^n$ are required as input. These alphabets may without harm be thought of as infinite, as in (1.17) and (1.18), even though all but the finitely many variables in $\mathbf{x_r}$ are to be ignored. Later, in Section 2.2, we shall define 'double quiver polynomials', with arguments $\mathbf{x} - \mathring{\mathbf{y}}$, indicating two sequences of alphabets as input; then the symbol $\mathring{\mathbf{x}}$ will take on additional meaning as the reversed sequence $\mathbf{x}^n, \ldots, \mathbf{x}^0$ in Section 2.3.

Our discussion of quiver polynomials will require supersymmetric Schur functions. Given (finite or infinite) alphabets $\mathcal{X}$ and $\mathcal{Y}$, define for $r \in \mathbb{N}$ the **homogeneous symmetric function** $h_r(\mathcal{X} - \mathcal{Y})$ in the difference $\mathcal{X} - \mathcal{Y}$ of alphabets by the generating series

$$\frac{\prod_{y \in \mathcal{Y}}(1 - zy)}{\prod_{x \in \mathcal{X}}(1 - zx)} \;=\; \sum_{r \geq 0} z^r h_r(\mathcal{X} - \mathcal{Y}),$$

in which $h_r(\mathcal{X} - \mathcal{Y}) = 0$ for $r < 0$. Then the **Schur function** $s_\lambda(\mathcal{X} - \mathcal{Y})$ in the difference $\mathcal{X} - \mathcal{Y}$ of alphabets is defined by the Jacobi–Trudi determinant

$$s_\lambda(\mathcal{X} - \mathcal{Y}) \;=\; \det\big(h_{\lambda_i - i + j}(\mathcal{X} - \mathcal{Y})\big).$$

The right-hand side is the determinant of an $\ell \times \ell$ matrix, where ℓ is the number of parts in the partition λ.

Buch and Fulton defined polynomials in [BF99] that also deserve rightly to be called 'quiver polynomials' (though they did not use this appellation). Given a sequence of maps between vector bundles $E_0, \ldots, E_n$ on a scheme, the polynomials appearing in their Main Theorem take as arguments the Chern classes of the bundles. Here, we shall express these polynomials formally as symmetric functions of the Chern roots

$$\mathbf{x_r} \;=\; \{x^i_j \mid i = 0, \ldots, n \text{ and } j = 1, \ldots, r_i\} \tag{1.15}$$

of the dual bundles $E_0^\vee, \ldots, E_n^\vee$. The Main Theorem of [BF99] is expressed in terms of integers $c_{\underline{\lambda}}(\mathbf{r})$ called **quiver constants**, each depending on a rank array $\mathbf{r}$ and a sequence $\underline{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ of n partitions. Let $\sum c_{\underline{\lambda}}(\mathbf{r}) s_{\underline{\lambda}}(\mathbf{x} - \mathring{\mathbf{x}})$ be the corresponding sum of products

$$s_{\underline{\lambda}}(\mathbf{x} - \mathring{\mathbf{x}}) \;=\; \prod_{i=1}^n s_{\lambda_j}(\mathbf{x}^{i-1} - \mathbf{x}^i) \tag{1.16}$$

of Schur functions in differences of consecutive alphabets from the sequence

$$\mathbf{x} \;=\; \mathbf{x}^0, \ldots, \mathbf{x}^n. \tag{1.17}$$

In this notation, each of the alphabets

$$\mathbf{x}^j \;=\; x_1^j, x_2^j, x_3^j, \ldots \tag{1.18}$$

is taken to be *infinite*, so that $s_{\underline{\lambda}}(\mathbf{x} - \mathring{\mathbf{x}})$ is a power series symmetric in each of $n+1$ sets of infinitely many variables. This notation allows us to evaluate $s_{\underline{\lambda}}$ on finite alphabets of any given size, by using $\mathbf{x_r}$ as in (1.12) and (1.13). It results in our notation $\sum c_{\underline{\lambda}}(\mathbf{r}) s_{\underline{\lambda}}(\mathbf{x_r} - \mathring{\mathbf{x}}_\mathbf{r})$ for the polynomials appearing in the Main Theorem of [BF99]. However, since the variables $\mathbf{x_r}$ in (1.15) are Chern roots of dual bundles, our $c_{\underline{\lambda}}(\mathbf{r})$ would be denoted by $c_{\underline{\lambda}'}(\mathbf{r})$ in [BF99], where $\underline{\lambda}' = (\lambda_1', \ldots, \lambda_n')$ is the sequence of conjugate (i.e., transposed) partitions. This follows because $s_\lambda(\mathcal{X} - \mathcal{Y}) = s_{\lambda'}((-\mathcal{Y}) - (-\mathcal{X}))$; see also [BF99, Section 4].

Before getting to Theorem 1.20, we must first say more about the geometry underlying multidegrees, which are algebraic substitutes for equivariant cohomology classes of sheaves on torus representations. Here, the torus is a maximal torus T inside GL acting on Hom. We choose T so the bases for $V_0, \ldots, V_n$ that were fixed in Section 1.1 consist of eigenvectors. Thus T consists of sequences of diagonal matrices, of sizes $r_0, \ldots, r_n$. The group $\mathbb{Z}^d$ in this context is the weight lattice of the torus T, and x_j^i is the weight corresponding to the action of the j^{th} diagonal entry in the i^{th} matrix. The action of T on Hom via the inclusion $T \subset GL$ induces the grading (1.14) on $\Bbbk[Hom]$.

The geometric data of a torus action on Hom determines an equivariant Chow ring $A_T^*(Hom)$. Let us recall briefly the relevant definitions from [EG98] in the context of a linear algebraic group G acting on a smooth scheme M. Let W be a representation of G containing an open subset $U \subset W$ on which G acts freely. For any integer k, we can choose W and U so that $W \smallsetminus U$ has codimension at least $k+1$ inside W [EG98, Lemma 9 in Section 6], and the construction is independent of our particular choices by [EG98, Definition–Proposition 1 and Proposition 4]. The **G-equivariant Chow ring** $A_G^*(M)$ has degree k piece

$$A_G^k(M) \;=\; A^k((M \times U)/G)$$

equal to the degree k piece of the ordinary Chow ring of the quotient of $M \times U$ modulo the diagonal action of G. We have used smoothness of M via [EG98, Corollary 2].

The equivariant Chow rings $A_T^*(M)$ for torus actions on vector spaces M all equal the integral symmetric algebra $\mathrm{Sym}^{\bullet}_{\mathbb{Z}}(\mathfrak{t}^*_{\mathbb{Z}})$ of the weight lattice $\mathfrak{t}^*_{\mathbb{Z}}$, independently of M. Thus, once we pick a basis $\mathcal{X} = x_1, \ldots, x_d$ for $\mathfrak{t}^*_{\mathbb{Z}}$, the class of any T-stable subvariety is uniquely expressed as a polynomial in $\mathbb{Z}[\mathcal{X}]$. More generally, every $\mathbb{Z}^d$-graded module Γ (equivalently, T-equivariant sheaf $\tilde{\Gamma}$) determines a polynomial $[\mathrm{cycle}(\Gamma)]_T$, where the **cycle** of Γ is the sum of its maximal-dimensional support components with coefficients given by the generic multiplicity of Γ along each.

Proposition 1.19. *The multidegree $\mathcal{C}(\Gamma; \mathcal{X})$ of a $\mathbb{Z}^d$-graded module Γ over $\Bbbk[\mathbf{f}]$ is the class $[\mathrm{cycle}(\Gamma)]_T$ in the equivariant Chow ring $A_T^*(M)$ of $M = \mathrm{Spec}(\Bbbk[\mathbf{f}])$.*

Proof. The function sending each graded module to its multidegree is characterized uniquely as having the 'Additivity', 'Degeneration', and 'Normalization' properties in [KM03a, Theorem 1.7.1]. Thus it suffices to show that the function sending each module to the equivariant Chow class of its cycle also has these properties. Additivity for Chow classes is by definition. Degeneration is the invariance of Chow classes under rational equivalence of cycles. Normalization, which says that a coordinate subspace $L \subseteq M$ has class equal to the product of the weights of the variables in its defining ideal, follows from [Bri97, Theorem 2.1] and [EG98, Proposition 4] by downward induction on the dimension of L. □

Theorem 1.20. *The quiver polynomial* $\mathcal{Q}_{\mathbf{r}}(\mathbf{x} - \mathring{\mathbf{x}})$ *in Definition 1.18 coincides with the degeneracy locus polynomial* $\sum c_{\underline{\lambda}}(\mathbf{r}) s_{\underline{\lambda}}(\mathbf{x}_{\mathbf{r}} - \mathring{\mathbf{x}}_{\mathbf{r}})$ *in the Main Theorem of* [BF99] *(although our notation for this polynomial differs from* [BF99]*; see the lines after* (1.18)*, above).*

Proof. Let $\mathrm{Gr}(\mathbf{r}, \ell)$ be the product $\mathrm{Gr}(r_0, \ell) \times \cdots \times \mathrm{Gr}(r_n, \ell)$ of Grassmannians of subspaces of dimensions $r_0, \ldots, r_n$ inside $\Bbbk^\ell$ for some large ℓ. This variety $\mathrm{Gr}(\mathbf{r}, \ell)$ comes endowed with $n+1$ universal vector bundles $\mathbf{V}_0, \ldots, \mathbf{V}_n$ of ranks $r_0, \ldots, r_n$. Denote by $\boldsymbol{Hom}$ the vector bundle $\prod_{j=1}^n \mathrm{Hom}(\mathbf{V}_{i-1}, \mathbf{V}_i)$ over $\mathrm{Gr}(\mathbf{r}, \ell)$. Pulling back $\mathbf{V}_i$ to $\boldsymbol{Hom}$ for all i yields vector bundles with a universal (or "tautological") sequence

$$\widetilde{\Phi}: \qquad \widetilde{\mathbf{V}}_0 \xrightarrow{\widetilde{\Phi}_1} \widetilde{\mathbf{V}}_1 \xrightarrow{\widetilde{\Phi}_2} \cdots \xrightarrow{\widetilde{\Phi}_{n-1}} \widetilde{\mathbf{V}}_{n-1} \xrightarrow{\widetilde{\Phi}_n} \widetilde{\mathbf{V}}_n.$$

of maps between them. Let $\boldsymbol{\Omega}_{\mathbf{r}} \subseteq \boldsymbol{Hom}$ denote the degeneracy locus of $\widetilde{\Phi}$ for ranks $\mathbf{r}$. If $k = d(\mathbf{r})$ is the codimension of $\boldsymbol{\Omega}_{\mathbf{r}}$ from Corollary 1.16, then we can assume ℓ has been chosen so large that the graded component $A^k(\boldsymbol{Hom})$ of the Chow ring of $\boldsymbol{Hom}$ consists of all polynomials of degree k in $\mathbb{Z}[\mathbf{x}_{\mathbf{r}}]$ symmetric in each of the $n+1$ alphabets $\mathbf{x}^i_{\mathbf{r}}$ from (1.13).

The class $[\boldsymbol{\Omega}_{\mathbf{r}}] \in A^k(\boldsymbol{Hom})$ is by definition the polynomial in the Main Theorem of [BF99]. On the other hand, $A^k(\boldsymbol{Hom})$ is by [EG98, Corollary 2] the degree k piece $A^k_{GL}(Hom)$ of the GL-equivariant Chow ring of Hom, and the ordinary class $[\boldsymbol{\Omega}_{\mathbf{r}}] \in A^k(\boldsymbol{Hom})$ coincides with the equivariant class $[\Omega_{\mathbf{r}}]_{GL} \in A^k_{GL}(Hom)$. The result now follows from Proposition 1.19, given that $A^*_{GL}(Hom)$ consists of the symmetric group invariants inside $A^*_T(Hom)$ by [EG98, Proposition 6]. □

Remark 1.21. Thinking of quiver polynomials as equivariant Chow or equivariant cohomology classes makes sense from the topological viewpoint taken originally by Thom [Tho55] in the context of what we now know as the Giambelli–Thom–Porteous formula. In the present context, a quiver of vector bundles on a space Y is induced by a unique homotopy class of maps from Y to the universal bundle Hom_{GL} with fiber Hom on the classifying space BGL of GL. The induced map $H^*_{GL}(\text{point}) = H^*(BGL) \to H^*(Y)$ on cohomology simply evaluates the universal degeneracy locus polynomial for $\mathbf{r}$, namely the quiver polynomial, on the Chern classes of the vector bundles in the quiver. Replacing the classifying space for GL by the corresponding Artin stack accomplishes the same thing in the Chow ring context instead of cohomology.

Section 2. Double quiver polynomials

2.1. Double Schubert polynomials

Let d be a positive integer, $\mathcal{X} = (x_1, x_2, \ldots, x_d)$ and $\mathcal{Y} = (y_1, y_2, \ldots, y_d)$ two sets of variables, and $w \in S_d$ a permutation. The double Schubert polynomial $\mathfrak{S}_w(\mathcal{X} - \mathcal{Y})$ of Lascoux and Schützenberger [LS82] is defined by downward induction on the length of w as follows. For the long permutation $w_0 \in S_d$ that reverses the order of $1, \ldots, d$, set

$$\mathfrak{S}_{w_0}(\mathcal{X} - \mathcal{Y}) \quad = \quad \prod_{i+j \leq d} (x_i - y_j).$$

If $w \neq w_0$, then choose q so that $w(q) < w(q+1)$, and define

$$\mathfrak{S}_w(\mathcal{X} - \mathcal{Y}) \quad = \quad \partial_q \mathfrak{S}_{w s_q}(\mathcal{X} - \mathcal{Y}).$$

The operator ∂_q is the q^{th} divided difference acting on the $\mathcal{Y}$ variables, so

$$\partial_q P(x_1, x_2, \ldots) \quad = \quad \frac{P(x_1, x_2, \ldots) - P(x_1, \ldots, x_{q-1}, x_{q+1}, x_q, x_{q+2}, \ldots)}{x_q - x_{q+1}} \tag{2.1}$$

for any polynomial P taking the $\mathcal{X}$ variables and possibly other variables for input.

In Section 6 we will require Schubert polynomials for *partial* permutations w (Definition 1.11). By convention, when we write $\mathfrak{S}_w$ for a partial permutation w of size $k \times \ell$, we shall mean $\mathfrak{S}_{\widetilde{w}}$, where $\widetilde{w}$ is a minimal-length completion of w to an actual permutation. Thus $\widetilde{w}$ can be be any permutation whose matrix agrees with w in the northwest $k \times \ell$ rectangle, and such that in the columns strictly to the right of ℓ, as well as in the rows strictly below k, the nonzero entries of $\widetilde{w}$ proceed from northwest to southeast (no nonzero entry is northeast of another). This convention is consistent with Definition 1.11, in view of the next result, because the matrix Schubert varieties $\overline{X}_w$ and $\overline{X}_{\widetilde{w}}$ have equal multidegrees, as they are defined by the *same* equations (though in a priori different polynomial rings).

The following statement appears (in similar language) as [KM03a, Theorem A]; an equivalent result over the complex numbers $\Bbbk = \mathbb{C}$ (given the translation between equivariant Chow and equivariant cohomology) is given in [FR02b]. These results are now seen to be essentially equivalent to [Ful92], via equivariant Chow groups as in the argument of Theorem 1.20.

Theorem 2.1. *Let w be a partial permutation matrix of size $k \times \ell$, so $\overline{X}_w$ is a subvariety of $M_{k\ell} = \operatorname{Spec}(\Bbbk[\mathbf{f}])$, where $\mathbf{f} = (f_{\alpha\beta})$ for $\alpha = 1, \ldots, k$ and $\beta = 1, \ldots, \ell$. If $f_{\alpha\beta}$ has ordinary weight $\deg(f_{\alpha\beta}) = x_\alpha - y_\beta$, then $\overline{X}_w$ has multidegree*

$$\mathcal{C}(\overline{X}_w; \mathcal{X}, \mathcal{Y}) \quad = \quad \mathfrak{S}_w(\mathcal{X} - \mathcal{Y})$$

equal to the double Schubert polynomial for w. (We remind the reader that the matrix for a partial permutation w has a 1 in row q and column p if and only if $w(q) = p$.)

When the partial permutation w is an honest permutation $v \in S_d$ with $d = r_0 + \cdots + r_n$, we often write $\mathfrak{S}_v(\mathcal{X} - \mathcal{Y}) = \mathfrak{S}_v(\mathbf{x_r} - \mathring{\mathbf{y}}_\mathbf{r})$ and break the variables into sequences of alphabets

$$\mathbf{x_r} = \mathbf{x}^0_\mathbf{r}, \ldots, \mathbf{x}^n_\mathbf{r} \quad \text{and} \quad \mathring{\mathbf{y}}_\mathbf{r} = \mathbf{y}^n_\mathbf{r}, \ldots, \mathbf{y}^0_\mathbf{r}, \tag{2.2}$$

$$\text{where} \quad \mathbf{x}^j_\mathbf{r} = x^j_1, \ldots, x^j_{r_j} \quad \text{and} \quad \mathbf{y}^j_\mathbf{r} = y^j_1, \ldots, y^j_{r_j}. \tag{2.3}$$

On the other hand, most *partial* permutations w that occur in the sequel will have size $r_{j-1} \times r_j$ for some $j \in \{1, \ldots, n\}$. In that case we will be interested in $\mathfrak{S}_w(\mathbf{x}^{j-1} - \mathbf{y}^j)$; as with quiver polynomials, we allow the input alphabets $\mathbf{x}^{j-1}$ and $\mathbf{y}^j$ to be infinite here, even though only the finitely many variables in $\mathbf{x}^{j-1}_\mathbf{r}$ and $\mathbf{y}^j_\mathbf{r}$ can actually appear.

2.2. Double quiver polynomials

The coordinate ring $\Bbbk[M_d]$ of the $d \times d$ matrices is multigraded by the group $\mathbb{Z}^{2d} = (\mathbb{Z}^{r_0} \oplus \cdots \oplus \mathbb{Z}^{r_n})^2$, which we take to have basis as in (2.2) and (2.3). In our context, it is most natural to index the variables $f^{ij}_{\alpha\beta}$ in the generic $d \times d$ matrix in a slightly unusual manner: $f^{ij}_{\alpha\beta} \in \Bbbk[M_d]$ occupies the spot in row α and column β within the rectangle at the intersection of the i^{th} block row and the j^{th} block column, where $i, j = 0, \ldots, n$ and we label block columns starting from the right. Set the ordinary weight of $f^{ij}_{\alpha\beta}$ equal to

$$\deg(f^{ij}_{\alpha\beta}) \quad = \quad x^i_\alpha - y^j_\beta$$

More pictorially, label the rows of the $d \times d$ grid with the $\mathbf{x}$ basis vectors in the order they are given, from top to bottom, and similarly label the columns with $\mathring{\mathbf{y}}_\mathbf{r} = \mathbf{y}^n_\mathbf{r}, \ldots, \mathbf{y}^0_\mathbf{r}$, from left to right (go left to right within each alphabet). Under our multigrading, the ordinary weight of a variable is its row basis vector minus its column basis vector. Therefore:

- Rows in the j^{th} block from the top are labeled $x^j_1, \ldots, x^j_r$ for $r = r_{jj}$, starting with x^j_1 in the highest row of that block.
- Columns in the j^{th} block *from the right* are labeled $y^j_1, \ldots, y^j_r$ for $r = r_{jj}$, starting with y^j_1 above the leftmost column of that block.

For notational clarity in examples, it is usually convenient to rename the alphabets $\mathbf{x}^0, \mathbf{x}^1, \mathbf{x}^2, \ldots$ using distinct Latin names, such as

$$\mathbf{x}^0 = \mathbf{a} = a_1, a_2, a_3, \ldots \quad \text{and} \quad \mathbf{x}^1 = \mathbf{b} = b_1, b_2, b_3, \ldots \quad \text{and} \quad \mathbf{x}^2 = \mathbf{c} = c_1, c_2, c_3, \ldots$$

and so on, rather than upper indices. Then, we rename the alphabets $\mathbf{y}^0, \mathbf{y}^1, \mathbf{y}^2, \ldots$ as

$$\mathbf{y}^0 = \dot{\mathbf{a}} = \dot{a}_1, \dot{a}_2, \dot{a}_3, \ldots \quad \text{and} \quad \mathbf{y}^1 = \dot{\mathbf{b}} = \dot{b}_1, \dot{b}_2, \dot{b}_3, \ldots \quad \text{and} \quad \mathbf{y}^2 = \dot{\mathbf{c}} = \dot{c}_1, \dot{c}_2, \dot{c}_3, \ldots$$

and so on, the same as the $\mathbf{x}$'s but with dots on top.

All of the notation above should be made clearer by the following example.

Example 2.2. Consider quivers on the vector spaces $V = (V_0, V_1, V_2)$ with dimensions $(r_0, r_1, r_2) = (2, 3, 1)$. The coordinate ring $\Bbbk[M_d]$ has variables $f^{ij}_{\alpha\beta}$ as they appear in the matrices below (the f variables are the same in both):

$$\begin{array}{c|c|ccc|cc|}
 & y^2_1 & y^1_1 & y^1_2 & y^1_3 & y^0_1 & y^0_2 \\ \hline
x^0_1 & f^{02}_{11} & f^{01}_{11} & f^{01}_{12} & f^{01}_{13} & f^{00}_{11} & f^{00}_{12} \\
x^0_2 & f^{02}_{21} & f^{01}_{21} & f^{01}_{22} & f^{01}_{23} & f^{00}_{21} & f^{00}_{22} \\ \hline
x^1_1 & f^{12}_{11} & f^{11}_{11} & f^{11}_{12} & f^{11}_{13} & f^{10}_{11} & f^{10}_{12} \\
x^1_2 & f^{12}_{21} & f^{11}_{21} & f^{11}_{22} & f^{11}_{23} & f^{10}_{21} & f^{10}_{22} \\
x^1_3 & f^{12}_{31} & f^{11}_{31} & f^{11}_{32} & f^{11}_{33} & f^{10}_{31} & f^{10}_{32} \\ \hline
x^2_1 & f^{22}_{11} & f^{21}_{11} & f^{21}_{12} & f^{21}_{13} & f^{20}_{11} & f^{20}_{12} \\ \hline
\end{array}
\quad = \quad
\begin{array}{c|c|ccc|cc|}
 & \dot{c}_1 & \dot{b}_1 & \dot{b}_2 & \dot{b}_3 & \dot{a}_1 & \dot{a}_2 \\ \hline
a_1 & f^{02}_{11} & f^{01}_{11} & f^{01}_{12} & f^{01}_{13} & f^{00}_{11} & f^{00}_{12} \\
a_2 & f^{02}_{21} & f^{01}_{21} & f^{01}_{22} & f^{01}_{23} & f^{00}_{21} & f^{00}_{22} \\ \hline
b_1 & f^{12}_{11} & f^{11}_{11} & f^{11}_{12} & f^{11}_{13} & f^{10}_{11} & f^{10}_{12} \\
b_2 & f^{12}_{21} & f^{11}_{21} & f^{11}_{22} & f^{11}_{23} & f^{10}_{21} & f^{10}_{22} \\
b_3 & f^{12}_{31} & f^{11}_{31} & f^{11}_{32} & f^{11}_{33} & f^{10}_{31} & f^{10}_{32} \\ \hline
c_1 & f^{22}_{11} & f^{21}_{11} & f^{21}_{12} & f^{21}_{13} & f^{20}_{11} & f^{20}_{12} \\ \hline
\end{array}$$

The ordinary weight of each f variable equals its row label minus its column label, and its exponential weight is the ratio instead of the difference. For example, the variable f^{01}_{23} has ordinary weight $x^0_2 - y^1_3 = a_2 - \dot{b}_3$ and exponential weight $x^0_2/y^1_3 = a_2/\dot{b}_3$. $\square$

The coordinate ring $\Bbbk[Y_0]$ of the variety Y_0 from Section 1.3 is not naturally multigraded by all of $\mathbb{Z}^{2d}$, but only by $\mathbb{Z}^d$, with the variable $f^{ij}_{\alpha\beta} \in \Bbbk[Y_0]$ having ordinary weight $x^i_\alpha - x^j_\beta$. This convention is consistent with the multigrading of $\Bbbk[\mathit{Hom}]$ in (1.14) under the homomorphism to $\Bbbk[Y_0]$ induced by the Zelevinsky map, since the variable $f^{i-1,i}_{\alpha\beta} \in \Bbbk[Y_0]$ maps to $f^i_{\alpha\beta} \in \Bbbk[\mathit{Hom}]$, and their ordinary weights $x^{i-1}_\alpha - x^i_\beta$ agree. Our $\mathbb{Z}^d$-grading of $\Bbbk[Y_0]$ is positive, so every quotient of $\Bbbk[Y_0]$ has a Hilbert series.

Example 2.3. In the situation of Example 2.2, the coordinate ring $\Bbbk[Y_0]$ has only the variables $f^{ij}_{\alpha\beta}$ that appear in the matrices below:

$$\begin{array}{c|c|ccc|cc|}
 & x^2_1 & x^1_1 & x^1_2 & x^1_3 & x^0_1 & x^0_2 \\ \hline
x^0_1 & f^{02}_{11} & f^{01}_{11} & f^{01}_{12} & f^{01}_{13} & 1 & \\
x^0_2 & f^{02}_{21} & f^{01}_{21} & f^{01}_{22} & f^{01}_{23} & & 1 \\ \hline
x^1_1 & f^{12}_{11} & 1 & & & & \\
x^1_2 & f^{12}_{21} & & 1 & & & \\
x^1_3 & f^{12}_{31} & & & 1 & & \\ \hline
x^2_1 & 1 & & & & & \\ \hline
\end{array}
\quad = \quad
\begin{array}{c|c|ccc|cc|}
 & c_1 & b_1 & b_2 & b_3 & a_1 & a_2 \\ \hline
a_1 & f^{02}_{11} & f^{01}_{11} & f^{01}_{12} & f^{01}_{13} & 1 & \\
a_2 & f^{02}_{21} & f^{01}_{21} & f^{01}_{22} & f^{01}_{23} & & 1 \\ \hline
b_1 & f^{12}_{11} & 1 & & & & \\
b_2 & f^{12}_{21} & & 1 & & & \\
b_3 & f^{12}_{31} & & & 1 & & \\ \hline
c_1 & 1 & & & & & \\ \hline
\end{array}$$

In this case, the variable f^{01}_{23} has ordinary weight $x^0_2 - x^1_3 = a_2 - b_3$ and exponential weight $x^0_2/x^1_3 = a_2/b_3$. $\square$

The multigrading giving rise to the ordinary quiver polynomials in Section 1 is derived from the action of GL on Hom. In the current context of GL_d acting on M_d, we identify $GL = \prod_{i=0}^n GL(V_i)$ as the Levi factor L, consisting of block diagonal matrices inside the parabolic subgroup $P \subset GL_d$, by sticking $GL(V_i)$ in the i^{th} diagonal block. This identifies the torus T of diagonal matrices in L with the torus T from Section 1.4.

Lemma 2.4. *The matrix Schubert variety $\overline{X}_{v(\mathbf{r})}$ is the closure inside M_d of the variety $P\mathcal{Z}(\Omega_\mathbf{r})$, by which we mean the image of $P \times \mathcal{Z}(\Omega_\mathbf{r})$ under multiplication.*

Proof. In general, if X_v is a Schubert subvariety of $P\backslash GL_d$, and $\tilde{X}_v$ is the preimage of X_v inside GL_d (under projection modulo P), then $\overline{X}_v$ equals the closure of $\tilde{X}_v$ inside M_d. Theorem 1.14 implies that $P\mathcal{Z}(\Omega_\mathbf{r})$ is dense inside the preimage of $X_{v(\mathbf{r})}$. $\square$

The quiver locus $\Omega_\mathbf{r}$ has only one copy of the torus $T \subset GL$ acting on it, as does its Zelevinsky image $\mathcal{Z}(\Omega_\mathbf{r})$ (conjugation by $T \subset L$). Multiplying by P on the left as in Lemma 2.4 frees up the left and right-hand tori (and even the left and right-hand Levi factors), allowing them to act independently. Consequently, we can define a *doubly* equivariant cohomology class on the smeared Zelevinsky image $\overline{X}_{v(\mathbf{r})}$ of $\Omega_\mathbf{r}$. In terms of multidegrees, we can now use the two sequences $\mathbf{x}$ and $\mathring{\mathbf{y}}$ of alphabets instead of only the $\mathbf{x}$ alphabets in ordinary quiver polynomials.

Definition 2.5. The **double quiver polynomial** is the ratio

$$\mathcal{Q}_\mathbf{r}(\mathbf{x} - \mathring{\mathbf{y}}) \quad = \quad \frac{\mathfrak{S}_{v(\mathbf{r})}(\mathbf{x_r} - \mathring{\mathbf{y}}_\mathbf{r})}{\mathfrak{S}_{v(Hom)}(\mathbf{x_r} - \mathring{\mathbf{y}}_\mathbf{r})}$$

of double Schubert polynomials in the concatenations of the two sequences of finite alphabets described in (2.2) and (2.3).

For the purposes of results such as Theorem 1.20 as well as Corollary 4.9 and Corollary 6.23, it is useful to think of the argument $\mathbf{x} - \mathring{\mathbf{y}}$ of the double quiver polynomial on the left side of Definition 2.5 as two *sequences* $\mathbf{x}$ and $\mathring{\mathbf{y}}$ of alphabets, even though the double Schubert polynomials on the right side more naturally take the argument $\mathbf{x_r} - \mathring{\mathbf{y}}_\mathbf{r}$ thought of simply as a pair of alphabets $\mathbf{x_r}$ and $\mathring{\mathbf{y}}_\mathbf{r}$. In addition, like ordinary quiver polynomials, it does no harm to think of double quiver polynomials as being evaluated on finite sequences of infinite alphabets, with all but the finitely many $\mathbf{x_r}$ and $\mathring{\mathbf{y}}_\mathbf{r}$ variables ignored.

The denominator $\mathfrak{S}_{v(Hom)}(\mathbf{x_r} - \mathring{\mathbf{y}}_\mathbf{r})$ should be regarded as a fudge factor, being simply the product of all weights $(x_* - y_*)$ of variables lying strictly above the block superantidiagonal; see also Remark 2.10. These variables lie in locations corresponding to $*$ entries in the diagram of every Zelevinsky permutation, so $\mathfrak{S}_{v(Hom)}$ obviously divides $\mathfrak{S}_{v(\mathbf{r})}$ (see Theorem 5.3).

The simple relation between double and ordinary quiver polynomials, to be presented in Theorem 2.9, justifies the notation $\mathcal{Q}_\mathbf{r}(\mathbf{x} - \mathring{\mathbf{x}})$ for the ordinary case.

2.3. Ratio formula for quiver polynomials

Our goal in this section is to express quiver polynomials as the specializations of double quiver polynomials obtained by setting $\mathbf{y}^j = \mathbf{x}^j$ for all j.

More generally, we shall express the quiver K-polynomial as a ratio of corresponding specialized **double Grothendieck polynomials**. These are usually defined by 'isobaric divided difference operators' (or 'Demazure operators'), which we review in Section 7.1.

However, since Grothendieck polynomials are tangential to the focus here, we do not review the background, which can be found in [LS82, Las90] or [KM03a], the latter being well-suited to the current applications. Readers unfamiliar with Grothendieck polynomials will lose nothing in the present context by taking the first sentence of Proposition 2.6 as a definition.

Double Grothendieck polynomials $\mathcal{G}_v(\mathcal{X}/\mathcal{Y})$ for permutations $v \in S_d$ take as arguments two alphabets $\mathcal{X}$ and $\mathcal{Y}$, each having size d. For purposes relating to quiver loci, $\mathcal{X}$ and $\mathcal{Y}$ are equated with sequences $\mathbf{x_r}$ and $\mathbf{y_r}$ of alphabets as in (2.2) and (2.3). Write

$$\mathring{\mathbf{x}}_{\mathbf{r}} \quad = \quad \mathbf{x}_{\mathbf{r}}^n, \ldots, \mathbf{x}_{\mathbf{r}}^0$$

to mean the alphabet obtained by block reversing the finite list $\mathbf{x_r}$ of alphabets from (2.2). It will arise when calculating the K-polynomials of opposite 'cells' in Schubert subvarieties of partial flag manifolds $P\backslash GL_d$, as we shall see shortly. Note that in the case of full flags, where $P = B$ is the Borel subgroup of lower-triangular matrices in GL_d, each alphabet in the list $\mathbf{x_r}$ consists of just one variable (as opposed to there being only one alphabet in the list), so the reversed sequence $\mathring{\mathbf{x}}_{\mathbf{r}}$ is really just a reversed alphabet in that case.

For notation, let $\mathcal{K}_M(\overline{X}; \mathcal{X}, \mathcal{Y})$ denote the K-polynomial of a multigraded subvariety $\overline{X} \subseteq M_d$ under the $\mathbb{Z}^{2d}$-grading from Section 2.2, and let $\mathcal{K}_Y(Z; \mathcal{X})$ denote the K-polynomial of a multigraded subvariety $Z \subseteq Y_0$ under the $\mathbb{Z}^d$-grading there.

Proposition 2.6. *The K-polynomial $\mathcal{K}_M(\overline{X}_v; \mathcal{X}, \mathcal{Y})$ of a matrix Schubert variety $\overline{X}_v \subseteq M_d$ is the double Grothendieck polynomial $\mathcal{G}_v(\mathcal{X}/\mathcal{Y})$. Intersecting $\overline{X}_{v(\mathbf{r})}$ with Y_0 yields the variety $\mathcal{Z}(\Omega_\mathbf{r})$, whose K-polynomial $\mathcal{K}_Y(\mathcal{Z}(\Omega_\mathbf{r}); \mathbf{x_r})$ is the specialization $\mathcal{G}_{v(\mathbf{r})}(\mathbf{x_r}/\mathring{\mathbf{x}}_{\mathbf{r}})$.*

Proof. The first sentence follows from [KM03a, Theorem A], one of the main results there. For the second sentence, observe that $\mathcal{Z}(\Omega_\mathbf{r})$ has the same codimension inside Y_0 as does $\overline{X}_{v(\mathbf{r})}$ inside M_d. Hence the equations defining Y_0 as a subvariety of M_d (namely $f - 1$ for diagonal f in antidiagonal blocks and $f - 0$ for the other variables in or below the main block antidiagonal) form a regular sequence on $\overline{X}_{v(\mathbf{r})}$. Coarsening the $\mathbb{Z}^{2d}$-grading on $\Bbbk[M_d]$ to the grading by $\mathbb{Z}^d$ in which $f_{\alpha\beta}^{ij}$ has ordinary weight $x_\alpha^i - x_\beta^j$, by setting $y_\beta^j = x_\beta^j$, makes the equations defining Y_0 become homogeneous, because the variables set equal to 1 have weight zero. Therefore, if $\mathcal{F}_{\bullet}$ is any $\mathbb{Z}^{2d}$-graded free resolution of $\Bbbk[\overline{X}_{v(\mathbf{r})}]$ over $\Bbbk[M_d]$, then $\mathcal{F}_{\bullet} \otimes_{\Bbbk[M_d]} \Bbbk[Y_0]$ is a $\mathbb{Z}^d$-graded free resolution of $\Bbbk[\mathcal{Z}(\Omega_\mathbf{r})]$ over $\Bbbk[Y_0]$. It follows that $\mathcal{K}_M(\overline{X}_{v(\mathbf{r})}; \mathbf{x_r}, \mathring{\mathbf{x}}_{\mathbf{r}}) = \mathcal{K}_Y(\mathcal{Z}(\Omega_\mathbf{r}); \mathbf{x_r})$, because the relevant $\mathbb{Z}^d$-graded Betti numbers coincide. Applying the first sentence of the Proposition and substituting $y_\beta^j = x_\beta^j$ now proves the second sentence. □

Theorem 2.7. *The K-polynomial $\mathcal{K}_{Hom}(\Omega_\mathbf{r}; \mathbf{x_r})$ of $\Omega_\mathbf{r}$ inside Hom is the ratio*

$$\mathcal{K}_{Hom}(\Omega_\mathbf{r}; \mathbf{x_r}) \quad = \quad \frac{\mathcal{G}_{v(\mathbf{r})}(\mathbf{x_r}/\mathring{\mathbf{x}}_{\mathbf{r}})}{\mathcal{G}_{v(Hom)}(\mathbf{x_r}/\mathring{\mathbf{x}}_{\mathbf{r}})}$$

of specialized double Grothendieck polynomials for $v(\mathbf{r})$ and $v(Hom)$.

Proof. The equality $H(\Omega_\mathbf{r}; \mathbf{x_r}) = \mathcal{K}_{Hom}(\Omega_\mathbf{r}; \mathbf{x_r}) H(Hom; \mathbf{x_r})$ of Hilbert series (which are well-defined by positivity of the grading of $\Bbbk[Hom]$ by $\mathbb{Z}^d$) follows from the definition of K-polynomial. For the same reason, we have $H(Hom; \mathbf{x_r}) = \mathcal{K}_Y(\mathcal{Z}(Hom); \mathbf{x_r}) H(Y_0; \mathbf{x_r})$ and also $H(\Omega_\mathbf{r}; \mathbf{x_r}) = \mathcal{K}_Y(\mathcal{Z}(\Omega_\mathbf{r}); \mathbf{x_r}) H(Y_0; \mathbf{x_r})$. Therefore

$$\mathcal{K}_Y(\mathcal{Z}(\Omega_\mathbf{r}); \mathbf{x_r}) H(Y_0; \mathbf{x_r}) \quad = \quad \mathcal{K}_{Hom}(\Omega_\mathbf{r}; \mathbf{x_r}) \mathcal{K}_Y(\mathcal{Z}(Hom); \mathbf{x_r}) H(Y_0; \mathbf{x_r}).$$

Canceling the factors of $H(Y_0;\mathbf{x_r})$ and dividing by $\mathcal{K}_Y(\mathcal{Z}(Hom);\mathbf{x_r})$ yields

$$\mathcal{K}_{Hom}(\Omega_{\mathbf{r}};\mathbf{x_r}) \quad = \quad \frac{\mathcal{K}_Y(\mathcal{Z}(\Omega_{\mathbf{r}});\mathbf{x_r})}{\mathcal{K}_Y(\mathcal{Z}(Hom);\mathbf{x_r})}.$$

Now apply Theorem 1.14 and Proposition 2.6 to complete the proof. □

Remark 2.8. Here is the geometric content of Proposition 2.6. The torus $T \times T$ of dimension $2d$ acts on M_d, the left factor by multiplication on the left, and the right factor by inverse multiplication on the right. Viewing $\mathcal{G}_v(\mathbf{x}/\mathbf{y})$ as an equivariant class for this action of $T \times T$, specializing the $\mathbf{y}$ variables to $\mathring{\mathbf{x}}$ does not arise by restricting the $T \times T$ action to some subtorus. Rather, it uses a different d-dimensional torus T entirely. The main point is that the opposite big cell is an orbit for an *opposite* parabolic group, whose torus has been conjugated by the identity-in-each-antidiagonal-block permutation $w_0\mathbf{w}_0$ from the usual one.

In more detail, the subvariety $Y_0 \subset GL_d$ is not stable under the usual action of $T \times T$ above, because we need to preserve the 1s in antidiagonal blocks. Instead, it is stable under the action of the diagonal torus T inside the copy of $T \times T$ whose left factor acts by usual multiplication on the left, but where a diagonal matrix τ in the right factor acts as right multiplication by $w_0\mathbf{w}_0\tau^{-1}\mathbf{w}_0w_0$. Thus the diagonal T acts with the weight on the variable at position (α,β) in block row i and block column j (from the *right*) being $x^i_\alpha - x^j_\beta$.

As a reality check, note that this action of the right-hand T factor is indeed the restriction of the "opposite" parabolic action on $P\backslash GL_d$ to its torus. Indeed, our "usual" parabolic acting on $P\backslash GL_d$ is the block upper-triangular $P_+ \subset GL_d$ acting by inverse multiplication on the right (we want $P \times P_+$ to act via a left group action), so the opposite parabolic is $w_0\mathbf{w}_0P_+\mathbf{w}_0w_0$ acting by inverse multiplication on the right.

One might observe that the torus in $w_0\mathbf{w}_0P_+\mathbf{w}_0w_0$ acts on $P\backslash GL_d$ simply by inverse right multiplication, not by twisted conjugation. All this means is that multiplying by a diagonal matrix $w_0\mathbf{w}_0\tau^{-1}\mathbf{w}_0w_0$ moves the cosets in $P\backslash GL_d$ correctly. However, the subvariety $Y_0 \subset GL_d$ consists of specific coset *representatives* for points in the opposite big cell of $P\backslash GL_d$, and we need to know which coset representative corresponds to the image of a matrix after right multiplication by $w_0\mathbf{w}_0\tau^{-1}\mathbf{w}_0w_0$. For this, we have to left-multiply by τ, bringing the antidiagonal blocks back to a list of identity matrices without altering the coset.

Next comes what we have been after from the beginning, the consequence of Theorem 2.7 on multidegrees (or equivariant cohomology or Chow groups, by Proposition 1.19).

Theorem 2.9. *The quiver polynomial $\mathcal{Q}_{\mathbf{r}}(\mathbf{x}-\mathring{\mathbf{x}})$ for a rank array $\mathbf{r}$ equals the ratio*

$$\mathcal{Q}_{\mathbf{r}}(\mathbf{x}-\mathring{\mathbf{x}}) \quad = \quad \frac{\mathfrak{S}_{v(\mathbf{r})}(\mathbf{x_r}-\mathring{\mathbf{x}}_{\mathbf{r}})}{\mathfrak{S}_{v(Hom)}(\mathbf{x_r}-\mathring{\mathbf{x}}_{\mathbf{r}})}$$

of specialized double Schubert polynomials. In other words, the ordinary quiver polynomial $\mathcal{Q}_{\mathbf{r}}(\mathbf{x}-\mathring{\mathbf{x}})$ is the $\mathring{\mathbf{y}} = \mathring{\mathbf{x}}$ specialization of the double quiver polynomial $\mathcal{Q}_{\mathbf{r}}(\mathbf{x}-\mathring{\mathbf{y}})$.

Proof. Clear denominators in Theorem 2.7, substitute $1-x$ for every occurrence of each variable x, take lowest degree terms, and divide the result by $\mathfrak{S}_{v(Hom)}(\mathbf{x_r}-\mathring{\mathbf{x}}_{\mathbf{r}})$. Note that $\mathfrak{S}_{v(Hom)}(\mathbf{x_r}-\mathring{\mathbf{x}}_{\mathbf{r}})$ is nonzero, being simply the product of the $\mathbb{Z}^d$-graded weights of the variables in $\Bbbk[Y_0]$ strictly above the block superantidiagonal. □

Remark 2.10. The proofs of both Proposition 2.6 and Theorem 2.7 could as well have been presented entirely in geometric language, referring to pullbacks and pushforwards in equivariant K-theory rather than ratios of numerators of Hilbert series. In this geometric language, the denominator in Theorem 2.9 results via the push-pull formula from the equivariant Euler class of the normal bundle to $\mathcal{Z}(Hom)$ in Y_0, or equivalently, the equivariant Euler class of the normal bundle to $\overline{X}_{v(Hom)}$ in M_d.

In view of Theorem 2.9, we consider Definition 2.5 to be our first positive formula for double quiver polynomials, since it expresses them in terms of known polynomials (Schubert polynomials) that depend on combinatorial data (the Zelevinsky permutation) associated to given ranks $\mathbf{r}$. This is not so bad: we could have (without motivation) defined double quiver polynomials starting from the Buch–Fulton formula [BF99] by using independent dual bundles in the second half of each argument in their Schur functions, and then Definition 2.5 would be a theorem (but not one in this paper; see Remark 6.25). As we have chosen to present things, the double version of the Buch–Fulton formula will come out as a theorem instead, in Theorems 7.10, 7.21, and 8.22; see also Corollary 8.23.

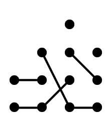
Section 3. Lacing diagrams

3.1. Geometry of lacing diagrams

In this section we use diagrams derived from those in [AD80] to describe the orbits of certain groups related to GL. Recall that we have fixed bases for the vector spaces $V_0, \ldots, V_n$, and this essentially amounted to fixing a maximal torus T inside the group GL acting on V. Suppose that, in this basis, the quiver representation $\phi \in Hom$ is represented by a sequence $\mathbf{w} = (w_1, \ldots, w_n)$ of n partial permutation matrices with respect to the above bases. We shall freely identify ϕ with $\mathbf{w}$.

Each list $\mathbf{w} = (w_1, \ldots, w_n) \in Hom$ of partial permutation matrices can be represented by a (nonembedded) graph in the plane called a **lacing diagram**. The vertex set of the graph consists of r_i bottom-justified dots in column i for $i = 0, \ldots, n$, with an edge connecting the dot at height α (from the bottom) in column $i-1$ with the dot at height β in column i if and only if the entry of w_i at (α, β) is 1. A **lace** is a connected component of a lacing diagram. An (i,j)-lace is one that starts in column i and ends in column j. We shall also identify $\mathbf{w} \in Hom$ with its associated lacing diagram.

Example 3.1. Here is a lacing diagram for the dimension vector $(2,3,4,3)$ and its corresponding list of partial permutation matrices.



$$\longleftrightarrow \quad \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right)$$

This lacing diagram has rank array $\mathbf{r}$ and lace array $\mathbf{s}$ from Example 1.5. □

Lemma 3.2. *Fix a rank array* $\mathbf{r}$ *that can occur for* V*, and let* $\mathbf{s}$ *be the lace array related to* $\mathbf{r}$ *by* (1.2)*. For a lacing diagram* $\mathbf{w} \in Hom$*, the following are equivalent:*

1. $\mathbf{w}$ *has rank array* $\mathbf{r}$.
2. *The quiver locus* $\Omega_{\mathbf{r}}$ *is the closure* $\overline{GL \cdot \mathbf{w}}$ *of the orbit of* GL *through* $\mathbf{w}$.
3. s_{ij} *is the number of* (i,j)*-laces in* $\mathbf{w}$ *for all* $0 \le i \le j \le n$.

Proof. This is a consequence of Proposition 1.2 plus Lemma 1.3 and its proof. □

Lemma 3.2 identifies orbits indexed by a collection of lacing diagrams. It will be important for us to consider another group action, for which each lacing diagram uniquely determines its own orbit. As a starting point, we have the following standard result; see

[Ful92] or [KM03a]. As a matter of notation, we write $w' \leq w$ for partial permutation matrices w and w' of the same shape if their associated permutations satisfy $\widetilde{w}' \leq \widetilde{w}$ in Bruhat order.

Lemma 3.3. *Let the product $B_-(k) \times B_+(\ell)$ of lower- and upper-triangular Borel groups inside $GL_k \times GL_\ell$ act on matrices $Z \in M_{k\ell}$ by $(b_-, b_+) \cdot Z = b_- Z b_+^{-1}$. The matrix Schubert variety $\overline{X}_w$ for a partial permutation matrix w is the closure in $M_{k\ell}$ of the orbit of $B_-(k) \times B_+(\ell)$ through w. Moreover, $\overline{X}_{w'} \subset \overline{X}_w$ if and only if $w' \leq w$.*

The observation in Lemma 3.3 easily extends to sequences of partial permutations—that is, lacing diagrams $\mathbf{w} = (w_1, \ldots, w_n)$—using the subgroup

$$B_+ \times B_- \quad = \quad \prod_{i=0}^{n} B_+(V_i) \times B_-(V_i)$$

of the group GL^2 from (1.1), where $B_+(V_i)$ and $B_-(V_i)$ are the upper and lower triangular Borel groups in $GL(V_i) = GL_{r_i}$, respectively. The action of $B_+ \times B_-$ on Hom is inherited from the GL^2 action after (1.1). We also need to consider the subgroup

$$B_+ \times_T B_- \quad = \quad \prod_{i=0}^{n} B_+(V_i) \times_{T(V_i)} B_-(V_i) \quad \subset \quad B_+ \times B_-, \tag{3.1}$$

where $B_+(V_i) \times_{T(V_i)} B_-(V_i)$ consists of those pairs $(b_+, b_-) \in B_+(V_i) \times B_-(V_i)$ in which the diagonals of b_+ and b_- are equal.

Proposition 3.4. *The groups $B_+ \times_T B_-$ and $B_+ \times B_-$ have the same orbits on Hom, and each orbit contains a unique lacing diagram $\mathbf{w}$. The closure in Hom of the orbit through $\mathbf{w}$ is the product*

$$\mathcal{O}(\mathbf{w}) \quad = \quad \prod_{j=1}^{n} \overline{X}_{w_j}$$

of matrix Schubert varieties, with $\overline{X}_{w_j}$ lying inside the factor $\mathrm{Hom}(V_{j-1}, V_j)$ of Hom.

Proof. By Lemma 3.3 the action of $B_+ \times B_-$ on Hom has orbit representatives given by lacing diagrams. Hence it suffices to show that every orbit of the subgroup $B_+ \times_T B_- \subset B_+ \times B_-$ acting on Hom contains a lacing diagram.

Note that $B_+ \times_T B_- = T \cdot U$ for the maximal torus $T \subset GL$ (diagonally embedded in GL^2) and the unipotent group $U = \prod_i U_+(V_i) \times U_-(V_i)$, where $U_+(V_i) \subset B_+(V_i)$ and $U_-(V_i) \subset B_-(V_i)$ are the unipotent radicals (unitriangular matrices). Acting on the factor $\mathrm{Hom}(V_{i-1}, V_i)$ by the group $U_-(V_{i-1}) \times U_+(V_i)$ via the action of GL^2 after (1.1), any matrix can be transformed into a monomial matrix (one having at most one nonzero element in each row and each column) by performing downward row elimination and rightward column elimination with no scaling of the pivot elements. Thus each orbit of U in Hom contains a list of monomial matrices $\phi = (\phi_1, \ldots, \phi_n)$.

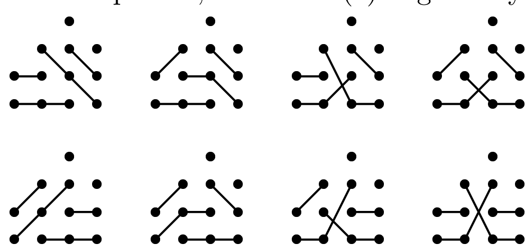
Let us view a list of monomial matrices as a lacing diagram whose edges have been labeled with nonzero elements of $\Bbbk$. The underlying lacing diagram $\mathbf{w}$ is determined by the list of partial permutation matrices obtained by changing each nonzero entry of ϕ to a 1, and if the coordinate $f^i_{\alpha\beta}$ is nonzero on ϕ, then its value $f^i_{\alpha\beta}(\phi)$ is the label on the edge connecting the height α vertex in column $i-1$ to the height β vertex in column i. It now suffices to find $\tau \in T$ with $\tau\phi = \mathbf{w}$. We construct τ by setting the j^{th} diagonal entry τ_i^j in its i^{th}

factor τ_i to be the product of the edge labels that are to the left of column i on the lace in ϕ containing the height j vertex in column i. □

The entry (i,j) of a partial permutation matrix is called an **inversion** if there is no 1 due north or due west of it, i.e., there is no 1 in position (i',j) for $i' < i$ or (i,j') for $j' < j$. Define the **length** $\ell(w)$ of a partial permutation matrix w to be the number of inversions of w. Equivalently, this is the length of any permutation of minimal length whose matrix contains w as its northwest corner. Given a list $\mathbf{w} = (w_1, \ldots, w_n)$ of partial permutation matrices, define the length of $\mathbf{w}$ to be $\ell(\mathbf{w}) = \sum_{j=1}^{n} \ell(w_j)$.

Definition 3.5. Denote by $W(\mathbf{r})$ the subset of lacing diagrams in *Hom* with rank array $\mathbf{r}$ and length $d(\mathbf{r}) = \mathrm{codim}(\Omega_{\mathbf{r}})$ from Corollary 1.16.

Example 3.6. For $\mathbf{r}$ as in Example 1.5, the set $W(\mathbf{r})$ is given by



The codimension of this quiver locus $\Omega_{\mathbf{r}}$ is 7, which will be reflected in the number of 'virtual crossings' in these diagrams, as defined in this next section. □

3.2. Minimum length lacing diagrams

In Theorem 3.8 of this section we give a combinatorial characterization of the lacing diagrams $W(\mathbf{r})$ from Definition 3.5.

For partial permutation matrix lists (lacing diagrams) $\mathbf{w}$ and $\mathbf{w}'$, write $\mathbf{w}' \leq \mathbf{w}$ if $w'_j \leq w_j$ in Bruhat order for all $1 \leq j \leq n$.

To compute the length of a lacing diagram we introduce the notion of a **crossing**. The vertex at height i (from the bottom) in column k is written as the ordered pair (i,k). For the purposes of the following definition one may imagine that every (a,b)-lace is extended so that its left endpoint is connected by an invisible edge to an invisible vertex $(\infty, a-1)$ and its right endpoint is connected by an invisible edge to an invisible vertex $(\infty, b+1)$. Two laces in a lacing diagram are said to **cross** if one contains an edge of the form $(i,k-1) \to (j,k)$ and the other an edge of the form $(i',k-1) \to (j',k)$ such that $i > i'$ and $j < j'$. Such a crossing shall be labeled by the triple (j,j',k). Looking at these two edges, this crossing is classified as follows.

Call the crossing **ordinary** if neither edge is invisible; **left-virtual** if the edge touching (j,k) is invisible but the other edge is not; **right-virtual** if the edge touching (j',k) is invisible but the other is not; and **doubly virtual** if both edges are invisible. Two laces have a doubly virtual crossing (necessarily unique) if and only if one is an (a,b)-lace and the other a $(b+1,c)$-lace for some $a \leq b < c$.

For the next proposition and theorem, a **crossing** refers to any of the above four kinds of crossings.

Proposition 3.7. *The number of crossings in any lacing diagram* $\mathbf{w} \in$ *Hom is* $\ell(\mathbf{w})$.

Proof. Fix $1 \leq k \leq n$. The inversions (i,j) of w_k come in four flavors:

1. w_k has a 1 in positions (i,j') and (i',j) with $j' > j$ and $i' > i$.

2. w_k has a 1 in position (i,j') with $j'>j$ but no 1 in the j^{th} column.
3. w_k has a 1 in position (i',j) with $i'>i$ but no 1 in the i^{th} row.
4. w_k has no 1 in either the i^{th} row or j^{th} column.

Consider the two laces in $\mathbf{w}$ containing the vertices $(i,k-1)$ and (j,k) respectively; they are distinct since (i,j) is an inversion of w_k. Then the four kinds of inversions correspond respectively to crossings (j,j',k) of these two laces that are ordinary, left-virtual, right-virtual, and doubly virtual respectively. □

It is a bit difficult to use this proposition to identify by eye the lacing diagrams in $W(\mathbf{r})$, because it is hard to make an accurate count of virtual crossings. The following theorem gives a more easily checked criterion (though it still involves virtual crossings).

Theorem 3.8. *Suppose the lacing diagram $\mathbf{w}$ has rank array $\mathbf{r}$. Then $\ell(\mathbf{w})\geq d(\mathbf{r})$, with equality if and only if*

(C1) *no pair of laces in $\mathbf{w}$ crosses more than once; and*
(C2) *two laces in $\mathbf{w}$ have no crossings if both start or both end in the same column.*

Theorem 3.8 follows immediately from Proposition 3.7 and the two following results, which may be of independent interest.

Proposition 3.9. *Suppose the lacing diagram $\mathbf{w}$ has rank array $\mathbf{r}$ but fails to satisfy (C1) or (C2). Then there is a lacing diagram $\mathbf{w}'$ with rank array $\mathbf{r}$ such that $\mathbf{w}'<\mathbf{w}$.*

Proof. Suppose first that $\mathbf{w}$ has a pair of laces L and L' that cross at least twice. Then undoing adjacent crossings in these two laces produces the desired lacing diagram $\mathbf{w}'$. More precisely, consider a pair of crossings (j,j',k) and (j_1,j_1',k_1) of L and L' with $k<k_1$ and k_1-k minimal. Let $\mathbf{w}'$ be the lacing diagram obtained from $\mathbf{w}$ by changing the vertex sets of L and L' by trading vertices in columns c such that $k\leq c<k_1$ (and leaving all other laces unchanged). It is straightforward to verify that $\mathbf{w}'$ has rank array $\mathbf{r}$, that $w_k'<w_k$, that $w_{k_1}'<w_{k_1}$, and that $w_c'=w_c$ for $c\notin\{k,k_1\}$.

Suppose $\mathbf{w}$ has an (a,b)-lace L and an (a,b')-lace L' that cross (the case that two crossing laces end in the same column is entirely similar). Let (j,j',k) be the crossing of L and L' with k minimal. As before we may change the vertex sets of L and L' by trading vertices in columns c such that $a\leq c<k$, resulting in the lacing diagram $\mathbf{w}'$. Then $\mathbf{w}'$ has rank array $\mathbf{r}$, with $w_k'<w_k$, and $w_c'=w_c$ for $a\leq c<k$. □

Proposition 3.10. *If $\mathbf{w}$ has rank array $\mathbf{r}$ then $\mathbf{w}$ has at least $d(\mathbf{r})$ crossings. Moreover if $\mathbf{w}$ satisfies (C1) and (C2) then $\mathbf{w}$ has exactly $d(\mathbf{r})$ crossings.*

Proof. Consider any (a,b)-lace L and (a',b')-lace L' in $\mathbf{w}$ such that $a<a'$ and $b<b'$. It is easy to check that the laces cross at least once if and only if $b\geq a'-1$. From this and Corollary 1.16 it follows that $\mathbf{w}$ has at least $d(\mathbf{r})$-crossings. Suppose $\mathbf{w}$ has rank array $\mathbf{r}$ and satisfies (C1) and (C2). Let L be an (a,b) lace and L' an (a',b')-lace with $a\leq a'$. If $a=a'$ the laces cannot cross by (C2), so suppose $a<a'$. If $b>b'$ then the laces cannot cross, for if they did then they would have to cross twice, violating (C1). If $b=b'$ then again by (C2) the laces cannot cross. Finally, for $b<b'$, the laces cross at least once and only if $b\geq a'-1$, and they cannot cross twice by (C1). This gives precisely the desired $d(\mathbf{r})$ crossings. □

Section 4. Degeneration of quiver loci

4.1. Orbit degenerations

Tracing the degeneration of a group acting on a space can allow one to degenerate the orbits of the group. More precisely, the next result says that under appropriate conditions, an action of a one-dimensional flat family $\overline{G}$ of groups on the general fiber X of a flat family $\overline{X}$ extends to a $\overline{G}$-action on all of $\overline{X}$. This implies, in particular, that the special fiber of $\overline{G}$ acts on the special fiber of $\overline{X}$.

Proposition 4.1. *Let the group scheme Γ_0 act on affine n-space $\mathbb{A}^n$ (over the integers, say). Fix a regular scheme $\overline{A}$ of dimension 1 along with a closed point $a \in \overline{A}$, and consider the family $\overline{V} = \mathbb{A}^n \times \overline{A}$. Suppose $\overline{G}$ is a flat closed subgroup scheme of $\overline{\Gamma} \cong \Gamma_0 \times \overline{A}$ over $\overline{A}$. Let V, G, and Γ denote the restrictions of $\overline{V}$, $\overline{G}$, and $\overline{\Gamma}$ to schemes over $A = \overline{A} \smallsetminus a$. If $X \subseteq V$ is a G-subscheme flat over A, then there is a unique equivariant completion of X to a closed $\overline{G}$-subscheme $\overline{X} \subseteq \overline{V}$ flat over $\overline{A}$.*

Proof. The flat family $X \subseteq V$ has a unique extension to a flat family $\overline{X}$ whose total space is closed inside $\overline{V}$; this is standard, taking $\overline{X}$ to be the closure of X inside $\overline{V}$. (The ideal sheaf $\mathcal{I}_{\overline{X}}$ is thus $\mathcal{I}_X \cap \mathcal{O}_{\overline{V}}$, the intersection taking place inside $\mathcal{O}_V$.) It remains to show that the map $\overline{G} \times_{\overline{A}} \overline{X} \to \overline{V}$ has image contained in $\overline{X}$ (more formally, the map factors through the inclusion $\overline{X} \hookrightarrow \overline{V}$).

The product $G \times_A X$ is a closed subscheme of $\Gamma \times_A V$, which is itself an open subscheme of $\overline{\Gamma} \times_{\overline{A}} \overline{V}$. As such we can take the closure $\overline{G \times_A X}$ inside $\overline{\Gamma} \times_{\overline{A}} \overline{V}$. The composite map

$$\overline{G \times_A X} \longrightarrow \overline{\Gamma} \times_{\overline{A}} \overline{V} \longrightarrow \overline{V}$$

lands inside $\overline{X}$, because (i) removing all the overbars in the above composite map yields a map whose image is precisely the G-scheme X, and (ii) $\overline{X}$ is closed in $\overline{V}$. It suffices to show now that $\overline{G \times_A X} = \overline{G} \times_{\overline{A}} \overline{X}$ as subschemes of $\overline{\Gamma} \times_{\overline{A}} \overline{V}$.

Clearly $\overline{G \times_A X} \subseteq \overline{G} \times_{\overline{A}} \overline{X}$, because $\overline{G} \times_{\overline{A}} \overline{X}$ is closed in $\overline{V}$ and contains $G \times_A X$. Moreover, $\overline{G} \times_{\overline{A}} \overline{X}$ is flat over $\overline{A}$ (because both $\overline{G}$ and $\overline{X}$ are) and agrees with $G \times_A X$ over A. Thus $\overline{G} \times_{\overline{A}} \overline{X}$ and $\overline{G \times_A X}$ both equal the unique extension of the family $G \times_A X \to A$ to a flat family over $\overline{A}$ that is closed inside $\overline{\Gamma} \times_{\overline{A}} \overline{V}$. □

In our case, the group degeneration is a group scheme $\widetilde{GL}$ over $\mathbb{A}^1$ (parametrized by the coordinate t on $\mathbb{A}^1$). $\widetilde{GL}$ is defined as follows via Gröbner degeneration of the diagonal embedding

$$\Delta : GL \hookrightarrow GL^2.$$

Let $\mathbf{g}_i = (g_i^{\alpha\beta})_{\alpha,\beta=1}^{r_i}$ be coordinates on $GL(V_i)$ for each index i, so the coordinate ring

$$\Bbbk[GL(V_i)^2] \quad = \quad \Bbbk[\overleftarrow{\mathbf{g}_i}, \overrightarrow{\mathbf{g}_i}][\overleftarrow{\det}^{-1}, \overrightarrow{\det}^{-1}]$$

is the polynomial ring with the determinants of the left and right pointing variables inverted. The ideal of the i^{th} component of the diagonal embedding,

$$I_{\Delta_i} \quad = \quad \langle \overleftarrow{g_i}^{\alpha\beta} - \overrightarrow{g_i}^{\alpha\beta} \mid \alpha, \beta = 1, \ldots, r_i \rangle,$$

simply sets the right and left pointing coordinates equal to each other in the i^{th} component. Choose a weight ω on the variables in the polynomial ring $\Bbbk[\overleftarrow{\mathbf{a}_i}, \overrightarrow{\mathbf{a}_i}]$ so that

$$\omega(\overleftarrow{g_i}^{\alpha\beta}) = \alpha - \beta \quad \text{and} \quad \omega(\overrightarrow{g_i}^{\alpha\beta}) = \beta - \alpha.$$

Using this weight we get a family

$$\widetilde{I}_{\Delta_i} \;=\; \langle \overleftarrow{g_i}^{\alpha\beta} - \overrightarrow{g_i}^{\alpha\beta} t^{2\alpha-2\beta} \mid \alpha \geq \beta \rangle \;+\; \langle \overleftarrow{g_i}^{\alpha\beta} t^{2\beta-2\alpha} - \overrightarrow{g_i}^{\alpha\beta} \mid \alpha \leq \beta \rangle$$

of ideals over $\mathbb{A}^1$, whose coordinate we call t.

Lemma 4.2. *$\widetilde{I}_\Delta = \sum_{i=0}^n \widetilde{I}_{\Delta_i}$ defines a flat group subscheme $\widetilde{GL}$ of $GL^2 \times \mathbb{A}^1$ whose special fiber at $t = 0$ is the group $GL(0) = B_+ \times_T B_-$ from* (3.1).

Proof. The special fiber $GL(0)$ over $t = 0$ is defined by $I_\Delta(0)$, which coincides with

$$\mathsf{in}_\omega(I_\Delta) \;=\; \sum_{i=0}^{n} \Big(\langle \overleftarrow{g_i}^{\alpha\beta} \mid \alpha > \beta \rangle + \langle \overrightarrow{g_i}^{\alpha\beta} \mid \alpha < \beta \rangle + \langle \overleftarrow{g_i}^{\alpha\alpha} - \overrightarrow{g_i}^{\alpha\alpha} \mid \alpha = 1, \ldots, r_i \rangle \Big)$$

by definition of the initial ideal in_ω for the weight ω [Eis95, Chapter 15]. Thus $\widetilde{GL}$ is a flat family (specifically, a Gröbner degeneration) over $\mathbb{A}^1$. Flatness still holds after inverting the left and right pointing determinants.

It remains to show that $\widetilde{GL}$ is a subgroup scheme over $\mathbb{A}^1$, i.e. that fiberwise multiplication in $GL^2 \times \mathbb{A}^1$ preserves $\widetilde{GL} \subset GL^2 \times \mathbb{A}^1$. This can be done set-theoretically because (for instance) all fibers of $\widetilde{GL}$ are geometrically reduced.

The fiber $GL(0)$ over $t = 0$ is the subgroup $B_+ \times_T B_- \subset GL^2$. On the other hand, the union of all fibers with $t \neq 0$ is isomorphic to $GL \times (\mathbb{A}^1 \smallsetminus \{0\})$ as a group subscheme of $GL^2 \times (\mathbb{A}^1 \smallsetminus \{0\})$, the isomorphism being

$$(\overleftarrow{\gamma}\,,\, \overrightarrow{\gamma}) \times \tau \quad\mapsto\quad (\tau_{\bullet} \overleftarrow{\gamma} \tau_{\bullet}^{-1}\,,\, \tau_{\bullet}^{-1} \overrightarrow{\gamma} \tau_{\bullet}) \tag{4.1}$$

where $\tau_{\bullet} \in T$ has $T(V_i)$ component τ_i with diagonal entries $1, \tau, \tau^2, \ldots, \tau^{r_i - 1}$. Indeed, the matrices $(\tau_{\bullet} \overleftarrow{\gamma} \tau_{\bullet}^{-1}\,,\, \tau_{\bullet}^{-1} \overrightarrow{\gamma} \tau_{\bullet})$ are precisely those satisfying the relations defining the ideal $\widetilde{I}_\Delta$, which can be rewritten more symmetrically as

$$I_\Delta(t \neq 0) \;=\; \langle t^{-\alpha}\, \overleftarrow{g_i}^{\alpha\beta}\, t^{\beta} - t^{\alpha}\, \overrightarrow{g_i}^{\alpha\beta}\, t^{-\beta} \mid i = 0, \ldots, n \rangle$$

away from $t = 0$. □

4.2. Quiver degenerations

The orbits we wish to degenerate are of course the quiver loci. Recall the notation from Section 1.1 regarding the indexing on variables $f^i_{\alpha\beta}$ in the coordinate ring $\mathbb{k}[\mathit{Hom}]$.

Definition 4.3. For $p(\mathbf{f}) \in \mathbb{k}[\mathit{Hom}]$ and a coordinate t on $\mathbb{A}^1 = \mathbb{k}$, let

$$p_t(\mathbf{f}) \;=\; \frac{p(t^{\alpha+\beta} f^i_{\alpha\beta})}{\text{as much } t \text{ as possible}}.$$

Set $\widetilde{I}_{\mathbf{r}} = \langle p_t(\mathbf{f}) \mid p(\mathbf{f}) \in I_{\mathbf{r}} \rangle \subseteq \mathbb{k}[\mathit{Hom} \times \mathbb{A}^1]$, and let $\widetilde{\Omega}_{\mathbf{r}} \subseteq \mathit{Hom} \times \mathbb{A}^1$ be the zero scheme of $\widetilde{I}_{\mathbf{r}}$. For $\tau \in \mathbb{k}$ let $\Omega_{\mathbf{r}}(\tau) \subset \mathit{Hom}$ be the zero scheme of the ideal $I_{\mathbf{r}}(\tau) \subset \mathbb{k}[\mathit{Hom}]$ obtained from $\widetilde{I}_{\mathbf{r}}$ by setting $t = \tau$. The special fiber $\Omega_{\mathbf{r}}(0) \subset \mathit{Hom}$ is called the **quiver degeneration**, and we refer to $\widetilde{\Omega}_{\mathbf{r}}$ as the **family** of $\Omega_{\mathbf{r}}(0)$.

Lemma 4.4. *The family $\widetilde{\Omega}_{\mathbf{r}} \subseteq \mathit{Hom} \times \mathbb{A}^1$ is flat over $\mathbb{A}^1$.*

Proof. The polynomial $p_t(\mathbf{f})$ is obtained from $p(\mathbf{f})$ by homogenizing with respect to the weight function on $\mathbb{k}[\mathit{Hom}]$ that assigns weight $-\alpha - \beta$ to the variable $f^i_{\alpha\beta}$. The resulting initial ideal is $I_{\mathbf{r}}(0)$, so [Eis95, Theorem 15.17] applies. □

Note that $I_{\mathbf{r}}(0)$ is rarely a monomial ideal. Thus $\Omega_{\mathbf{r}}(1) \rightsquigarrow \Omega_{\mathbf{r}}(0)$ is only a "partial" Gröbner degeneration of $\Omega_{\mathbf{r}} = \Omega_{\mathbf{r}}(1)$.

Proposition 4.5. *The group scheme $\widetilde{GL}$ over $\mathbb{A}^1$ acts (fiberwise over $\mathbb{A}^1$) on the flat family $\widetilde{\Omega}_{\mathbf{r}}$ over $\mathbb{A}^1$.*

Proof. This will follow from Proposition 4.1 applied to

$$\mathbb{A}^n = V, \quad \Gamma_0 = GL^2, \quad \overline{A} = \mathbb{A}^1, \quad y = 0, \quad \overline{G} = \widetilde{GL}, \quad \text{and} \quad \overline{X} = \widetilde{\Omega}_{\mathbf{r}}$$

by Lemmas 4.4 and 4.2, as soon as we show that $\widetilde{GL}(\tau \neq 0)$ acts on $\widetilde{\Omega}_{\mathbf{r}}(\tau \neq 0)$, these being the restrictions of $\widetilde{GL}$ and $\widetilde{\Omega}_{\mathbf{r}}$ to families over $\mathbb{A}^1 \smallsetminus \{0\}$.

Resume the notation from the last paragraph of the proof of Lemma 4.2. Observe that $\Omega_{\mathbf{r}} \times (\mathbb{A}^1 \smallsetminus \{0\}) \to \widetilde{\Omega}_{\mathbf{r}}(\tau \neq 0)$ is an isomorphism of varieties under the map

$$(\phi, \tau) \;=\; ((\phi_i)_{i=1}^n, \tau) \;\mapsto\; (\tau_{i-1}^{-1}\phi_i\tau_i^{-1}) \;=:\; \tau_\bullet^{-1}\phi\tau_\bullet^{-1}. \tag{4.2}$$

Now combine Eq. (4.1) and (4.2),

$$\begin{aligned}(\tau_\bullet \overleftarrow{\gamma}\tau_\bullet^{-1}, \tau_\bullet^{-1}\overrightarrow{\gamma}\tau_\bullet)\cdot(\tau_\bullet^{-1}\phi\,\tau_\bullet^{-1}) \;&=\; (\tau_{i-1}^{-1}\overrightarrow{\gamma}_{i-1}\phi_i\overleftarrow{\gamma}_i^{-1}\tau_i^{-1})\\ &=\; \tau_\bullet^{-1}[(\overleftarrow{\gamma}, \overrightarrow{\gamma})\cdot\phi]\tau_\bullet^{-1},\end{aligned}$$

and use that $(\overleftarrow{\gamma}, \overrightarrow{\gamma})\cdot\phi \in \Omega_{\mathbf{r}}$ to get the desired result. □

Theorem 4.6, the main result of Section 4, relates the quiver degeneration $\Omega_{\mathbf{r}}(0)$ to matrix Schubert varieties as in Proposition 3.4.

Theorem 4.6. *Each irreducible component of $\Omega_{\mathbf{r}}(0)$ is a possibly nonreduced product $\mathcal{O}(\mathbf{w})$ of matrix Schubert varieties for a lacing diagram $\mathbf{w} \in Hom$.*

Proof. By Proposition 4.5, the special fiber $\Omega_{\mathbf{r}}(0)$ is stable under the action of $GL(0) = B_+ \times_T B_-$ on Hom, and hence is a union of orbit closures for this group. By Proposition 3.4 these orbit closures are direct products of matrix Schubert varieties. □

Remark 4.7. The methods of the current and the previous sections can be applied with no essential changes to the more general case of nilpotent cyclic quivers.

4.3. Multidegrees of quiver degenerations

Knowing multidegrees of quiver degenerations gives us multidegrees of quiver loci:

Lemma 4.8. *The quiver degeneration $\Omega_{\mathbf{r}}(0)$ has the same multidegree as $\Omega_{\mathbf{r}}$.*

Proof. By Lemma 4.4 we may apply 'Degeneration' in [KM03a, Theorem 1.7.1]. □

Now we can conclude from Theorem 4.6 that the quiver polynomial has a positive expression in terms of products of double Schubert polynomials. This will provide a "lower bound" on the quiver polynomial $\mathcal{Q}_{\mathbf{r}}(\mathbf{x} - \mathring{\mathbf{x}})$, in Proposition 4.11.

Corollary 4.9. *Let $\mathfrak{S}_{\mathbf{w}}(\mathbf{x} - \mathring{\mathbf{x}})$ for a lacing diagram $\mathbf{w}$ denote the corresponding product $\prod_{j=1}^n \mathfrak{S}_{w_j}(\mathbf{x}^{j-1} - \mathbf{x}^j)$ of double Schubert polynomials. Then*

$$\mathcal{Q}_{\mathbf{r}}(\mathbf{x} - \mathring{\mathbf{x}}) \;=\; \sum_{\mathbf{w} \in Hom} c_{\mathbf{w}}(\mathbf{r})\,\mathfrak{S}_{\mathbf{w}}(\mathbf{x} - \mathring{\mathbf{x}}),$$

where $c_{\mathbf{w}}(\mathbf{r})$ is the multiplicity of the component $\mathcal{O}(\mathbf{w})$ in $\Omega_{\mathbf{r}}(0)$. In particular, $c_{\mathbf{w}}(\mathbf{r}) \geq 0$.

Proof. In light of Lemma 4.8 and the additivity of multidegrees under taking unions of components of equal dimension [KM03a, Theorem 1.7.1], the quiver polynomial $\mathcal{Q}_{\mathbf{r}}(\mathbf{x} - \mathring{\mathbf{x}})$, which is by definition the multidegree of the quiver locus $\Omega_{\mathbf{r}}$, breaks up as a sum over lacing diagrams by Theorem 4.6. The summands are products of double Schubert polynomials by Theorem 2.1. □

To make Corollary 4.9 more explicit, it is necessary to identify which lacing diagrams $\mathbf{w}$ give components $\mathcal{O}(\mathbf{w})$, and to find their multiplicities. We shall accomplish this goal in Theorem 6.16, whose proof requires the less precise statement in Proposition 4.11, below, as a stepping stone (in the end, it turns out to have identified all the components, not just some of them). First, let us observe that the quiver degeneration $\Omega_{\mathbf{r}}(0)$ contains every lacing diagram with rank array $\mathbf{r}$.

Lemma 4.10. *If the lacing diagram* $\mathbf{w}$ *has rank array* $\mathbf{r}$*, then* $\mathbf{w} \in \Omega_{\mathbf{r}}(\tau)$ *for all* $\tau \in \Bbbk$*.*

Proof. Let $\mathbf{w} \in \mathit{Hom}$ have ranks $\mathbf{r}$. It suffices to show that $\mathbf{w} \in \Omega_{\mathbf{r}}(\tau)$ for $\tau \in \mathbb{A}^1 \smallsetminus 0$; indeed, it follows that it holds for all τ because the family $\widetilde{\Omega}_{\mathbf{r}}$ is closed in $\mathit{Hom} \times \mathbb{A}^1$.

By Lemma 3.2, $\mathbf{w}$ lies inside $\Omega_{\mathbf{r}}$. Using the notation of (4.1) and (4.2), for every $\tau \in \mathbb{A}^1 \smallsetminus 0$ the list $\tau\boldsymbol{.}\mathbf{w}\tau\boldsymbol{.}$ of monomial matrices also lies in $\Omega_{\mathbf{r}}$. Now consider the curve

$$(\tau\boldsymbol{.}\mathbf{w}\tau\boldsymbol{.}) \times \tau$$

inside $\Omega_{\mathbf{r}} \times (\mathbb{A}^1 \smallsetminus 0)$. Under the map given in (4.2), this curve is sent to the constant curve $\mathbf{w} \times (\mathbb{A}^1 \smallsetminus 0)$ inside $\widetilde{\Omega}_{\mathbf{r}}(\tau \neq 0)$, proving the lemma. □

Proposition 4.11. *Among the irreducible components of the quiver degeneration* $\Omega_{\mathbf{r}}(0)$ *are the orbit closures* $\mathcal{O}(\mathbf{w})$ *for* $\mathbf{w} \in W(\mathbf{r})$ *(Definition 3.5). Hence* $c_{\mathbf{w}}(\mathbf{r}) \geq 1$ *for* $\mathbf{w} \in W(\mathbf{r})$*.*

Proof. Let $\mathbf{w} \in W(\mathbf{r})$. Then $\mathcal{O}(\mathbf{w}) \subset \Omega_{\mathbf{r}}(0)$, by Lemma 4.10 along with Propositions 3.4 and 4.5. Consider a component of $\Omega_{\mathbf{r}}(0)$ containing $\mathcal{O}(\mathbf{w})$; it must have the form $\mathcal{O}(\mathbf{w}')$ by Theorem 4.6. But $\mathcal{O}(\mathbf{w}) \subset \mathcal{O}(\mathbf{w}')$ implies that $w'_j \leq w_j$ in the strong Bruhat order for all j, by Lemma 3.3. Thus $\ell(\mathbf{w}) \geq \ell(\mathbf{w}')$. But $\ell(\mathbf{w})$ is the codimension of $\mathcal{O}(\mathbf{w})$ in Hom for all $\mathbf{w}$, and $d(\mathbf{r})$ is the codimension of the flat degeneration $\Omega_{\mathbf{r}}(0)$ of $\Omega_{\mathbf{r}}$ in Hom. It follows that $\mathbf{w}' = \mathbf{w}$. □

4.4. Rank stability of components

Later, in Section 6, it will be crucial to understand how the components of $\Omega_{\mathbf{r}}(0)$ and their multiplicities behave when a constant m is added to all the ranks in $\mathbf{r}$.

For notation, given $m \in \mathbb{N}$ and a partial permutation w, let $m + w$ be the new partial permutation obtained by letting w act on $m + \mathbb{Z}_{>0} = \{m+1, m+2, \ldots\}$ instead of $\mathbb{Z}_{>0} = \{1, 2, \ldots\}$ in the obvious manner. In particular $m + w$ fixes $1, 2, \ldots, m$. For a list $\mathbf{w} = (w_1, \ldots, w_n)$ of partial permutations, set $m+\mathbf{w} = (m+w_1, \ldots, m+w_n)$. If m is a nonnegative integer and $\mathbf{r} = (r_{ij})$ is a rank array, let $m + \mathbf{r}$ be the new rank array obtained by adding m to every rank r_{ij} in the array $\mathbf{r}$. This transformation is accomplished simply by adding m to the bottom entry s_{0n} in the lace array $\mathbf{s}$ appearing in (1.2).

In dealing with stability of components of quiver degenerations $\Omega_{\mathbf{r}}(0)$, we take our cue from the set $W(\mathbf{r})$, whose behavior in response to uniformly increasing ranks we deduce immediately from Theorem 3.8.

Corollary 4.12. *There is a bijection* $W(\mathbf{r}) \to W(m + \mathbf{r})$ *given by adding* m $(0, n)$*-laces along the bottom.*

The corresponding statement for the components of $\Omega_{\mathbf{r}}(0)$ is also true; moreover, the multiplicities of these components are also stable. In fact, the entire geometry of the family $\widetilde{\Omega}_{m+\mathbf{r}}$ is essentially the same as the geometry of the family $\widetilde{\Omega}_{\mathbf{r}}$.

Proposition 4.13. *The orbit closure* $\mathcal{O}(\mathbf{w})$ *is a component of the quiver degeneration* $\Omega_{\mathbf{r}}(0)$ *if and only if* $\mathcal{O}(m + \mathbf{w})$ *is a component of the quiver degeneration* $\Omega_{m+\mathbf{r}}(0)$*, and the multiplicities at their generic points are equal:* $c_{\mathbf{w}}(\mathbf{r}) = c_{m+\mathbf{w}}(m + \mathbf{r})$*. Furthermore, every component of* $\Omega_{m+\mathbf{r}}(0)$ *has the form* $\mathcal{O}(m + \mathbf{w})$ *for some* $\mathbf{w} \in \Omega_{\mathbf{r}}(0)$*.*

Proof. By induction on m, it suffices to prove the case $m = 1$ (the case $m = 0$ being trivial).

Fix a rank array $\mathbf{r}$. Let Hom' denote the vector space of matrix lists $\psi = (\psi_1, \ldots, \psi_n)$ in which ψ_i has size $(1 + r_{i-1}) \times (1 + r_i)$ for $i = 1, \ldots, n$. Just as GL acts on Hom, let GL' act on Hom' via the diagonal embedding $\Delta : GL' \hookrightarrow (GL')^2$.

Next we define two subgroups R' and C' of GL'. The "row" group R' consists of matrix lists $\rho' = (\rho'_0, \ldots, \rho'_n) \in GL'$ such that ρ'_0 is the identity matrix, while ρ'_i for $i \geq 1$ has 1's on its diagonal and is otherwise nonzero only in its bottom row. Similarly, the "column" group C' consists of matrix lists $\kappa' = (\kappa'_0, \ldots, \kappa'_n) \in GL'$ such that κ'_n is the identity matrix, while κ'_i for $i \leq n-1$ has 1's on its diagonal and is otherwise nonzero only in its rightmost column. By conjugation as in (4.1), we get two corresponding group schemes $\widetilde{C}'$ and $\widetilde{R}'$ over the affine line $\mathbb{A} = \mathbb{A}^1(\Bbbk)$, just as we get $\widetilde{GL}'$ from GL'.

Let $\widetilde{Hom}$, $\widetilde{\Bbbk}^*$, and $\widetilde{Hom}'$ denote the products of the corresponding (un-tilded) spaces with the line $\mathbb{A}$, where $\Bbbk^* = \mathbb{A}^1 \smallsetminus \{0\}$ is the multiplicative group of $\Bbbk$. In addition, for $\lambda = (\lambda_1, \ldots, \lambda_n) \in (\Bbbk^*)^n$, write $\phi \oplus \lambda \in Hom'$ for the matrix list whose i^{th} component has ϕ_i in the northwest corner and λ_i in the southeast corner. Now define the morphism

$$\theta : \widetilde{C}' \times_{\mathbb{A}} \widetilde{R}' \times_{\mathbb{A}} \widetilde{Hom} \times_{\mathbb{A}} \widetilde{(\Bbbk^*)^n} \quad \longrightarrow \quad \widetilde{Hom}'$$

by letting θ take the fiber over $\tau \in \mathbb{A}$ to its image via

$$\theta : (\kappa', \rho', \phi, \lambda) \quad \longmapsto \quad \kappa' \cdot \rho' \cdot (\phi \oplus \lambda).$$

The dots after κ' and ρ' denote the action of $\widetilde{GL}'$ on $\widetilde{Hom}'$.

Our first task is to show that θ is an isomorphism onto its (open) image. This is completely elementary, but since it will be useful to know the image of θ precisely, let us say how to construct the inverse. Some notation will help. For $\rho' \in \widetilde{R}'(\tau)$, write

$$\rho'_i \quad = \quad (\mathbf{1} + \overleftarrow{\rho}_i, \mathbf{1} + \overrightarrow{\rho}_i),$$

where $\mathbf{1}$ is the identity matrix of size $1 + r_i$. Thus, for example, $\overleftarrow{\rho}_i$ has at most one nonzero row, namely its bottom row, and $\mathbf{1} + \overleftarrow{\rho}_i$ acts via its inverse, which equals $\mathbf{1} - \overleftarrow{\rho}_i$. Similarly, for $\kappa' \in \widetilde{C}'(\tau)$, write $\kappa'_i = (\mathbf{1} + \overleftarrow{\kappa}_i, \mathbf{1} + \overrightarrow{\kappa}_i)$. We identify $\lambda \in (\Bbbk^*)^n$ with the matrix list $0 \oplus \lambda \in Hom'$, writing $|\lambda_i|$ if we wish to refer to the sole nonzero entry of λ_i. Finally, we identify Hom with the subvariety of Hom' consisting of all matrix lists ψ such that each ψ_i is zero in its rightmost column as well as its bottom row; thus $\phi \in Hom$ is identified with "$\phi \oplus 0$" (we only defined $\phi \oplus \lambda$ for nonzero λ). In this notation,

$$\text{(4.3)} \quad \theta(\kappa', \rho', \phi, \lambda) = \psi, \quad \text{where} \quad \psi_i = \begin{bmatrix} \phi_i + \overrightarrow{\kappa}_{i-1} v_i & \overrightarrow{\kappa}_{i-1}\lambda_i - (\phi_i + \overrightarrow{\kappa}_{i-1} v_i)\overleftarrow{\kappa}_i \\ v_i & \lambda_i - v_i \overleftarrow{\kappa}_i \end{bmatrix}$$
$$\text{and} \quad v_i = \overrightarrow{\rho}_{i-1}\phi_i - \lambda_i \overleftarrow{\rho}_i.$$

As in the above display, each matrix ψ_i in a matrix list $\psi \in Hom'$ can be expressed as $\psi_i = NW(\psi_i) + NE(\psi_i) + SE(\psi_i) + SW(\psi_i)$, where

$$NW(\psi_i) = \begin{bmatrix} * & 0 \\ 0 & 0 \end{bmatrix}, \quad NE(\psi_i) = \begin{bmatrix} 0 & * \\ 0 & 0 \end{bmatrix}, \quad SE(\psi_i) = \begin{bmatrix} 0 & 0 \\ 0 & * \end{bmatrix}, \quad \text{and} \quad SW(\psi_i) = \begin{bmatrix} 0 & 0 \\ * & 0 \end{bmatrix}.$$

The $*$ blocks agree with ψ_i in rectangles of size $(r_{i-1} \times r_i)$, $(r_{i-1} \times 1)$, (1×1), and $(1 \times r_i)$, in the order listed above. The morphism θ is inverted by an easy seven-step program. For the first five steps, proceed from $i = n$ down to $i = 1$; each loop here assumes that $\overleftarrow{\kappa}_i$ has been defined in the previous loop, starting from $\overleftarrow{\kappa}_n = 0$.

1. Set $v_i = SW(\psi_i)$.
2. Set $\lambda_i = SE(\psi_i) + v_i \overleftarrow{\kappa}_i$.
3. Set $\overrightarrow{\kappa}_{i-1} = \frac{1}{|\lambda_i|}(NE(\psi_i) + NW(\psi_i)\overleftarrow{\kappa}_i)$.

4. Set $\overleftarrow{\kappa}_{i-1} = \tau_{i-1}^2 \overrightarrow{\kappa}_{i-1}\tau_{i-1}^{-2}$ if $\tau \neq 0$ and $\overleftarrow{\kappa}_{i-1} = 0$ if $\tau = 0$ (τ_i is defined in (4.1)).
5. Set $\phi_i = NW(\psi_i) - \overrightarrow{\kappa}_{i-1}v_i$.

To complete the inversion, proceed in a loop from $i = 1$ up to $i = n$, starting with $\overrightarrow{\rho}_0 = 0$:

6. Set $\overleftarrow{\rho}_i = -\frac{1}{|\lambda_i|}(v_i - \overrightarrow{\rho}_{i-1}\phi_i)$.
7. Set $\overrightarrow{\rho}_i = \tau_i^{-2}\overleftarrow{\rho}_i\tau_i^2$ if $\tau \neq 0$ and $\overrightarrow{\rho}_i = 0$ if $\tau = 0$.

The conjugation in Step 7 multiplies each entry of $\overleftarrow{\rho}_i$ by a strictly positive power of τ, so the definition of $\overrightarrow{\rho}_i$ is polynomial in τ, even near $\tau = 0$; the same goes for $\overleftarrow{\kappa}_{i-1}$. Step 2 defines the λ coordinates as rational functions on Hom'. In fact, $\lambda_i\lambda_{i+1}\cdots\lambda_n$ is a polynomial function on $\widetilde{Hom}'$ for each $i = 1, \ldots, n$. The algorithm produces the inverse of the morphism θ on the locus in $\widetilde{Hom}'$ where these polynomials are nonzero, regardless of which fiber τ is being considered. The elementary details of this verification are omitted.

Let $\theta_{\mathbf{r}}$ be the restriction of θ to the subvariety where the $\widetilde{Hom}$ component is replaced by $\widetilde{\Omega}_{\mathbf{r}}$. Our next task is to prove that $\theta_{\mathbf{r}}$ has image $\widetilde{\Omega}^{\circ}_{1+\mathbf{r}}$, by which we mean the intersection of $\widetilde{\Omega}_{1+\mathbf{r}}$ with the image of θ, where $\lambda_1\cdots\lambda_n$ is nonzero. As θ is an isomorphism, the image of $\theta_{\mathbf{r}}$ is Zariski closed in the image of θ and reduced, since its source is. Moreover, the dimension of the source of $\theta_{\mathbf{r}}$ is $\dim(\widetilde{\Omega}_{\mathbf{r}}) + \dim(R') + \dim(C') + 1 = \dim(\widetilde{\Omega}_{1+\mathbf{r}})$. Hence it is enough to show that the image of $\theta_{\mathbf{r}}$ is contained in $\widetilde{\Omega}_{1+\mathbf{r}}$. Again by Zariski closedness we need only check this over fibers $\tau \neq 0$. Finally, since θ was constructed to be equivariant for the conjugation action defining $\widetilde{\Omega}_{\mathbf{r}}$, $\widetilde{R}'$, $\widetilde{C}'$, and $\widetilde{\Omega}_{1+\mathbf{r}}$, we need only check the fiber $\tau = 1$. This final step is elementary: if $\phi \in \Omega_{\mathbf{r}}$, then $\phi \oplus \lambda \in \Omega_{1+\mathbf{r}}$ by definition; our task is done because $\theta_{\mathbf{r}}$ spreads $\phi \oplus \lambda$ around inside of Hom' via the action of GL', which preserves $\Omega_{1+\mathbf{r}}$.

Our third task is to identify the components (and their multiplicities) in the zero fiber of the image of $\theta_{\mathbf{r}}$, which we have shown is $\Omega^{\circ}_{1+\mathbf{r}}(0)$. The isomorphism of $\widetilde{C}' \times_{\mathbb{A}} \widetilde{R}' \times_{\mathbb{A}} \widetilde{\Omega}_{\mathbf{r}} \times_{\mathbb{A}} \widetilde{(\mathbb{k}^*)^n}$ with $\widetilde{\Omega}^{\circ}_{1+\mathbf{r}}$ immediately implies that the components of $\Omega^{\circ}_{1+\mathbf{r}}(0)$ are in bijection with those of $\Omega_{\mathbf{r}}(0)$, and that the corresponding multiplicities are equal. To complete this task, it therefore remains only to be more explicit about the bijection.

Let $\theta_{\mathbf{r}}(0)$ be the restriction of $\theta_{\mathbf{r}}$ to the fiber over $\tau = 0$. Referring to (4.3), the fact that $\overleftarrow{\kappa}_i$ is zero over $\tau = 0$ immensely simplifies $\theta_{\mathbf{r}}(0)$. Writing $[v_i, \lambda_i]$ for the bottom row of ψ_i and $\left[\begin{smallmatrix}\kappa_{i-1}\\ 1\end{smallmatrix}\right]$ for the single column κ_{i-1} with an extra 1 at the bottom, the effect of $\theta_{\mathbf{r}}(0)$ is to add the rank 1 matrix $[v_i, \lambda_i] \cdot \left[\begin{smallmatrix}\kappa_{i-1}\\ 1\end{smallmatrix}\right]$ to ϕ_i.

Let $\mathcal{O}(\mathbf{w}')$ be the component of $\Omega_{1+\mathbf{r}}(0)$ corresponding via $\theta_{\mathbf{r}}(0)$ to a component $\mathcal{O}(\mathbf{w})$ of $\Omega_{\mathbf{r}}(0)$; thus $\mathcal{O}(\mathbf{w}')$ is Zariski closed in all of Hom', not just where $\lambda_1\cdots\lambda_n \neq 0$. Since v_i, λ_i, and κ_{i-1} can be chosen arbitrarily as long as $\lambda_i \neq 0$, the matrix Schubert variety $\overline{X}_{w_i'}$ (Definition 1.11) appearing as a factor of $\mathcal{O}(\mathbf{w}')$ contains the sum of the $r_{i-1} \times r_i$ matrix Schubert variety $\overline{X}_{w_i}$ with the variety of all $(1 + r_{i-1}) \times (1 + r_i)$ matrices of rank at most 1. But $\operatorname{rank}(Z_{q\times p}) \leq \operatorname{rank}((1 + w_i)_{q\times p})$ for all matrices Z in this sum, and equality is attained whenever a sufficiently generic matrix of rank 1 is added to a full-rank matrix in $\overline{X}_{w_i}$. It follows that $\mathcal{O}(\mathbf{w}')$ contains $\mathcal{O}(1 + \mathbf{w})$, so $\mathcal{O}(\mathbf{w}') = \mathcal{O}(1 + \mathbf{w})$ by dimension considerations.

Our final task is to verify that no components of $\Omega_{1+\mathbf{r}}(0)$ have been lost by restricting to $\widetilde{\Omega}^{\circ}_{1+\mathbf{r}}$ instead of $\widetilde{\Omega}_{1+\mathbf{r}}$. More precisely, we need to prove that no component of $\Omega_{1+\mathbf{r}}(0)$ lies in the zero set of the polynomial $\lambda_1\cdots\lambda_n$. Our task is made easier by the fact that the zero set of $\lambda_1\cdots\lambda_n$ over $\tau = 0$ agrees with that of the product $f = \prod_{i=1}^n f^{\,i}_{1+r_{i-1},\,1+r_i}$ of the southeast entries of the matrices $\mathbf{f}^i$ of coordinate variables (Definition 1.1); this fact is immediate from Step 2, above, which implies that $\lambda_i = f^{\,i}_{1+r_{i-1},\,1+r_i}$ over $\tau = 0$.

The key point in checking the nonzerodivisor property is the following general statement.

Lemma 4.14. *Suppose that $I \subseteq \Bbbk[\mathbf{f}^1, \ldots, \mathbf{f}^n]$ is a prime ideal generated by polynomials homogeneous in each of the finite alphabets $\mathbf{f}^1, \ldots, \mathbf{f}^n$. Assume that I contains none of the ideals $\langle \mathbf{f}^1 \rangle, \ldots, \langle \mathbf{f}^n \rangle$ generated by a complete alphabet. If P is a minimal prime of some initial monomial ideal $\mathsf{in}(I)$ of I, then P contains none of the ideals $\langle \mathbf{f}^1 \rangle, \ldots, \langle \mathbf{f}^n \rangle$.*

Proof. Let $\mathbb{A} = \mathbb{A}_1 \times \cdots \times \mathbb{A}_n$, where $\mathbb{A}_j = \mathrm{Spec}(\Bbbk[\mathbf{f}^j])$. Using the homogeneity hypothesis, the zero scheme $\Omega \subseteq \mathbb{A}$ of I has a multidegree $\mathcal{C}(\Omega) \in \mathbb{Z}[x_1, \ldots, x_n]$, where x_i corresponds to ordinary total degree in the variables $\mathbf{f}^i$. The zero set of each minimal prime of $\mathsf{in}(I)$ is a linear subspace $L \subseteq \mathbb{A}$ of the same dimension as Ω itself [KS95], with multidegree $\mathcal{C}(L)$ and multiplicity $\mu(L)$ in the zero scheme of $\mathsf{in}(I)$. By additivity [KM03a, Theorem 1.7.1], $\mathcal{C}(\Omega)$ is a sum of terms $\mu(L) \cdot \mathcal{C}(L)$, one for every such L.

On the other hand, the hypotheses on I imply that it defines a closed subvariety of the corresponding product $\mathbb{P} = \mathbb{P}_1 \times \cdots \times \mathbb{P}_n$ of projective spaces. Viewing the Gröbner degeneration as taking place inside of $\mathbb{P}$, an irreducible component $L \subseteq \mathbb{A}$ of the (reduced) zero set of $\mathsf{in}(I)$ contributes a component to the special fiber if and only if its defining ideal contains none of the ideals $\langle \mathbf{f}^1 \rangle, \ldots, \langle \mathbf{f}^n \rangle$. Since the multidegree $\mathcal{C}(\Omega)$ equals the subsum of the terms $\mu(L) \cdot \mathcal{C}(L)$ coming from the components L contributing to the projective special fiber, we conclude that every component L must contribute. $\square$

The special fiber of any Gröbner degeneration of $\Omega_{1+\mathbf{r}}(0)$ can be realized as the special fiber of a Gröbner degeneration of $\Omega_{1+\mathbf{r}}$ itself. Therefore, Gröbner degenerating $\Omega_{1+\mathbf{r}}(0)$ to the zero scheme Ω' of a monomial ideal, we find by Lemma 4.14 applied to $I = I_{1+\mathbf{r}}$ that no component of Ω'—and hence no component $\mathcal{O}(\mathbf{w}')$ of $\Omega_{1+\mathbf{r}}(0)$—is defined by an ideal containing $\langle \mathbf{f}^i \rangle$. Therefore the partial permutation w_i' is nonzero for all $i = 1, \ldots, n$ if $\mathcal{O}(\mathbf{w}')$ is a component of $\Omega_{1+\mathbf{r}}(0)$. The condition $w_i' \neq 0$ means that $\overline{X}_{w_i'}$ has dimension ≥ 1. The southeast corner variable is a nonzerodivisor on every positive-dimensional matrix Schubert variety. Hence the product f is a nonzerodivisor on every component of $\Omega_{1+\mathbf{r}}(0)$. By the argument preceding Lemma 4.14, we conclude that the same holds for $\lambda_1 \cdots \lambda_n$. $\square$

Remark 4.15. It would be possible at this juncture to justify replacing the double Schubert polynomials in Corollary 4.9 with double Stanley symmetric functions, since our results through Proposition 4.13 suffice for this purpose. (Notably, the ratio formula does not enter into these arguments, which appear in the proofs of Theorem 6.16 and Corollary 6.23, below.) Hence we could immediately deduce the positivity of the Buch–Fulton quiver constants $c_{\underline{\lambda}}(\mathbf{r})$ using the Schur-positivity of Stanley symmetric functions. However, we postpone this discussion until after we have reviewed Stanley symmetric functions and their Schur-positivity in Section 6, because we shall in any case prove more precise statements about the quiver constants in Sections 7 and 8.

Section 5. Pipe dreams for Zelevinsky permutations

5.1. Pipe dream formula for double quiver polynomials

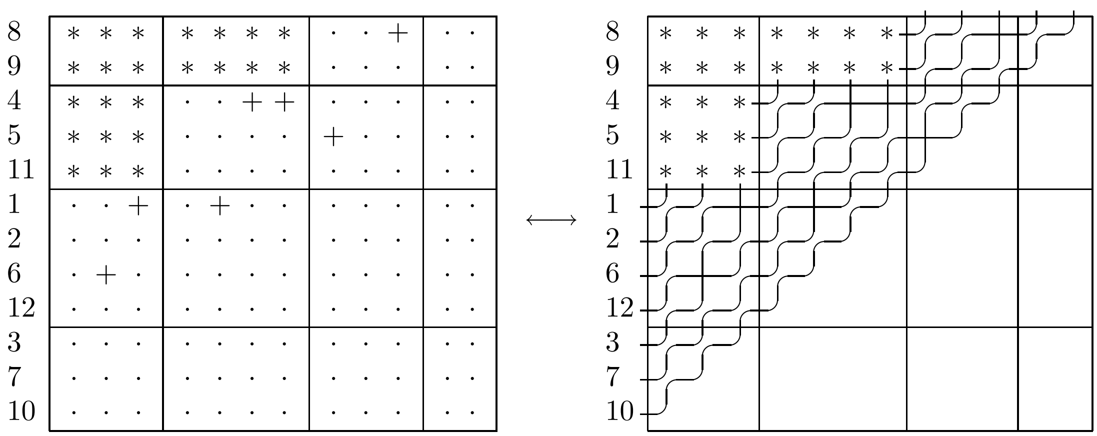
Schubert polynomials possess a beautiful combinatorial description, proved independently in [BJS93, FS94], in terms of diagrams defined in [FK96] that we call 'reduced pipe dreams', although they have also been called 'rc-graphs' in the literature [BB93, Len02]. These diagrams are the main characters in our next combinatorial formula, Theorem 5.5, which is little more than a corollary of the ratio formula. Although it is also of independent interest, Theorem 5.5 is an essential ingredient for proving our remaining formulae. From a purely combinatorial perspective, we find the direct connection between reduced pipe dreams for Zelevinsky permutations and lacing diagrams particularly appealing (Section 5.2).

Consider a square grid of size $\ell \times \ell$, with the box in row q and column p labeled (q, p), as in $\ell \times \ell$ matrix. If each box in the grid is covered with a square tile containing either $+$ or $\lrcorner\ulcorner$, then one can think of the tiled grid as a network of pipes.

Definition 5.1. A **pipe dream** is a subset of the $\ell \times \ell$ grid, identified as the set of crosses in a tiling by **crosses** $+$ and **elbow joints** $\lrcorner\ulcorner$. A pipe dream is **reduced** if each pair of pipes crosses at most once. The set $\mathcal{RP}(v)$ of reduced pipe dreams for a permutation $v \in S_\ell$ is the set of reduced pipe dreams D such that the pipe entering row q flows northeast to exit from column $v(q)$.

We usually draw crossing tiles as some sort of cross, either '+' or '$+$', although we shall frequently use an asterisk $*$ to denote a $+$ strictly above the block superantidiagonal. We often leave elbow tiles blank or denote them by dots, to make the diagrams less cluttered. We shall only be interested in pipe dreams whose $+$ entries lie in the triangular region strictly above the main antidiagonal (in spots (q, p) with $q + p \leq \ell$), with elbow joints elsewhere in the square grid of size ℓ.

Example 5.2. Using $\ell = 12$, a typical reduced pipe dream for the Zelevinsky permutation v in Example 1.9 (v is written down the left side) looks like



when the $*$'s remain as in the diagram. Although each $*$ represents a $+$ in every pipe dream for v, the $*$'s will be just as irrelevant here as they were for the diagram. □

Each pipe dream D yields a monomial $(\mathcal{X} - \mathcal{Y})^D$, defined as the product over all $+$ entries in D of $(x_+ - y_+)$, where x_+ sits at the left end of the row containing $+$, and y_+ sits atop the column containing $+$. At this level of generality, $\mathcal{X}$ and $\mathcal{Y}$ are simply alphabets (of size ℓ or of infinite size; it does not matter). Here is the fundamental result relating pipe dreams to Schubert polynomials; its 'single' form was proved independently in [BJS93, FS94], but it is the 'double' form that we need.

Theorem 5.3. [FK96] *The double Schubert polynomial for v equals the sum*

$$\mathfrak{S}_v(\mathcal{X} - \mathcal{Y}) \quad = \sum_{D \in \mathcal{RP}(v)} (\mathcal{X} - \mathcal{Y})^D$$

of all monomials associated to reduced pipe dreams for v.

In pictures of pipe dreams D for Zelevinsky permutations, we retain the notation from Section 2 regarding block matrices of total size d. For the purposes of seeing where crosses lie, we will not need the '$*$' entries above the block superantidiagonal of D. In other words, we shall consider the crosses in the pipe dream $D \smallsetminus D_{Hom}$ instead of D, where D_{Hom} is identified with the reduced pipe dream consisting entirely of $*$ locations (the unique reduced pipe dream for the dominant Zelevinsky permutation of Hom).

Example 5.4. The pipe dream from Example 5.2 has row and column labels

$$
\begin{array}{c|ccc|cccc|ccc|cc|}
 & y^3_1 & y^3_2 & y^3_3 & y^2_1 & y^2_2 & y^2_3 & y^2_4 & y^1_1 & y^1_2 & y^1_3 & y^0_1 & y^0_2 \\
\hline
x^0_1 & * & * & * & * & * & * & * & \cdot & \cdot & + & \cdot & \cdot \\
x^0_2 & * & * & * & * & * & * & * & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
x^1_1 & * & * & * & \cdot & \cdot & + & + & \cdot & \cdot & \cdot & \cdot & \cdot \\
x^1_2 & * & * & * & \cdot & \cdot & \cdot & \cdot & + & \cdot & \cdot & \cdot & \cdot \\
x^1_3 & * & * & * & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
x^2_1 & \cdot & \cdot & + & \cdot & + & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
x^2_2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
x^2_3 & \cdot & + & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
x^2_4 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
x^3_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
x^3_2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
x^3_3 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\end{array}
\quad = \quad
\begin{array}{c|ccc|cccc|ccc|cc|}
 & \dot d_1 & \dot d_2 & \dot d_3 & \dot c_1 & \dot c_2 & \dot c_3 & \dot c_4 & \dot b_1 & \dot b_2 & \dot b_3 & \dot a_1 & \dot a_2 \\
\hline
a_1 & * & * & * & * & * & * & * & \cdot & \cdot & + & \cdot & \cdot \\
a_2 & * & * & * & * & * & * & * & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
b_1 & * & * & * & \cdot & \cdot & + & + & \cdot & \cdot & \cdot & \cdot & \cdot \\
b_2 & * & * & * & \cdot & \cdot & \cdot & \cdot & + & \cdot & \cdot & \cdot & \cdot \\
b_3 & * & * & * & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
c_1 & \cdot & \cdot & + & \cdot & + & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
c_2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
c_3 & \cdot & + & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
c_4 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
d_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
d_2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
d_3 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\end{array}
$$

where we use alphabets $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ interchangeably with $\mathbf{x}^0, \mathbf{x}^1, \mathbf{x}^2, \mathbf{x}^3$, and also $\dot{\mathbf{d}}, \dot{\mathbf{c}}, \dot{\mathbf{b}}, \dot{\mathbf{a}}$ interchangeably with $\mathbf{y}^3, \mathbf{y}^2, \mathbf{y}^1, \mathbf{y}^0$. The monomial for the above pipe dream is

$$(a_1 - \dot b_3)(b_1 - \dot c_3)(b_1 - \dot c_4)(b_2 - \dot b_1)(c_1 - \dot d_3)(c_1 - \dot c_2)(c_3 - \dot d_2),$$

ignoring all $*$ entries as required. Removing the dots yields this pipe dream's contribution to the ordinary quiver polynomial. $\square$

Now we have enough ingredients for our second positive combinatorial formula for double quiver polynomials, the first being Definition 2.5.

Theorem 5.5. *The double quiver polynomial for ranks* $\mathbf{r}$ *equals the sum*

$$\mathcal{Q}_{\mathbf{r}}(\mathbf{x} - \mathring{\mathbf{y}}) \quad = \sum_{D \in \mathcal{RP}(v(\mathbf{r}))} (\mathbf{x}_{\mathbf{r}} - \mathring{\mathbf{y}}_{\mathbf{r}})^{D \smallsetminus D_{Hom}}$$

over all reduced pipe dreams D *for the Zelevinsky permutation* $v(\mathbf{r})$ *of the monomials* $(\mathbf{x}_{\mathbf{r}} - \mathring{\mathbf{y}}_{\mathbf{r}})^{D \smallsetminus D_{Hom}}$ *made from crosses occupying the block antidiagonal and block superantidiagonal (that is, ignoring* $*$ *entries occupying boxes strictly above the block superantidiagonal).*

Proof. This follows from Definition 2.5 and Theorem 5.3, using the fact that every pipe dream $D \in \mathcal{RP}(v(\mathbf{r}))$ contains the subdiagram D_{Hom}, and that $\mathcal{RP}(v(Hom))$ consists of the single pipe dream D_{Hom}. $\square$

In other words, the double quiver polynomial $\mathcal{Q}_{\mathbf{r}}(\mathbf{x} - \mathring{\mathbf{y}})$ is the sum over monomials for all "skew reduced pipe dreams" $D \smallsetminus D_{Hom}$ with $D \in \mathcal{RP}(v(\mathbf{r}))$.

Since it will be important in Section 5.3 (and adds even more concreteness to Theorem 5.3), let us provide one way of generating all reduced pipe dreams for v.

Definition 5.6 ([BB93])**.** A **chutable rectangle** is a connected $2 \times k$ rectangle C inside a pipe dream D such that $k \geq 2$ and every location in C is a $+$ except the following three: the northwest, southwest, and southeast corners. Applying a **chute move** to D is accomplished by placing a $+$ in the southwest corner of a chutable rectangle C and removing the $+$ from the northeast corner of the same C.

Heuristically, a chute move therefore looks like:

Proposition 5.7. [BB93] *There is a unique* ***top reduced pipe dream*** D_{top} *for* v*, in which no* ╋ *has any elbows* ╯╭ *due north of it. Every reduced pipe dream in* $\mathcal{RP}(v)$ *can be obtained by starting with* D_{top} *and applying some sequence of chute moves.*

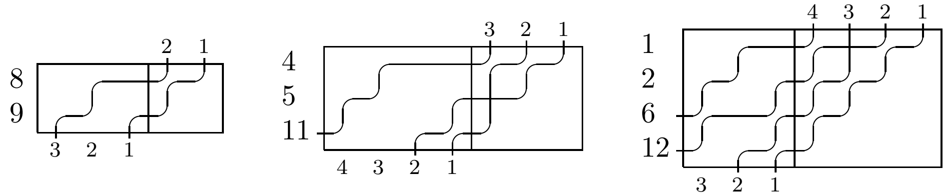
5.2. Pipes to laces

Pipe dreams for Zelevinsky permutations give rise to lacing diagrams.

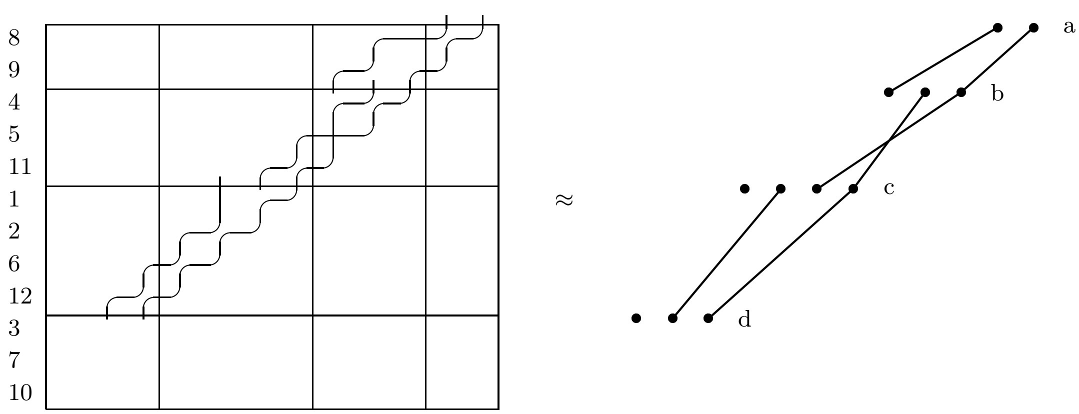
Definition 5.8. The j^{th} antidiagonal block is the block of size $r_j \times r_j$ along the main antidiagonal in the j^{th} block row. Given a reduced pipe dream D for the Zelevinsky permutation $v(\mathbf{r})$, define the partial permutation $w_j = w_j(D)$ sending p to q if the pipe entering the p^{th} column from the *right* of the $(j-1)^{\text{st}}$ antidiagonal block enters the j^{th} antidiagonal block in its q^{th} column from the *right*. Set $\mathbf{w}(D) = (w_1, \ldots, w_n)$. Equivalently, $\mathbf{w}(D)$ is the lacing diagram determined by D.

Example 5.9. The partial permutations arising from the reduced pipe dream in Examples 5.2 and 5.4 come from the following partial reduced pipe dreams,

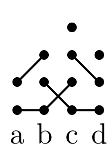
(5.1)



where they send each number along the top to the number along the bottom connected to it by a pipe (if such a pipe exists), or to nowhere. It is easy to see the lacing diagram from these pictures. Indeed, removing all segments of all pipes not contributing to one of the partial permutations leaves some pipes



that can be interpreted directly as the desired lacing diagram



by shearing to make the rightmost dots in each row line up vertically, and then reflecting through the diagonal line $\diagdown$ of slope -1. □

Warning: the $d \times d$ 'pipe' representation of the lacing diagram (at left above) does *not* determine a reduced pipe dream uniquely, because although the ordinary and singly virtual crossings have been placed in the $d \times d$ grid, the doubly virtual crossings still need to be placed (in the lower two superantidiagonal blocks).

Theorem 5.10. *Every reduced pipe dream $D \in \mathcal{RP}(v(\mathbf{r}))$ gives rise to a minimally crossing lacing diagram $\mathbf{w}(D)$ with rank array $\mathbf{r}$; that is, $\mathbf{w}(D) \in W(\mathbf{r})$.*

Proof. Each $\times$ entry in the permutation matrix for $v(\mathbf{r})$ corresponds to a pipe in D entering due north of it and exiting due west of it. The permutation $v(\mathbf{r})$ was specifically constructed to have exactly s_{ij} entries $\times$ (for $i, j = 0, \ldots, n$) in the intersection of the i^{th} block row and the j^{th} block column from the right. □

We shall see in Corollary 6.18 that in fact every minimally crossing lacing diagram arises in this manner from a reduced pipe dream.

5.3. Rank stability of pipe dreams

Just as components in the quiver degeneration are stable for uniformly increasing ranks $\mathbf{r}$ (Proposition 4.13), so are the sets of pipe dreams for $v(\mathbf{r})$. Before stating the precise result, we analyze the changes to $v(\mathbf{r})$ and its diagram when 1 is added to all ranks in $\mathbf{r}$.

Lemma 5.11. *The passage from $\mathbf{r}$ to $1+\mathbf{r}$ enlarges every block submatrix in the permutation matrix for $v(\mathbf{r})$ by one row and one column. Similarly, the diagram of $v(1+\mathbf{r})$ has the same rectangular collections of cells in the southeast corners of the same blocks as does the diagram of $v(\mathbf{r})$. (For an example, see Fig. 2.)*

10	∗	∗	∗	∗	∗	∗	∗	∗	∗	×	·	·	·	·	·	·
11	∗	∗	∗	∗	∗	∗	∗	∗	∗	·	×	·	·	·	·	·
12	∗	∗	∗	∗	∗	∗	∗	∗	∗	·	·	×	·	·	·	·
5	∗	∗	∗	∗	×	·	·	·	·	·	·	·	·	·	·	·
6	∗	∗	∗	∗	·	×	·	·	·	·	·	·	·	·	·	·
7	∗	∗	∗	∗	·	·	×	·	·	·	·	·	·	·	·	·
14	∗	∗	∗	∗	·	·	·	□	□	·	·	·	□	×	·	·
1	×	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·
2	·	×	·	·	·	·	·	·	·	·	·	·	·	·	·	·
3	·	·	×	·	·	·	·	·	·	·	·	·	·	·	·	·
8	·	·	·	□	·	·	·	×	·	·	·	·	·	·	·	·
15	·	·	·	□	·	·	·	·	□	·	·	·	□	·	×	·
4	·	·	·	×	·	·	·	·	·	·	·	·	·	·	·	·
9	·	·	·	·	·	·	·	·	×	·	·	·	·	·	·	·
13	·	·	·	·	·	·	·	·	·	·	·	·	×	·	·	·
16	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	×

FIGURE 2. The permutation $v(1+\mathbf{r})$ and its diagram, for $v(\mathbf{r})$ as in Example 1.9

Proof. The lace array giving rise to $1+\mathbf{r}$ is the same as the array $\mathbf{s}$ giving rise to $\mathbf{r}$ except that s_{0n} must be replaced by $1+s_{0n}$. The rest is straightforward. □

Suppose $D \in \mathcal{RP}(v(\mathbf{r}))$, and split D into horizontal strips (block rows). In each strip, chute every $+$ upward (that is, apply inverse chute moves) as far as possible. By Proposition 5.7, this yields a top reduced pipe dream in each horizontal strip. Since every relevant $+$ in a pipe dream $D \in \mathcal{RP}(v(\mathbf{r}))$ is confined to the antidiagonal blocks and those immediately to their left, we consider the j^{th} horizontal strip of a pipe dream to consist only of those two blocks, with a single vertical dividing line.

Example 5.12. For example, in our running example of a reduced pipe dream (Examples 5.2, 5.4, and 5.9), the top pipe dream in the second (that is, $j = 2$) horizontal strip is depicted on the left side of Fig. 3. The corresponding top pipe dream (see Proposition 5.15)

	$\dot d_1$	$\dot d_2$	$\dot d_3$	$\dot c_1$	$\dot c_2$	$\dot c_3$	$\dot c_4$
c_1	·	·	+	·	+	·	·
c_2	·	·	+	·	·	·	·
c_3	·	·	·	·	·	·	·
c_4	·	·	·	·	·	·	·

	$\dot d_1$	$\dot d_2$	$\dot d_3$	$\dot d_4$	$\dot c_1$	$\dot c_2$	$\dot c_3$	$\dot c_4$	$\dot c_5$
c_1	·	·	·	+	·	+	·	·	·
c_2	·	·	·	+	·	·	·	·	·
c_3	·	·	·	·	·	·	·	·	·
c_4	·	·	·	·	·	·	·	·	·
c_5	·	·	·	·	·	·	·	·	·

FIGURE 3. Top pipe dreams in $D \in \mathcal{RP}(v)$, for $v = v(\mathbf{r})$ or $v(1+\mathbf{r})$

in that horizontal strip for $v(1 + \mathbf{r})$ is on the right. □

Lemma 5.13. *Suppose $D \in \mathcal{RP}(v(\mathbf{r}))$ consists of miniature top reduced pipe dreams in its horizontal strips. Inside the restriction of D to its superantidiagonal block in block row j, the 2×2 configurations $\begin{smallmatrix}+&+\\+&\cdot\end{smallmatrix}$ and $\begin{smallmatrix}*&*\\+&\cdot\end{smallmatrix}$ in consecutive rows and columns never occur.*

Proof. Suppose there is such a 2×2 configuration. Since the restriction of D to the superantidiagonal block in block row j is a top pipe dream, all the cells in D due north of the 2×2 configuration are either $+$ or $*$, and all of these represent crossings in D. Therefore the southwest $+$ in the 2×2 configuration represents the crossing of two pipes exiting through adjacent columns in block column $j + 1$ from the right. This is impossible because the transpose of D is a reduced pipe dream for the permutation $v(\mathbf{r})^{-1}$, which is the Zelevinsky permutation $v(\mathbf{r}')$ for the transpose ranks $\mathbf{r}'$ and hence can have no descents within a block row. □

The easy proof of the next lemma is omitted.

Lemma 5.14. *Adding a row of elbows across the top (or a column of elbows to the left) of any reduced pipe dream for a permutation w yields a reduced pipe dream for $1 + w$.* □

Here is the rank stability statement for pipe dreams, to be applied in Section 6.

Proposition 5.15. *Let* $\mathrm{top}(\mathbf{r}) \subseteq \mathcal{RP}(v(\mathbf{r}))$ *denote the set of reduced pipe dreams D for $v(\mathbf{r})$ consisting of a miniature top pipe dream D_j in the j^{th} horizontal strip, for $j = 0, \ldots, n-1$. There is a canonical bijection* $\mathrm{top}(\mathbf{r}) \cong \mathrm{top}(m + \mathbf{r})$ *for all $m \geq 0$. Moreover, corresponding miniature top pipe dreams D_j and D'_j are translates of each other, in the same relative position to the vertical dividing line in horizontal strip j.*

Proof. It suffices by induction to prove the case $m = 1$. Starting with $D \in \mathrm{top}(\mathbf{r})$, inflate the blocks as in Lemma 5.11, and construct a new pipe dream D' by placing each miniature top pipe dream D_j in the same location relative to the vertical dividing line in its inflated strip. First we claim that $D' \in \mathcal{RP}(v(1+\mathbf{r}))$, for then automatically $D' \in \mathrm{top}(1+\mathbf{r})$. Then we claim that every reduced pipe dream in $\text{top}(1 + \mathbf{r})$ comes from some $D \in \text{top}(\mathbf{r})$ in this manner.

What we show precisely for the first claim is that for every $\times$ in the permutation matrix for $v(1 + \mathbf{r})$, the pipe (in D') entering into its row exits through its column. Given a $\times$ entry of $v(1 + \mathbf{r})$, exactly one of the following three statements must hold, by construction of Zelevinsky permutations: (i) the $\times$ lies in the northwest corner of some superantidiagonal block; (ii) the $\times$ lies in the southeast corner of the whole matrix; or (iii) there is a corresponding $\times$ in $v(\mathbf{r})$.

For an $\times$ entry of type (i), the pipe entering its row encounters precisely one elbow, located in the same position as $\times$, that turns the pipe north to exit through its column. For the $\times$ of type (ii), observe that every entry along the superantidiagonal (not the block superantidiagonal) is an elbow, because of the extra row added into each block row (Lemma 5.11). Thus its pipe proceeds from the southwest corner to the northeast with no obstructions.

For an $\times$ entry of type (iii), we claim that the pipe $\mathcal{P}'$ starting in the row of $\times$ enters the first antidiagonal block (through which it passes) in the same location relative to the block's northwest corner as does the pipe $\mathcal{P}$ for the corresponding $\times$ in $v(\mathbf{r})$. If the $\times$ lies in the bottom block row, then this is trivial, and if $\mathcal{P}'$ never enters an antidiagonal block, then this is vacuous. Otherwise, the $\times$ itself lies one row farther south from the top of its block row, and we must show why this difference is recovered by $\mathcal{P}'$. Think of each D'_j as having a blank column (no $+$ entries) along the western edge of its strip, and a blank row along the southern edge of its strip. The pipe $\mathcal{P}'$ recovers one unit of height as it passes through each such region, and it passes through exactly one on the way to its first antidiagonal block.

Suppose that both $\mathcal{P}$ and $\mathcal{P}'$ enter an antidiagonal block at the same vertical distance from the block's northwest corner. Since D' agrees with D in each antidiagonal block (as measured from their northwest corners), the pipe $\mathcal{P}'$ exits the antidiagonal block in the same position relative to its entry point as does $\mathcal{P}$. Therefore we can use induction on the number of antidiagonal blocks traversed by $\mathcal{P}$ to show that the pipe $\mathcal{P}'$ exits *every* antidiagonal block in the same location relative to its northwest corner as does $\mathcal{P}$. If the $\times$ lies in the rightmost block column, then $\mathcal{P}'$ has already finished in the correct column. Otherwise, $\mathcal{P}'$ must recover one more horizontal unit, which it does by traversing the blank row and column in the top horizontal strip.

For the second claim, we need only show that for any given $D' \in \text{top}(1 + \mathbf{r})$, each horizontal strip in D' has a column of elbows along its western edge, a row of elbows along its southern edge, and an antidiagonal's worth of elbows on the superantidiagonal, for then each miniature top pipe dream D'_j fits inside the deflated block matrix for $v(\mathbf{r})$ to make $D \in \text{top}(\mathbf{r})$. The antidiagonal elbows come from the pipe added to make $s_{0n} \geq 1$: by Theorem 5.10 there is a pipe going from the southwest block to the northeast block, and by Theorem 3.8 this pipe crosses no others. In particular, the southeast corner of each superantidiagonal block in D' is an elbow. Lemma 5.13 implies the southern elbow row, and its transpose (applied to $v(1 + \mathbf{r})^{-1} = v(1 + \mathbf{r}')$ for the transpose $\mathbf{r}'$ of $\mathbf{r}$) implies the western elbow column. □

Interpreting Proposition 5.15 on lacing diagrams for pipe dreams yields another useful stability statement that we shall apply in Section 6; compare Corollary 4.12.

Corollary 5.16. *Let $W_{\mathcal{RP}}(\mathbf{r}) = \{\mathbf{w}(D) \mid D \in \mathcal{RP}(v(\mathbf{r}))\}$ be the set of lacing diagrams obtained by decomposition of reduced pipe dreams for $v(\mathbf{r})$ into horizontal strips as in Definition 5.8 and Theorem 5.10. Then $W_{\mathcal{RP}}(m + \mathbf{r}) = \{m + \mathbf{w} \mid \mathbf{w} \in W_{\mathcal{RP}}(\mathbf{r})\}$ is obtained by shifting each partial permutation list in $W_{\mathcal{RP}}(\mathbf{r})$ up by m.*

Proof. Let $D \in \text{top}(\mathbf{r})$ correspond to $D' \in \text{top}(1+\mathbf{r})$. The miniature top pipe dream D'_j in the j^{th} horizontal strip of D' is one row farther from the bottom row of that strip than D_j is from the bottom row in its strip, by Lemma 5.11 and Proposition 5.15. Reading (partial) permutations in each horizontal strip with the southeast corner as the origin, the result follows from Lemma 5.14. □

Section 6. Component formulae

Our goal in this section is to prove that stable limits of double quiver polynomials exist for uniformly increasing ranks (Theorem 6.11), and to deduce from it the various forms of the component formula (Corollary 6.17, Theorem 6.20, and Corollary 6.23). Along the way, we complete the combinatorial characterization of the geometric components in quiver degenerations (Theorem 6.16). Since the proofs gather together most of what we have done, we begin this section with an interlude: a summary of the roles played by the results thus far in the coming sections.

A combinatorial hierarchy. It will be helpful to have in mind a natural ordering of the four formulae, different from the order in which we presented them in the Introduction (which was determined by our proofs). The combinatorial objects indexing the summands in the four formulae can all be described in terms of the set $\mathcal{RP}(v(\mathbf{r}))$ of reduced pipe dreams for the Zelevinsky permutation $v(\mathbf{r})$. More precisely, there are four partitions of the set $\mathcal{RP}(v(\mathbf{r}))$ such that each type of combinatorial object corresponds to an equivalence class in one of these partitions. The hierarchy is as follows.

1. Pipe formula: equivalence classes are singletons.
2. Tableau formula: equivalence classes are *key classes*. There is a bijection from pipe dreams in $\mathcal{RP}(v(\mathbf{r}))$ to 'compatible pairs' $(\mathbf{a}, \mathbf{i})$ in which $\mathbf{a}$ is a reduced word for the Zelevinsky permutation and $\mathbf{i}$ is a 'compatible sequence' of positive integers [BB93]. The set of compatible pairs can be partitioned according to the Coxeter–Knuth classes of the reduced words $\mathbf{a}$. The generating function of each equivalence class with respect to the weight of the compatible sequences $\mathbf{i}$ is known as a 'key polynomial' or 'Demazure character of type A' [RS95b]. These polynomials are crucial to our derivation of the tableau formula (Section 7).
3. Component formula: equivalence classes are strip classes, indexed by lacing diagrams. One of our favorite combinatorial results in this paper is the combination of Theorem 5.10 and Corollary 6.18. These say that out of every reduced pipe dream for $v(\mathbf{r})$ comes a minimal length lacing diagram with rank array $\mathbf{r}$. Section 5.2 describes the simple construction, which divides each pipe dream into horizontal strips (block rows). Two pipe dreams are equivalent if they produce the same lacing diagram—that is, if their pipes carry out the same partial permutations in each strip.
4. Ratio formula: only one equivalence class.

Roughly speaking, summing monomials $(\mathbf{x} - \mathring{\mathbf{x}})^{D \smallsetminus D_{Hom}}$ over an equivalence class in one of these four partitions yields a summand in one of the four formulae. This is content-free for the ratio and pipe formulae. Moreover, we do not explicitly use this idea for the tableau formula, although it is faintly evident in the proof of Proposition 7.13. That proof is also suggestive of the essentially true statement that key equivalence refines lacing equivalence, which is the fundamental (though not explicit) principle behind our derivation of the peelable tableau formula from the component formula. For the component formula itself, summation over a lacing equivalence class is an essential operation, but it only *roughly* yields a lacing diagram summand. This is the first of many situations where stable limits come to the rescue.

Limits and stabilization. There are two primary aspects to our proof of the ordinary component formula: we must show that

- components in the degeneration $\Omega_{\mathbf{r}}(0)$ have multiplicity 1, and they are exactly the closures of the orbits through minimal length lacing diagrams with ranks $\mathbf{r}$; and
- the summands can be taken to be products of Stanley symmetric functions rather than products of double Schubert polynomials.

The Schubert version for ordinary quiver polynomials follows immediately from the first of the two statements above, by Corollary 4.9. But keep in mind that we need to prove a component formula for *double* quiver polynomials to be able to deduce the tableau formula for quiver constants, as outlined at the end of the Introduction.

The two aspects itemized above split the proof into two essentially disjoint parts: a geometric argument producing a lower bound, and a combinatorial argument producing an upper bound. The conclusion comes by noticing that the lower bound is visibly at least as high as the upper bound, so that both must be equal. Let us make this vague description more precise, starting with the geometric lower bound.

The mere existence of the flat degeneration in Section 4.2 automatically implies that the ordinary quiver polynomial $\mathcal{Q}_{\mathbf{r}}(\mathbf{x} - \mathring{\mathbf{x}})$ is a sum of double Schubert products $\mathfrak{S}_{\mathbf{w}}(\mathbf{x} - \mathring{\mathbf{x}})$, with positive integer coefficients possibly greater than 1 (Corollary 4.9). This is because the components in the degeneration are products of matrix Schubert varieties, possibly nonreduced a priori (Theorem 4.6). Moreover, it is easy to verify that the orbit through each lacing diagram in $W(\mathbf{r})$ appears with multiplicity at least 1 (Proposition 4.11).

One of the fundamental observations in [BF99] gained by viewing quiver polynomials as ordinary classes of degeneracy loci for vector bundle maps on arbitrary schemes is that nothing vital changes when the ranks of all the bundles are increased by 1, or by any fixed integer m. The key observation for us along these lines is that this stability under uniformly increasing ranks can be proved directly for the total families of quiver locus degenerations—and hence for components in the special fiber, including their scheme structures—by a direct geometric argument (Proposition 4.13). As a consequence, the previous paragraph, which a priori only gives existence of a Schubert polynomial component formula along with lower bounds on its coefficients, in fact proves the *same* existence and lower bounds for an ordinary Stanley function component formula, simply by taking limits (see Remark 4.15).

The combinatorial argument producing an upper bound is based on the partition of the pipe dreams in $\mathcal{RP}(v(\mathbf{r}))$ into strip classes, and it automatically works in the 'double' setting, with $\mathbf{x}$ and $\mathbf{y}$ alphabets. Summing $(\mathbf{x} - \mathring{\mathbf{y}})^{D \smallsetminus D_{Hom}}$ over reduced pipe dreams D with strip equivalence class $\mathbf{w}$ yields a polynomial that, while not actually equal to $\mathfrak{S}_{\mathbf{w}}(\mathbf{x} - \mathring{\mathbf{y}})$ (Remark 6.26), agrees with $\mathfrak{S}_{\mathbf{w}}(\mathbf{x} - \mathring{\mathbf{y}})$ in all coefficients on monomials all of whose variables appear "not too late" in the alphabets $\mathbf{x}^0, \ldots, \mathbf{x}^n$ and $\mathbf{y}^0, \ldots, \mathbf{y}^n$; see Corollary 6.10. This observation depends subtly on the combinatorics of reduced pipe dreams, and on the symmetry of $\mathcal{Q}_{\mathbf{r}}(\mathbf{x} - \mathring{\mathbf{y}})$ in each of its $2n+2$ alphabets as in Proposition 6.9.

Just as in the geometric argument, we have a stability statement. Here, it says that strip classes are stable under uniform increase in rank (Corollary 5.16). Consequently, summing over all strip classes and taking the limit as the ranks uniformly become infinite yields Theorem 6.11, the "existence and upper bound theorem" for the resulting *double quiver functions* $\mathcal{F}_{\mathbf{r}}(\mathbf{x} - \mathring{\mathbf{y}})$. That the sum is over strip classes, whose lacing diagrams automatically lie inside $W(\mathbf{r})$ by Theorem 5.10, implies the upper bound: the coefficients on $F_{\mathbf{w}}(\mathbf{x} - \mathring{\mathbf{y}})$ is 1 if $\mathbf{w}$ indexes a strip class, and zero otherwise.

After specializing the strip class component formula in Theorem 6.11 to $\mathring{\mathbf{y}} = \mathring{\mathbf{x}}$, the geometric lower bound and the combinatorial upper bound are forced to match. This argument comprises the proof of Theorem 6.16 on the combinatorics and geometry of the components in quiver degenerations, and it also implies the bijection between strip classes and lacing diagrams of minimal length (Corollary 6.18). The stable double component formula (Theorem 6.20) is an immediate consequence.

To convey a better feel for what limits accomplish here, let us summarize the fundamental principle that drives most of our logic. There are a number of geometric, algebraic, and combinatorial objects that we are unable to get a precise handle on directly. These objects include the components of quiver degenerations, as well as their multiplicities; the sum of all monomials associated to pipe dreams in a fixed strip class; and the set of lacing diagrams indexing strip classes for a fixed Zelevinsky permutation. (In an earlier draft, peelable tableaux were on this list, as well; see Corollary 8.21 and the comment following it.) The key realization is that, even though we fail to identify these objects precisely, we can prove their stability under uniformly increasing ranks, and in the limit it becomes possible to identify the desired objects exactly. Hence our failures occurred only because we fixed the ranks to begin with.

Let us note that our failures in fixed finite rank are not always artifacts of our proof technique. For example, the sum of all monomials for pipe dreams in a fixed strip class $\mathbf{w}$ really *isn't* equal to the Stanley product $F_{\mathbf{w}}$ or the Schubert product $\mathfrak{S}_{\mathbf{w}}$; it *only* becomes so in the limit.

6.1. Double Stanley symmetric functions

Symmetric functions are formal power series in infinite alphabets, which may be specialized to polynomials in alphabets of finite size. We prepare conventions for such specializations, since they will be crucial to our line of reasoning.

Convention 6.1. If a polynomial requiring r or more variables for input is evaluated on an alphabet with $r' < r$ letters, then the remaining $r - r'$ variables are to be set to zero. Conversely, if a polynomial requiring no more than r variables for input is evaluated on an alphabet of size $r' > r$, then the last $r' - r$ variables are ignored.

Example 6.2. Consider the four $\mathbf{bc}$ blocks obtained from the big pipe dream in Example 5.4:

$$\begin{array}{c|cccc|ccc|}
 & y_1^2 & y_2^2 & y_3^2 & y_4^2 & y_1^1 & y_2^1 & y_3^1 \\ \cline{2-8}
x_1^1 & \cdot & \cdot & + & + & \cdot & \cdot & \cdot \\
x_2^1 & \cdot & \cdot & \cdot & \cdot & + & \cdot & \cdot \\
x_3^1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cline{2-8}
x_1^2 & \cdot & + & \cdot & \cdot & \cdot & \cdot & \cdot \\
x_2^2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
x_3^2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
x_4^2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cline{2-8}
\end{array}$$

This is a reduced pipe dream for the permutation $w = 1253746$. The double Schubert polynomial $\mathfrak{S}_{1253746}(\mathbf{x}^1, \mathbf{x}^2 - \mathbf{y}^2, \mathbf{y}^1)$ is a sum of 'monomials' of the form $(x - y)$, one for each reduced pipe dream $D \in \mathcal{RP}(1253746)$. Setting $\mathbf{x}^2 = \mathbf{y}^1 = 0$ in $\mathfrak{S}_{1253746}(\mathbf{x}^1, \mathbf{x}^2 - \mathbf{y}^2, \mathbf{y}^1)$ yields $\mathfrak{S}_{1253746}(\mathbf{x}^1 - \mathbf{y}^2)$. In the expression of this polynomial as a sum of 'monomials', reduced pipe dreams with crosses outside the upper-left block contribute differently to $\mathfrak{S}_w(\mathbf{x}^1 - \mathbf{y}^2)$ than they did to $\mathfrak{S}_w(\mathbf{x}^1, \mathbf{x}^2 - \mathbf{y}^2, \mathbf{y}^1)$. The pipe dream depicted above, for example, contributes $(x_1^1 - y_3^2)(x_1^1 - y_4^2)(x_2^1 - y_1^1)(x_1^2 - y_2^2)$ to $\mathfrak{S}_w(\mathbf{x}^1, \mathbf{x}^2 - \mathbf{y}^2, \mathbf{y}^1)$, but only $(x_1^1 - y_3^2)(x_1^1 - y_4^2)(x_2^1)(-y_2^2)$ to $\mathfrak{S}_w(\mathbf{x}^1 - \mathbf{y}^2)$. □

Suppose that $\mathbf{x} = \mathbf{x}^0, \mathbf{x}^1, \ldots, \mathbf{x}^n$ and $\mathring{\mathbf{y}} = \mathbf{y}^n, \ldots, \mathbf{y}^1, \mathbf{y}^0$ are two ordered finite lists of infinite alphabets. A sequence $p_m(\mathbf{x}, \mathring{\mathbf{y}})$ of polynomials in these lists (or any infinite alphabet) is said to **converge** to a power series $p(\mathbf{x}, \mathring{\mathbf{y}})$ if the coefficient on any fixed monomial is eventually constant as a function of m. Equivalently, we say the **limit exists**, and write

$$\lim_{m\to\infty} p_m(\mathbf{x}, \mathring{\mathbf{y}}) \quad = \quad p(\mathbf{x}, \mathring{\mathbf{y}}).$$

Here, $p(\mathbf{x}, \mathring{\mathbf{y}})$ is allowed to be an arbitrary formal sum of monomials with integer coefficients in variables from the union of alphabets in the lists $\mathbf{x}$ and $\mathring{\mathbf{y}}$.

Recall from Section 4.4 the notation $m + w$ for nonnegative integers m and partial permutations w.

Proposition 6.3. *Given a partial permutation w and an infinite alphabet $\mathcal{X}$, the limit*

$$F_w(\mathcal{X}) \quad = \quad \lim_{m\to\infty} \mathfrak{S}_{m+w}(\mathcal{X})$$

exists and is symmetric in $\mathcal{X}$. (Only finitely many $\mathcal{X}$ variables appear in each $\mathfrak{S}_{m+w}(\mathcal{X})$.)

Definition 6.4. The limit $F_w(\mathcal{X})$ is called the **Stanley symmetric function** or **stable Schubert polynomial** for w. (N.B. In the notation of [Sta84] the permutation w^{-1} is used for indexing instead of w).

Proof. After harmlessly extending w to an honest permutation in the minimal way (which leaves the Schubert polynomials unchanged), see [Mac91, Section VII]. □

Stanley's original definition of F_w was combinatorial, and motivated by the fact that the coefficient of any squarefree monomial in F_w is the number of reduced decompositions of the permutation w. The fact that Stanley's formulation of F_w is a stable Schubert polynomial in the above sense follows immediately from the formula for the Schubert polynomial given in [BJS93, Theorem 1.1].

Being a symmetric function, F_w can be written as a sum

$$F_w(\mathcal{X}) \quad = \quad \sum_\lambda \alpha_w^\lambda s_\lambda(\mathcal{X}) \tag{6.1}$$

of Schur functions s_λ with coefficients α_w^λ, which we call **Stanley coefficients**. Stanley credits Edelman and Greene [EG87] with the first proof that these coefficients α_w^λ are *nonnegative*. There are various descriptions for α_w^λ, all combinatorial: reduced word tableaux [LS89, EG87], the leaves of shape λ in the transition tree for w [LS85], certain promotion sequences [Hai92], and peelable tableaux [RS95a]. The coefficients α_w^λ are also special cases of the quiver constants $c_{\underline{\lambda}}(\mathbf{r})$ from Theorem 1.20 [Buc01a]. We shall prove the converse in Theorem 7.14, below: the quiver constants $c_{\underline{\lambda}}(\mathbf{r})$ are special cases of Stanley coefficients α_w^λ.

Proposition 6.5. *Given a partial permutation w and infinite alphabets $\mathcal{X}, \mathcal{Y}$, the limit*

$$F_w(\mathcal{X} - \mathcal{Y}) \quad = \quad \lim_{m\to\infty} \mathfrak{S}_{m+w}(\mathcal{X} - \mathcal{Y})$$

of double Schubert polynomials exists and is symmetric separately in $\mathcal{X}$ and $\mathcal{Y}$.

Definition 6.6. The limit $F_w(\mathcal{X} - \mathcal{Y})$ is the **double Stanley symmetric function** or **stable double Schubert polynomial**.

Proof. Again, we may as well assume w is an honest permutation. One may reduce to the "single" stable Schubert case by applying the coproduct formula [Mac91, (6.3)]

$$\mathfrak{S}_w(\mathcal{X}-\mathcal{Y}) \;=\; \sum_{u,v}(-1)^{\ell(v)}\mathfrak{S}_u(\mathcal{X})\mathfrak{S}_{v^{-1}}(\mathcal{Y})$$

where the sum runs over the pairs of permutations u, v such that $vu = w$ and the sum $\ell(u) + \ell(v)$ of the lengths of v and u equals the length $\ell(w)$ of w. □

For a sequence $\mathbf{w} = w_1, \ldots, w_n$ of partial permutations of size $r_0 \times r_1, \ldots, r_{n-1} \times r_n$, and two sequences $\mathbf{x} = \mathbf{x}^0, \ldots, \mathbf{x}^n$ and $\mathring{\mathbf{y}} = \mathbf{y}^n, \ldots, \mathbf{y}^0$ of infinite alphabets, denote by

$$\mathfrak{S}_{\mathbf{w}}(\mathbf{x}-\mathring{\mathbf{y}}) \;=\; \prod_{j=1}^{n}\mathfrak{S}_{w_j}(\mathbf{x}^{j-1}_{\mathbf{r}}-\mathbf{y}^{j}_{\mathbf{r}})$$

the product of double Schubert polynomials in consecutive $\mathbf{x}$ and $\mathbf{y}$ alphabets. As in Definitions 1.18 and 2.5, only finitely many variables actually appear in $\mathfrak{S}_{\mathbf{w}}(\mathbf{x}-\mathring{\mathbf{y}})$. Write

$$F_{\mathbf{w}}(\mathbf{x}-\mathring{\mathbf{y}}) \;=\; \prod_{j=1}^{n}F_{w_j}(\mathbf{x}^{j-1}-\mathbf{y}^{j}) \tag{6.2}$$

for the corresponding product of double Stanley symmetric functions, where now all of the alphabets are infinite. If, in (6.2), we need to specify finite alphabets (2.3), when ranks $\mathbf{r}$ have been set so that $\mathbf{x}^j$ and $\mathbf{y}^j$ have size r_j, then we shall write explicitly $F_{\mathbf{w}}(\mathbf{x}_{\mathbf{r}}-\mathring{\mathbf{y}}_{\mathbf{r}})$. Keep in mind that the power series in (6.2) remains unchanged when $\mathbf{w} = (w_1, \ldots, w_n)$ is replaced by $m + \mathbf{w}$ for $m \in \mathbb{N}$.

6.2. Double quiver functions

Stanley symmetric functions arise in our context by taking limits of double quiver polynomials via the pipe formula, Theorem 5.5. Each reduced pipe dream D for the Zelevinsky permutation $v(\mathbf{r})$ determines a polynomial $\mathfrak{S}_{\mathbf{w}(D)}(\mathbf{x}-\mathring{\mathbf{y}})$ by Definition 5.8, to which D contributes a "monomial", as in Section 5.1. Here, in Proposition 6.9, we shall see that each reduced pipe dream for $v(\mathbf{r})$ can contribute a different monomial by way of a crucial symmetry, as follows.

Definition 6.7. The **reverse monomial** associated to a $d \times d$ pipe dream D is the product $(\tilde{\mathbf{x}}_{\mathbf{r}}-\tilde{\mathbf{y}}_{\mathbf{r}})^D$ over all $+$ entries in D of $(\tilde{x}_+ - \tilde{y}_+)$, where the variable $\tilde{x}_+$ sits at the left end of the row containing $+$ after reversing each of the $n+1$ alphabets in $\mathbf{x}_{\mathbf{r}}$, and $\tilde{y}_+$ sits atop the column containing $+$ after reversing each of the $n+1$ alphabets in $\mathring{\mathbf{y}}_{\mathbf{r}}$.

Example 6.8. Reversing each of the row and column alphabets in Example 5.4 gives

$$\begin{array}{c|ccc|cccc|ccc|cc|}
 & y^3_3 & y^3_2 & y^3_1 & y^2_4 & y^2_3 & y^2_2 & y^2_1 & y^1_3 & y^1_2 & y^1_1 & y^0_2 & y^0_1 \\ \hline
x^0_2 & * & * & * & * & * & * & * & \cdot & \cdot & + & \cdot & \cdot \\
x^0_1 & * & * & * & * & * & * & * & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline
x^1_3 & * & * & * & \cdot & \cdot & + & + & \cdot & \cdot & \cdot & \cdot & \cdot \\
x^1_2 & * & * & * & \cdot & \cdot & \cdot & \cdot & + & \cdot & \cdot & \cdot & \cdot \\
x^1_1 & * & * & * & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline
x^2_4 & \cdot & \cdot & + & \cdot & + & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
x^2_3 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
x^2_2 & \cdot & + & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
x^2_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline
x^3_3 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
x^3_2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
x^3_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline
\end{array}
\quad = \quad
\begin{array}{c|ccc|cccc|ccc|cc|}
 & \dot{d}_3 & \dot{d}_2 & \dot{d}_1 & \dot{c}_4 & \dot{c}_3 & \dot{c}_2 & \dot{c}_1 & \dot{b}_3 & \dot{b}_2 & \dot{b}_1 & \dot{a}_2 & \dot{a}_1 \\ \hline
a_2 & * & * & * & * & * & * & * & \cdot & \cdot & + & \cdot & \cdot \\
a_1 & * & * & * & * & * & * & * & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline
b_3 & * & * & * & \cdot & \cdot & + & + & \cdot & \cdot & \cdot & \cdot & \cdot \\
b_2 & * & * & * & \cdot & \cdot & \cdot & \cdot & + & \cdot & \cdot & \cdot & \cdot \\
b_1 & * & * & * & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline
c_4 & \cdot & \cdot & + & \cdot & + & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
c_3 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
c_2 & \cdot & + & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
c_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline
d_3 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
d_2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
d_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline
\end{array}$$

This reduced pipe dream contributes the reverse monomial

$$(a_2 - \dot{b}_1)(b_3 - \dot{c}_2)(b_3 - \dot{c}_1)(b_2 - \dot{b}_3)(c_4 - \dot{d}_1)(c_4 - \dot{c}_3)(c_2 - \dot{d}_2)$$

to the double quiver polynomial $\mathcal{Q}_{\mathbf{r}}(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} - \dot{\mathbf{d}}, \dot{\mathbf{c}}, \dot{\mathbf{b}}, \dot{\mathbf{a}})$. Reversing the alphabets in the argument of this polynomial leaves it invariant, because by Proposition 6.9 this polynomial already equals the sum of all reverse monomials from reduced pipe dreams for $v(\mathbf{r})$. □

In conjunction with Corollary 6.10, the next observation will allow us to take limits of double quiver polynomials effectively, via Proposition 6.5.

Proposition 6.9. *The double quiver polynomial $\mathcal{Q}_{\mathbf{r}}(\mathbf{x} - \mathring{\mathbf{y}})$ is symmetric separately in each of the alphabets $\mathbf{x}^i_{\mathbf{r}}$ as well as in each of the alphabets $\mathbf{y}^j_{\mathbf{r}}$. In particular,*

$$\mathcal{Q}_{\mathbf{r}}(\mathbf{x} - \mathring{\mathbf{y}}) \quad = \sum_{D \in \mathcal{RP}(v(\mathbf{r}))} (\tilde{\mathbf{x}}_{\mathbf{r}} - \tilde{\mathbf{y}}_{\mathbf{r}})^{D \smallsetminus D_{Hom}}.$$

where $(\tilde{\mathbf{x}}_{\mathbf{r}} - \tilde{\mathbf{y}}_{\mathbf{r}})^{D \smallsetminus D_{Hom}}$ is the reverse monomial associated to the $d \times d$ pipe dream $D \smallsetminus D_{Hom}$.

Proof. Matrix Schubert varieties for Zelevinsky permutations are preserved by the action of block diagonal matrices—including block diagonal permutation matrices—on the right and left. Now apply the symmetry to Theorem 5.5. □

Next we record an important consequence of Proposition 5.15.

Corollary 6.10. *There is a fixed integer ℓ, independent of m, such that setting the last ℓ variables to zero in every finite alphabet from the lists $\mathbf{x}_{m+\mathbf{r}}$ and $\mathring{\mathbf{y}}_{m+\mathbf{r}}$ kills the reverse monomial for every reduced pipe dream $D \in \mathcal{RP}(v(m+\mathbf{r}))$ containing at least one cross $+$ in an antidiagonal block.*

Proof. Suppose D is a reduced pipe dream for $v(m + \mathbf{r})$. Divide D into horizontal strips (block rows) as in Section 5.3, and consider the reduced pipe dream D' in the same chute class as D that consists of a top pipe dream in each strip. The miniature top pipe dream in horizontal strip j has an eastern half D'_j in the j^{th} antidiagonal block. Inside the j^{th} antidiagonal block, consider the highest antidiagonal A_j above which every $+$ in D'_j lies. Thus A_j bounds an isosceles right triangular region $\triangledown_j(D)$ in the northwest corner of the j^{th} antidiagonal block that contains every $+$ in D'_j.

We claim that $\triangledown_j(D)$ in fact contains every $+$ of D itself that lies in the j^{th} antidiagonal block. Indeed, suppose that D contains a $+$ in the j^{th} antidiagonal block. Using (any sequence of) upward chute moves to get from D to D', the antidiagonal $+$ we are considering must end up inside $\triangledown_j(D)$, because chuting any $+$ up pushes it north and east. Our claim follows because upward chutes push each $+$ at least as far east as north.

Now define $\ell(D)$ to be the maximum of the leg lengths of the isosceles right triangles $\triangledown_j(D)$. Chuting up as in the previous paragraph, both the row variable $\tilde{x}_+$ and the column variable $\tilde{y}_+$ of each antidiagonal $+$ in D must lie within $\ell(D)$ of the end of its finite alphabet. Therefore, pick ℓ to be the maximum of the numbers $\ell(D)$ for reduced pipe dreams $D \in \mathcal{RP}(v(m + \mathbf{r}))$. The key (and final) point is that ℓ does not depend on m for $m \geq 0$, by Proposition 5.15. □

The main result of this section concerning double quiver polynomials $\mathcal{Q}_{\mathbf{r}}(\mathbf{x} - \mathring{\mathbf{y}})$ says that limits exist as the ranks $\mathbf{r}$ get replaced by $m + \mathbf{r}$ for $m \to \infty$. We consider this an algebraic rather than a combinatorial result (the latter being Theorem 6.20), since we are unable at this stage to identify the set $W_{\mathcal{RP}}(\mathbf{r})$ from Corollary 5.16 in a satisfactory way. Note that $W_{\mathcal{RP}}(\mathbf{r})$ is a set of distinct partial permutation lists—not a multiset—contained in the set $W(\mathbf{r})$ of minimum length lacing diagrams with ranks $\mathbf{r}$, by Theorem 5.10. We shall see in Corollary 6.18 that in fact $W_{\mathcal{RP}}(\mathbf{r}) = W(\mathbf{r})$; but until then it is important for the proofs to distinguish $W_{\mathcal{RP}}(\mathbf{r})$ from $W(\mathbf{r})$.

Theorem 6.11. *The limit of double quiver polynomials $\mathcal{Q}_{m+\mathbf{r}}(\mathbf{x}-\mathring{\mathbf{y}})$ for m approaching ∞ exists and equals the multiplicity-free sum*

$$\mathcal{F}_{\mathbf{r}}(\mathbf{x}-\mathring{\mathbf{y}}) \;:=\; \lim_{m\to\infty} \mathcal{Q}_{m+\mathbf{r}}(\mathbf{x}-\mathring{\mathbf{y}}) \;=\; \sum_{\mathbf{w}\in W_{\mathcal{RP}}(\mathbf{r})} F_{\mathbf{w}}(\mathbf{x}-\mathring{\mathbf{y}})$$

of products of double Stanley symmetric functions. The limit power series $\mathcal{F}_{\mathbf{r}}(\mathbf{x}-\mathring{\mathbf{y}})$ is symmetric separately in each of the $2n+2$ infinite alphabets $\mathbf{x}^0,\ldots,\mathbf{x}^n$ and $\mathbf{y}^n,\ldots,\mathbf{y}^0$.

Definition 6.12. $\mathcal{F}_{\mathbf{r}}(\mathbf{x}-\mathring{\mathbf{y}})$ is called the **double quiver function** for the rank array $\mathbf{r}$.

Proof. For all $m \geq 0$, the set $W_{\mathcal{RP}}(m+\mathbf{r})$ is obtained from $W_{\mathcal{RP}}(\mathbf{r})$ by replacing each permutation list $\mathbf{w}$ with $m+\mathbf{w}$, by Corollary 5.16. For each partial permutation list $\mathbf{w} \in W_{\mathcal{RP}}(\mathbf{r})$, consider the set $\mathcal{RP}_{\mathbf{w}}(v(m+\mathbf{r}))$ of reduced pipe dreams D for $v(m+\mathbf{r})$ whose lacing diagrams $\mathbf{w}(D)$ equal $m+\mathbf{w}$. The sum of all reverse monomials for pipe dreams $D \in \mathcal{RP}_{\mathbf{w}}(v(m+\mathbf{r}))$ is a product of polynomials, each factor coming from the miniature pipe dreams in a single horizontal strip. The double quiver polynomial $\mathcal{Q}_{m+\mathbf{r}}(\mathbf{x}-\mathring{\mathbf{y}})$ is the sum of these products, the sum being over lacing diagrams $\mathbf{w} \in W_{\mathcal{RP}}(\mathbf{r})$. A similar statement can be made after setting the last ℓ variables in each alphabet to zero as in Corollary 6.10, although now $\mathcal{Q}_{\mathbf{r}}(\mathbf{x}-\mathring{\mathbf{y}})$ and the polynomials for each lacing diagram $\mathbf{w} \in W_{\mathcal{RP}}(\mathbf{r})$ have changed a little (but only in those terms involving a variable near the end of some alphabet).

We claim that now, after setting the last few variables in each alphabet to zero, the j^{th} polynomial in the product for any fixed diagram $\mathbf{w}$ equals the result of setting the last few variables (at most ℓ) to zero in the honest double Schubert polynomial $\mathfrak{S}_{m+w_j}(\mathbf{x}^{j-1}-\mathbf{y}^j)$. For this, note that every reduced pipe dream D for $v(\mathbf{r})$ contributing a nonzero reverse monomial now has $+$ locations only in superantidiagonal blocks. Hence we can read the restriction of D to the j^{th} superantidiagonal block as a reduced pipe dream with its southeast corner as the origin. Since the variables in this one-block rectangle that we have set to zero are far from the southeast corner, our claim is proved.

Taking the limit as $m \to \infty$ produces the desired product of double Stanley symmetric functions by Proposition 6.5. $\square$

The limit in Theorem 6.11 stabilizes in the following strong sense. The analogous stabilization for the limit defining Stanley symmetric functions from Schubert polynomials (Proposition 6.3) almost never occurs, since the finite variable specialization is symmetric, while Schubert polynomials rarely are, even considering only variables that actually occur.

Corollary 6.13. *For all $m \gg 0$, the double quiver polynomial $\mathcal{Q}_{m+\mathbf{r}}(\mathbf{x}-\mathring{\mathbf{y}})$ actually equals the specialization $\mathcal{F}_{\mathbf{r}}(\mathbf{x}_{m+\mathbf{r}}-\mathring{\mathbf{y}}_{m+\mathbf{r}})$ of the double quiver function.*

Proof. The double quiver polynomial $\mathcal{Q}_{m+\mathbf{r}}(\mathbf{x}-\mathring{\mathbf{y}})$ has fixed degree and is symmetric in each of its finite alphabets for all m (Proposition 6.9). Hence we need only show that for any fixed monomial of degree at most $\deg(\mathcal{Q}_{m+\mathbf{r}}(\mathbf{x}-\mathring{\mathbf{y}}))$, applying a symmetry yields a monomial whose coefficient equals the coefficient on the corresponding monomial in $\mathcal{F}_{\mathbf{r}}(\mathbf{x}-\mathring{\mathbf{y}})$. This follows immediately from Theorem 6.11, because for $m \gg 0$, all monomials involving variables whose lower indices are at most $\deg(\mathcal{Q}_{m+\mathbf{r}}(\mathbf{x}-\mathring{\mathbf{y}}))$ have the correct coefficients. $\square$

The previous result led us to suspect that the '$\gg$' sign may be replaced by '$\geq$'.

Conjecture 6.14. *$\mathcal{Q}_{m+\mathbf{r}}(\mathbf{x} - \mathring{\mathbf{y}}) = \mathcal{F}_{\mathbf{r}}(\mathbf{x}_{m+\mathbf{r}} - \mathring{\mathbf{y}}_{m+\mathbf{r}})$ for all $m \geq 0$.*

After we made Conjecture 6.14 in the first draft of this paper, A. Buch responded by proving it in [Buc03]. We have no need for the full strength of Conjecture 6.14 in this paper, since Corollary 6.13 suffices; see Proposition 7.13, below, and the last line of its proof. We shall comment on related issues in Remarks 6.24, 6.25, and 6.26.

We shall see later, after a number of intermediate steps leading to Theorem 7.10, that $\mathcal{F}_{\mathbf{r}}(\mathbf{x} - \mathring{\mathbf{y}})$ is equal to the power series obtained by replacing the second argument $\mathring{\mathbf{x}}$ of the Buch–Fulton power series $\sum_{\underline{\lambda}} c_{\underline{\lambda}}(\mathbf{r}) s_{\underline{\lambda}}(\mathbf{x} - \mathring{\mathbf{x}})$ with $\mathring{\mathbf{y}}$, or equivalently by the appropriate specialization of the abstract symmetric functions P_r in [Buc01a] to pairs of infinite alphabets in each factor. As a consequence, the double quiver functions turn out to equal the sums of lowest-degree terms in the power series P_r defined by Buch in [Buc02, Section 4]. But until Theorem 7.10, the term 'double quiver function' will refer to the limit in Theorem 6.11.

6.3. All components are lacing diagram orbit closures

Recall from Theorem 2.9 that the quiver polynomial $\mathcal{Q}_{\mathbf{r}}(\mathbf{x} - \mathring{\mathbf{x}})$ is obtained from the double quiver polynomial $\mathcal{Q}_{\mathbf{r}}(\mathbf{x} - \mathring{\mathbf{y}})$ by setting $\mathbf{y}^j = \mathbf{x}^j$ for all j, so $\mathring{\mathbf{y}} = \mathring{\mathbf{x}}$. Here, we shall need to specialize products $F_{\mathbf{w}}(\mathbf{x} - \mathring{\mathbf{y}})$ and $\mathfrak{S}_{\mathbf{w}}(\mathbf{x} - \mathring{\mathbf{y}})$ of double Schubert and Stanley symmetric functions by setting $\mathring{\mathbf{y}} = \mathring{\mathbf{x}}$, to get $F_{\mathbf{w}}(\mathbf{x} - \mathring{\mathbf{x}})$ and $\mathfrak{S}_{\mathbf{w}}(\mathbf{x} - \mathring{\mathbf{x}})$. Note that the alphabet $\mathbf{x}^j$ obtained by specializing $\mathbf{y}^j$ never interferes in a catastrophic cancelative manner with the original alphabet $\mathbf{x}^j$, because for any lacing diagram $\mathbf{w}$, the j^{th} factors of $\mathfrak{S}_{\mathbf{w}}(\mathbf{x} - \mathring{\mathbf{x}})$ and $F_{\mathbf{w}}(\mathbf{x} - \mathring{\mathbf{x}})$ involve the distinct alphabets $\mathbf{x}^{j-1}$ and $\mathbf{x}^j$.

Lemma 6.15. *Any nonempty positive sum of products $F_{\mathbf{w}}(\mathbf{x} - \mathring{\mathbf{x}})$ of double Stanley symmetric functions in differences of consecutive alphabets is nonzero.*

Proof. One may use the double version of (6.1) to write each term $F_{\mathbf{w}}(\mathbf{x} - \mathring{\mathbf{x}})$ as a nonnegative sum of products of Schur functions in differences of consecutive alphabets. Such products are linearly independent by the remark in [BF99, Section 2.2]. □

Now we can finally identify the components of quiver degenerations.

Theorem 6.16. *The components of the quiver degeneration $\Omega_{\mathbf{r}}(0)$ are exactly the orbit closures $\mathcal{O}(\mathbf{w})$ for $\mathbf{w} \in W(\mathbf{r})$. Furthermore, $\Omega_{\mathbf{r}}(0)$ is generically reduced along each component $\mathcal{O}(\mathbf{w})$, so its multiplicity $c_{\mathbf{w}}(\mathbf{r})$ there equals 1.*

Proof. Applying Corollary 4.9 to $\mathcal{Q}_{m+\mathbf{r}}(\mathbf{x} - \mathring{\mathbf{x}})$ instead of $\mathcal{Q}_{\mathbf{r}}(\mathbf{x} - \mathring{\mathbf{x}})$ yields

$$\mathcal{Q}_{m+\mathbf{r}}(\mathbf{x} - \mathring{\mathbf{x}}) \quad = \sum_{\mathbf{w} \in Hom} c_{\mathbf{w}}(\mathbf{r}) \mathfrak{S}_{m+\mathbf{w}}(\mathbf{x} - \mathring{\mathbf{x}})$$

by Proposition 4.13. Taking the limit as $m \to \infty$ in the above equation, we find that

$$\mathcal{F}_{\mathbf{r}}(\mathbf{x} - \mathring{\mathbf{x}}) \quad = \sum_{\mathbf{w} \in Hom} c_{\mathbf{w}}(\mathbf{r}) \mathcal{F}_{\mathbf{w}}(\mathbf{x} - \mathring{\mathbf{x}}), \tag{6.3}$$

the right side by Proposition 6.5 and the left by specializing Theorem 6.11 to $\mathring{\mathbf{y}} = \mathring{\mathbf{x}}$.

On the other hand, the right side of Theorem 6.11 becomes a subsum of (6.3) after specializing to $\mathring{\mathbf{y}} = \mathring{\mathbf{x}}$, by Theorem 5.10 and Proposition 4.11. The remaining terms from (6.3) sum to zero, so there can be no remaining terms by Lemma 6.15. We conclude that in fact $W_{\mathcal{RP}}(\mathbf{r}) = W(\mathbf{r})$ and $c_{\mathbf{w}}(\mathbf{r}) = 1$ for all $\mathbf{w} \in W(\mathbf{r})$. □

Corollary 6.17 (Component formula for quiver polynomials—Schubert version). *The quiver polynomial* $\mathcal{Q}_{\mathbf{r}}(\mathbf{x} - \mathring{\mathbf{x}})$ *equals the sum*

$$\mathcal{Q}_{\mathbf{r}}(\mathbf{x} - \mathring{\mathbf{x}}) = \sum_{\mathbf{w} \in W(\mathbf{r})} \mathfrak{S}_{\mathbf{w}}(\mathbf{x} - \mathring{\mathbf{x}})$$

of products of double Schubert polynomials (in consecutive alphabets $\mathbf{x} = \mathbf{x}^0, \ldots, \mathbf{x}^n$*) indexed by minimum length lacing diagrams with rank array* $\mathbf{r}$*.*

We have chosen to present this consequence of Theorem 6.16 and Corollary 4.9 now, even though it is a specialization of the stable double component formula in the next section, to emphasize that it has a direct geometric interpretation: the right-hand side is the sum of the equivariant cohomology classes of components in the quiver degeneration. The double formula to come in Theorem 6.20 currently lacks such a geometric interpretation, and moreover the reason why it implies Corollary 6.17 is somewhat subtle; see Corollary 6.23 and Remark 6.26. The subtlety in this argument is related to the fact, discussed in Remark 6.26, that replacing $\mathring{\mathbf{x}}$ with $\mathring{\mathbf{y}}$ in Corollary 6.17 *always* yields a false statement whenever there is more than one term on the right-hand side.

In the course of proving Theorem 6.16, we reached a notable combinatorial result.

Corollary 6.18. *Every minimal length lacing diagram for a rank array* $\mathbf{r}$ *occurs as the lacing diagram derived from a reduced pipe dream for the Zelevinsky permutation* $v(\mathbf{r})$*:*

$$W(\mathbf{r}) = W_{\mathcal{RP}}(\mathbf{r}).$$

Remark 6.19. Theorem 5.10 and Corollary 6.18 give another explanation (and proof) for Theorem 3.8. That laces of $\mathbf{w} \in W(\mathbf{r})$ can cross at most once is exactly the statement that pipes in reduced pipe dreams for $v(\mathbf{r})$ cross at most once. That laces starting or ending in the same column do not cross at all follows from the fact that Zelevinsky permutations and their inverses have no descents within block rows.

6.4. Stable double component formula

Theorem 6.20. *The double quiver function can be expressed as the sum*

$$\mathcal{F}_{\mathbf{r}}(\mathbf{x} - \mathring{\mathbf{y}}) = \sum_{\mathbf{w} \in W(\mathbf{r})} F_{\mathbf{w}}(\mathbf{x} - \mathring{\mathbf{y}})$$

of products of double Stanley symmetric functions indexed by minimal lacing diagrams $\mathbf{w}$*.*

Proof. Using Corollary 6.18, we can replace the sum in Theorem 6.11 over $W_{\mathcal{RP}}(\mathbf{r})$ by one over $W(\mathbf{r})$. □

Remark 6.21. Definition 5.8 reads pipes as if they flow northeast to southwest (compare Definition 5.1) because our proof of Theorem 6.11 requires that we read antidiagonal blocks in Zelevinsky pipe dreams as miniature pipe dreams with their southeast corners as their origins. This method suggests a direct combinatorial proof of Corollary 6.18, without appealing to the geometric lower bound on multiplicities of components in quiver degenerations afforded by Corollary 4.9. This idea has since been carried out in [Yon03]. Thus the component formula in Theorem 6.20 can be proved in a purely combinatorial way using Theorem 6.11, if one assumes the ratio formula (whose proof requires geometry).

Example 6.22. In the case of *Fulton rank conditions*, the ranks $r_0, r_1, r_2, r_3, \ldots$ equal $1, 2, 3, \ldots, n-1, n, n, n-1, \ldots, 3, 2, 1$ and $\mathbf{r} = \mathbf{r}_w$ is specified by a permutation w in S_{n+1}. Theorem 6.20 contains a combinatorial formula found independently in [BKTY02]. This is because the Zelevinsky permutation $v(\mathbf{r}_w)$ has as many diagonal $\times$ entries as will fit in superantidiagonal blocks, and the southeast corner (last $n+1$ block rows and block columns) is a block version of the permutation w itself (rotated by 180°). See [Yon03] for details. □

Corollary 6.23 (Component formula for quiver polynomials—Stanley version)**.** *The quiver polynomial can be expressed as the specialization*

$$\mathcal{Q}_{\mathbf{r}}(\mathbf{x}-\mathring{\mathbf{x}}) \;=\; \mathcal{F}_{\mathbf{r}}(\mathbf{x}_{\mathbf{r}}-\mathring{\mathbf{x}}_{\mathbf{r}}) \;=\; \sum_{\mathbf{w}\in W(\mathbf{r})} F_{\mathbf{w}}(\mathbf{x}_{\mathbf{r}}-\mathring{\mathbf{x}}_{\mathbf{r}})$$

*of the **quiver function** $\mathcal{F}_{\mathbf{r}}(\mathbf{x}-\mathring{\mathbf{x}})$ to a sequence $\mathbf{x}_{\mathbf{r}}$ of finite alphabets of size $r_0,\ldots,r_n$.*

Proof. Since Theorem 1.20 holds for any ranks $\mathbf{r}$, it holds in particular for $m+\mathbf{r}$ when $m \gg 0$. Moreover, it is shown in [BF99] that

$$c_{\underline{\lambda}}(m+\mathbf{r}) \;=\; c_{\underline{\lambda}}(\mathbf{r}) \tag{6.4}$$

for all $\underline{\lambda}$ and $m \geq 0$. Specializing Theorem 6.11 to $\mathring{\mathbf{y}} = \mathring{\mathbf{x}}$, we may take the limit there with $\sum c_{\underline{\lambda}}(\mathbf{r}) s_{\underline{\lambda}}(\mathbf{x}_{m+\mathbf{r}} - \mathring{\mathbf{x}}_{m+\mathbf{r}})$ in place of $\mathcal{Q}_{m+\mathbf{r}}(\mathbf{x}-\mathring{\mathbf{x}})$, by Theorem 2.9 and Theorem 1.20. We conclude that

$$\mathcal{F}_{\mathbf{r}}(\mathbf{x}-\mathring{\mathbf{x}}) \;=\; \sum_{\underline{\lambda}} c_{\underline{\lambda}}(\mathbf{r}) s_{\underline{\lambda}}(\mathbf{x}-\mathring{\mathbf{x}}) \tag{6.5}$$

as power series in the ring of functions that are symmetric in each of the infinite sets $\mathbf{x}^0,\ldots,\mathbf{x}^n$ of variables. Specializing each $\mathbf{x}^j$ in (6.5) to have r_j nonzero variables yields $Q_{\mathbf{r}}(\mathbf{x}-\mathring{\mathbf{x}})$ by Theorem 1.20, and it yields $\sum_{\mathbf{w}\in W(\mathbf{r})} F_{\mathbf{w}}(\mathbf{x}_{\mathbf{r}}-\mathring{\mathbf{x}}_{\mathbf{r}})$ by Theorem 6.20. □

Remark 6.24. As a consequence of Corollary 6.23, there is no need to define 'stable quiver polynomials', just as there is no need to define 'stable Schur polynomials'. The fact that Conjecture 6.14 is actually true, as proved in [Buc03], means that we are correct to avoid the term 'stable double quiver polynomial' in favor of the more apt 'double quiver function'.

Remark 6.25. We chose our definition of the double quiver polynomial $\mathcal{Q}_{\mathbf{r}}(\mathbf{x}-\mathring{\mathbf{y}})$ via the ratio formula from among four possibilities that we considered, three of which turned out to be pairwise distinct. In hindsight we could equally well have chosen the candidate obtained by replacing $\mathring{\mathbf{x}}$ with $\mathring{\mathbf{y}}$ in the expression $\sum c_{\underline{\lambda}}(\mathbf{r}) s_{\underline{\lambda}}(\mathbf{x}_{\mathbf{r}}-\mathring{\mathbf{x}}_{\mathbf{r}})$ from Theorem 1.20, or equivalently by (6.5), in the expression $\sum F_{\mathbf{w}}(\mathbf{x}_{\mathbf{r}}-\mathring{\mathbf{x}}_{\mathbf{r}})$ from Corollary 6.23. However, it was the algebraic and combinatorial properties derived from the ratio formula that made $\mathcal{Q}_{\mathbf{r}}(\mathbf{x}-\mathring{\mathbf{y}})$ useful to us. Moreover, our methods did not prove the fact that this other candidate is equal to our double quiver polynomials; this is the content of Conjecture 6.14, which we now know to hold only because of its subsequent proof by Buch [Buc03].

One of our other two candidates for double quiver polynomial was obtained by replacing $\mathring{\mathbf{x}}$ with $\mathring{\mathbf{y}}$ on the right-hand side of Corollary 6.17. As we explain in Remark 6.26, below, this turned out to be less natural than we had expected from seeing the geometric degeneration of quiver loci to unions of products of matrix Schubert varieties.

The remaining candidate for double quiver polynomial was actually an ordinary quiver polynomial, but for a quiver of twice the length as the original. Given a lacing diagram $\mathbf{w}$ with rank array $\mathbf{r}$, the "doubled" ranks $\mathbf{r}^2$ were defined as the rank array of a "doubled" lacing diagram $\mathbf{w}^2$. To get $\mathbf{w}^2$ from $\mathbf{w}$, elongate each column of dots in $\mathbf{w}$ to a "ladder" with two adjacent columns (of the same height) and horizontal rungs connecting them. Replacing the even-indexed alphabets in $\mathcal{Q}_{\mathbf{r}^2}(\mathbf{x}-\mathring{\mathbf{x}})$ with $\mathbf{y}$ alphabets yielded our final candidate.

It is readily verified that the three candidates for double quiver polynomial differ when $n=3$, the dimension vector is $(1,2,1)$, and the Zelevinsky permutation is $2143 \in S_4$.

Remark 6.26. We now have formulae for quiver polynomials $\mathcal{Q}_{\mathbf{r}}(\mathbf{x} - \mathring{\mathbf{x}})$ in terms of

- Stanley symmetric functions $F_{\mathbf{w}}(\mathbf{x}_{\mathbf{r}} - \mathring{\mathbf{x}}_{\mathbf{r}})$ in finite alphabets, and
- Schubert polynomials $\mathfrak{S}_{\mathbf{w}}(\mathbf{x} - \mathring{\mathbf{x}})$, which a priori involve finite alphabets.

We also have a formula for double quiver functions $\mathcal{F}_{\mathbf{r}}(\mathbf{x} - \mathring{\mathbf{y}})$ in terms of double Stanley symmetric functions $F_{\mathbf{w}}(\mathbf{x} - \mathring{\mathbf{y}})$, in infinite alphabets. However, the double quiver polynomial $\mathcal{Q}_{\mathbf{r}}(\mathbf{x} - \mathring{\mathbf{y}})$ *never* equals the sum $\sum \mathfrak{S}_{\mathbf{w}}(\mathbf{x} - \mathring{\mathbf{y}})$ of double Schubert polynomials for minimum length lacing diagrams $\mathbf{w} \in W(\mathbf{r})$, unless the sum only has one term, even though

- setting $\mathring{\mathbf{y}} = \mathring{\mathbf{x}}$ in this sum yields Corollary 6.17, and
- taking limits for uniformly growing ranks yields Theorem 6.20.

This failure does not disappear by restricting to ranks $m + \mathbf{r}$ for $m \gg 0$, either. Indeed, $\mathfrak{S}_{v(Hom)}(\mathbf{x} - \mathring{\mathbf{y}})\mathfrak{S}_{\mathbf{w}}(\mathbf{x} - \mathring{\mathbf{y}})$ equals the double Schubert polynomial $\mathfrak{S}_{\overline{\mathbf{w}}}(\mathbf{x} - \mathring{\mathbf{y}})$ for a certain permutation $\overline{\mathbf{w}} \in S_d$ constructed from $\mathbf{w}$. Hence the linear independence of double Schubert polynomials prevents a direct double generalization of Corollary 6.17, after multiplying such a doubled version by $\mathfrak{S}_{v(Hom)}(\mathbf{x} - \mathring{\mathbf{y}})$. We view this as evidence for the naturality of double quiver polynomials as we defined them (rather than as $\sum_{\mathbf{w} \in W(\mathbf{r})} \mathfrak{S}_{\mathbf{w}}(\mathbf{x} - \mathring{\mathbf{y}})$, for example), and even stronger evidence for the naturality of their limits, the double quiver functions. The naturality of double quiver functions also follows from their role in [BF99], where the double quiver functions should be viewed as functionals on lists of bundles that take $E_0, \ldots, E_n$ to the degeneracy locus formula. The recent proof of our Conjecture 6.14 in [Buc03] cements our belief in the naturality of double quiver polynomials.

The permutation $\overline{\mathbf{w}}$ is constructed as follows. Let $\mathbf{w}'$ be the $d \times d$ partial permutation matrix obtained by placing each w_i in the northwest corner of the superantidiagonal block in row $i - 1$. Then $\mathbf{w}'$ can be completed uniquely to a permutation $\overline{\mathbf{w}} \in S_d$ by adding nonzero entries in or below the block antidiagonal in such a way that all cells in the diagram of $\overline{\mathbf{w}}$ lie in or above the block superantidiagonal.

Geometrically, $\mathfrak{S}_{v(Hom)}(\mathbf{x}-\mathring{\mathbf{y}})\mathfrak{S}_{\mathbf{w}}(\mathbf{x}-\mathring{\mathbf{y}})$ equals the double Schubert polynomial $\mathfrak{S}_{\overline{\mathbf{w}}}(\mathbf{x}-\mathring{\mathbf{y}})$ because of a direct connection between the orbit closure $\mathcal{O}(\mathbf{w})$ inside *Hom* and the matrix Schubert variety $\overline{X}_{\overline{\mathbf{w}}}$ inside M_d. Moreover, it can be shown that there is a flat (but not Gröbner) degeneration of the matrix Schubert variety $\overline{X}_{v(\mathbf{r})}$ to a generically reduced union $\overline{X}_{v(\mathbf{r})}(0)$ of matrix Schubert varieties for the permutations $\overline{\mathbf{w}}$ associated to lacing diagrams $\mathbf{w} \in W(\mathbf{r})$. However, this flat family is only equivariant for the multigrading by $\mathbb{Z}^d$, not $\mathbb{Z}^{2d}$.

Section 7. Quiver constants

Using the component formula for double quiver functions, we show that the quiver constants $c_{\underline{\lambda}}(\mathbf{r})$ arise as coefficients in the expansion of the Zelevinsky Schubert polynomial $\mathfrak{S}_{v(\mathbf{r})}$ into Demazure characters, and consequently the quiver constants $c_{\underline{\lambda}}(\mathbf{r})$ equal the corresponding Stanley coefficients α_w^λ. Using the interpretation of Stanley coefficients as enumerating peelable tableaux [RS95a, RS98], we deduce a combinatorial formula for quiver constants.

7.1. Demazure characters

We recall here the type A_∞ Demazure characters [Dem74] following [Mac91]. Using the ordinary divided difference ∂_i from (2.1), the **Demazure operator** π_i, also called the **isobaric divided difference**, is the linear operator on $\mathbb{Z}[x_1, x_2, \ldots]$ defined by

$$\pi_i f \quad = \quad \partial_i(x_i f).$$

Let σ_i be the transposition that exchanges x_i and x_{i+1}. The operator π_i is idempotent and, like ∂_i, commutes with multiplication by any σ_i-symmetric polynomial:

$$\partial_i(fg) = f\partial_i(g) \quad \text{and} \quad \pi_i(fg) = f\pi_i(g) \qquad \text{if } \sigma_i f = f. \tag{7.1}$$

Since $\pi_i 1 = 1$ it follows that

$$\pi_i f \;=\; f \qquad \text{if } \sigma_i f = f. \tag{7.2}$$

Given a permutation $w \in \bigcup_n S_n$, let $w = \sigma_{i_1} \cdots \sigma_{i_\ell}$ be any reduced decomposition of w, that is, a factorization of w into a minimum number of simple reflections σ_i. Define the operator $\pi_w = \pi_{i_1} \cdots \pi_{i_\ell}$. It is independent of the factorization.

Let $\beta = (\beta_1, \ldots, \beta_k) \in \mathbb{N}^k$ be a sequence of nonnegative integers, and β_+ the sequence obtained by sorting β into weakly decreasing order. If $w \in S_k$ is the shortest permutation satisfying $\beta = w\beta_+$, then define the **Demazure character** $\kappa_\beta \in \mathbb{Z}[x_1, \ldots, x_k]$ by

$$\kappa_\beta \;=\; \pi_w \mathbf{x}^{\beta_+}.$$

For a partition $\lambda \in \mathbb{N}^k$ and $w_0^{(k)} \in S_k$ the longest permutation, we have

$$\kappa_\lambda \;=\; \mathbf{x}^\lambda \tag{7.3}$$

$$\kappa_{w_0^{(k)}\lambda} \;=\; s_\lambda(x_1, x_2, \ldots, x_k). \tag{7.4}$$

Alternatively one may give the recursive definition

$$\kappa_\beta \;=\; \begin{cases} \mathbf{x}^\beta & \text{if } \beta_1 \geq \beta_2 \geq \cdots \geq \beta_k \\ \pi_i \kappa_{\sigma_i \beta} & \text{if } \beta_i < \beta_{i+1}. \end{cases} \tag{7.5}$$

It can be shown that

$$\pi_i \kappa_\beta \;=\; \begin{cases} \kappa_\beta & \text{if } \beta_i \leq \beta_{i+1} \\ \kappa_{\sigma_i \beta} & \text{if } \beta_i \geq \beta_{i+1}. \end{cases} \tag{7.6}$$

Remark 7.1. The Demazure characters $\{\kappa_\beta \mid \beta \in \mathbb{N}^k\}$ form a $\mathbb{Z}$-basis of the polynomial ring $\mathbb{Z}[x_1, x_2, \ldots, x_k]$ since

$$\kappa_\beta \;=\; \mathbf{x}^\beta + \text{terms lower in reverse lexicographic order.}$$

Demazure characters interpolate between dominant monomials and Schur polynomials. They have a tableau realization, due to Lascoux and Schützenberger [LS90]. Before describing it, we review some conventions regarding tableaux; for more background and unexplained terminology, see [Ful97].

Our (semistandard) Young tableaux will be drawn with origin at the northwest corner, weakly increasing along each row, and strictly increasing down each column. A **word** is a sequence of positive integers called "letters". A **column word** is one that strictly decreases. In what follows, each tableau t is identified with its **column reading word** $t_1 t_2 \cdots$, where t_i is the column word obtained by reading the letters upward in the i^{th} column of t. Given a word u, denote by $[u]$ the unique tableau equivalent to u under the *Knuth-equivalence relations* on words: $acb \sim cab$ for $a \leq b < c$ and $bac \sim bca$ for $a < b \leq c$ [Ful97, Section 2.1].

The **weight** of a tableau t is the sequence $\mathrm{wt}(t) = (\beta_1, \beta_2, \ldots)$, where β_j is the number of entries equal to j in t. For each **composition** $\beta = (\beta_1, \beta_2, \ldots)$, by which we mean a sequence of nonnegative integers that is eventually zero, the **key tableau** $\mathrm{key}(\beta)$ of weight β is the unique tableau of shape β_+ and weight β.

Suppose t has shape λ, so that its j^{th} column t_j has length λ'_j for $1 \leq j \leq r = \lambda_1$, where λ' is the partition **conjugate** to λ. Given any reordering $\gamma = (\gamma_1, \ldots, \gamma_r)$ of these column lengths, there is a unique word $u = u(t, \gamma)$ with a factorization $u = u_1 u_2 \cdots u_r$ into column words u_j of length γ_j, such that $[u] = t$. Such a word is called **frank** in [LS90].

The leftmost column factor u_1 in the word $u(t, \gamma)$ actually depends only on t and γ_1. Define $\mathrm{lc}_j(t) = u_1$ to be the j^{th} **left column word** of t when γ_1 equals the length λ'_j of the j^{th} column of t. Gathering together all left column words produces the **left key** of t, which is the tableau $K_-(t)$ whose j^{th} column is given by $\mathrm{lc}_j(t)$ for $j = 1, \ldots, r$.

Similarly, the rightmost column factor u_r in $u(t, \gamma)$ depends only on t and γ_r. Define $\mathrm{rc}_j(t) = u_r$ to be the j^{th} **right column word** of t when $\gamma_r = \lambda'_j$, and construct the tableau $K_+(t)$, called the the **right key** of t, whose j^{th} column is given by $\mathrm{rc}_j(t)$ for $j = 1, \ldots, r$. The left and right keys of t are key tableaux of the same shape as t.

For tableaux s and t of the same shape, write $s \leq t$ to mean that every entry of s is less than or equal to the corresponding entry of t. In [LS90], the equation

$$\kappa_\beta = \sum_{K_+(t) \leq \mathrm{key}(\beta)} \mathbf{x}^{\mathrm{wt}(t)} \tag{7.7}$$

is given as a formula for the Demazure character. This implies two positivity properties:

$$\kappa_\beta \in \mathbb{N}[\mathbf{x}] \qquad \text{for all compositions } \beta, \text{ and} \tag{7.8}$$

$$\kappa_{w\lambda} - \kappa_{v\lambda} \in \mathbb{N}[\mathbf{x}] \qquad \text{if } v \leq w \text{ and } \lambda \text{ is a partition.} \tag{7.9}$$

It is well-known that

$$s_\lambda(\mathbf{x}) = \sum_{\substack{\text{tableaux } t \\ \text{shape}(t) = \lambda}} \mathbf{x}^{\mathrm{wt}(t)} \tag{7.10}$$

is a formula for the Schur function s_λ in an infinite alphabet $\mathbf{x}$. From (7.7) and (7.10) it follows that

$$s_{\beta_+} = \lim_{m \to \infty} \kappa_{(0^m, \beta)}, \tag{7.11}$$

where $(0^m, \beta)$ is the composition obtained from β by prepending m zeros.

7.2. Schubert polynomials as sums of Demazure characters

We now recall the expansion of a Schubert polynomial as a positive sum of Demazure characters and deduce some special properties of this expansion for Zelevinsky permutations.

Theorem 7.2 ([LS89],[RS95a, Theorem 29])**.** *For any (partial) permutation w, there is a multiset $M(w)$ of compositions such that*

$$\mathfrak{S}_w(\mathbf{x}) = \sum_{\beta \in M(w)} \kappa_\beta.$$

The Schubert polynomial $\mathfrak{S}_w(\mathbf{x})$ here is obtained by setting $\mathring{\mathbf{y}} = \mathbf{0}$ in the double Schubert polynomial $\mathfrak{S}_w(\mathbf{x} - \mathring{\mathbf{y}})$ from Section 2.1. The multiset $M(w)$ has been described explicitly using reduced word tableaux in [LS89] and peelable tableaux in [RS95a]. (We shall define the latter in Section 7.4.) The expansion in Theorem 7.2 is a refinement of (6.1), in the sense that taking limits in Theorem 7.2, expresses the Stanley coefficient α_w^λ in terms of $M(w)$:

Corollary 7.3. *The coefficient α_w^λ in the expansion of a Stanley symmetric function in Schur functions is the number of compositions $\beta \in M(w)$ such that $\beta_+ = \lambda$.*

Proof. By the definition of the Stanley symmetric function and (7.11), it is enough to show that there is a bijection $M(w) \mapsto M(1+w)$ given by $\beta \mapsto (0,\beta)$. Supposing that w lies in S_k, the definition of $\mathfrak{S}_w$ in Section 2.1 says that $\mathfrak{S}_w(\mathbf{x}) = \partial_{w'}\mathfrak{S}_{w_0^{(k)}}(\mathbf{x})$, where $w' = w^{-1}w_0^{(k)}$ and $w_0^{(k)} \in S_k$ is the longest permutation. Note that

$$1 + w \quad = \quad \sigma_1\sigma_2\cdots\sigma_k w \sigma_k \cdots \sigma_2\sigma_1,$$

and that $w_0^{(k+1)} = \sigma_1\sigma_2\cdots\sigma_i w_0^{(k)}$. Also, in the expression $(1+w)^{-1}w_0^{(k+1)} = \sigma_1\sigma_2\cdots\sigma_k w'$, multiplying by each reflection σ_i lengthens the permutation. It follows that

$$\begin{aligned}
\mathfrak{S}_{1+w} &= \partial_{(1+w)^{-1}w_0^{(k+1)}}\big(\mathfrak{S}_{w_0^{(k+1)}}(\mathbf{x})\big)\\
&= \partial_{\sigma_1\sigma_2\cdots\sigma_k w'}\big(\mathfrak{S}_{w_0^{(k+1)}}(\mathbf{x})\big)\\
&= \partial_1\partial_2\cdots\partial_k\partial_{w'}\big(x_1\cdots x_k\cdot\mathfrak{S}_{w_0^{(k)}}(\mathbf{x})\big)\\
&= \partial_1 x_1\partial_2 x_2\cdots\partial_k x_k\partial_{w'}\big(\mathfrak{S}_{w_0^{(k)}}(\mathbf{x})\big)\\
&= \pi_1\cdots\pi_k\mathfrak{S}_w
\end{aligned}$$

using the definitions, (7.1), and (2.1).

Let β be a composition in $M(w)$. Since w lies in the symmetric group S_k on k letters, the Schubert polynomial $\mathfrak{S}_w(\mathbf{x})$ involves at most the variables $x_1,\ldots,x_{k-1}$. Hence only the first $k-1$ parts of β can be nonzero. By (7.6), $\pi_1\cdots\pi_k\kappa_\beta = \kappa_{(0,\beta)}$, as a zero in the $(k+1)^{\text{st}}$ position is swapped to the first position. Since the Demazure characters form a basis (Remark 7.1), the map $\beta \mapsto (0,\beta)$ is a bijection $M(w) \mapsto M(1+w)$. □

We now recall the dominance bounds on the shapes that occur in the Schur function expansion of Stanley symmetric functions. For any diagram D, by which we mean a set of pairs of positive integers, let $D\!\uparrow$ be the diagram obtained from D by top-justifying the cells in each column. Similarly, let $D\!\leftarrow$ be obtained by left-justifying each row of D. The **dominance partial order** on partitions of the same size, written as $\lambda \trianglerighteq \mu$, is defined by the condition that $\lambda_1 + \cdots + \lambda_i \geq \mu_1 + \cdots + \mu_i$ for all i.

Proposition 7.4 ([Sta84])**.** *If the Stanley coefficient α_w^λ is nonzero, then*

$$D(w)\!\leftarrow\uparrow \;\trianglelefteq\; \lambda \;\trianglelefteq\; D(w)\!\uparrow\leftarrow .$$

We require two more lemmata.

Lemma 7.5. *If $w(i) < w(i+1)$ then $\beta_i \leq \beta_{i+1}$ for every composition $\beta \in M(w)$.*

Proof. Suppose $w(i) < w(i+1)$. Since $\sigma_i \circ \partial_i = \partial_i$, it follows from the definition of Schubert polynomial that $\mathfrak{S}_w(\mathbf{x})$ is σ_i-invariant. Applying π_i to Theorem 7.2 and using (7.2) along with (7.6), it follows that $\sum_{\beta\in M(w)}\kappa_\beta = \sum_{\beta\in M(w)}\kappa_{\beta'}$, where $\beta' = \beta$ if $\beta_i \leq \beta_{i+1}$ and $\beta' = \sigma_i\beta$ otherwise. But $\kappa_{\beta'} - \kappa_\beta$ is a polynomial with nonnegative integer coefficients by (7.9). It follows that $\beta' = \beta$ and hence that $\beta_i \leq \beta_{i+1}$ for all $\beta \in M(w)$. □

Lemma 7.6. *Suppose γ is a composition such that $\mathbf{x}^\gamma$ divides $\mathfrak{S}_w(\mathbf{x})$. Then $\gamma \leq \beta$ for all compositions $\beta \in M(w)$.*

Proof. Suppose $\mathbf{x}^\gamma$ divides $\mathfrak{S}_w(\mathbf{x})$. Then $\mathbf{x}^\gamma$ divides the sum of Demazure characters on the right-hand side of Theorem 7.2. By (7.8) it follows that $\mathbf{x}^\gamma$ divides every monomial of every Demazure character κ_β for $\beta \in M(w)$. By Remark 7.1 the reverse lexicographic leading monomial in κ_β is $\mathbf{x}^\beta$, so $\gamma \leq \beta$. □

We now apply the above results to Zelevinsky permutations.

Proposition 7.7. *For a fixed dimension vector* $(r_0, r_1, \ldots, r_n)$*, identify the diagrams* D_{Hom} *and* $D(\Omega_0)$ *from Definition 1.10 with partitions of those shapes. Let* $\mathbf{r}$ *be a rank array with the above dimension vector. Then for all compositions* $\beta \in M(v(\mathbf{r}))$,

$$D_{Hom} \subset \beta_+ \subset D(\Omega_0) \quad \textit{and} \tag{7.12}$$

$$\beta \;=\; \mathbf{w}_0\beta_+, \tag{7.13}$$

where $\mathbf{w}_0$ *is the block long permutation, reversing the row indices within each block row.*

Proof. Let $\beta \in M(v(\mathbf{r}))$. Equation (7.13) is a consequence of Lemma 7.5 and Proposition 1.6, as Zelevinsky permutations have no descents in each block row.

Both of the partition diagrams $D_{\mathbf{r}}\!\upharpoonleft\!\!\leftarrow$ and $D_{\mathbf{r}}\!\leftarrow\!\!\upharpoonright$ are contained in $D(\Omega_0)$, by Definition 1.10 along with (1.3), (1.4), (1.7), and (1.8). Proposition 7.4 implies that $\beta_+ \subset D(\Omega_0)$.

Since $D_{Hom} \subset D_{\mathbf{r}}$, the monomial $\mathbf{x}^{D_{Hom}}$ divides $\mathfrak{S}_{v(\mathbf{r})}$. By Lemma 7.6, $D_{Hom} \leq \beta$. But the partition D_{Hom} is constant in each block row. With (7.13), it follows that $D_{Hom} \subset \beta_+$. $\square$

Finally, we observe that each Demazure character appearing in $\mathfrak{S}_{v(\mathbf{r})}$ is the monomial $\mathbf{x}^{D_{Hom}}$ times a product of Schur polynomials.

Definition 7.8. Assume that $\lambda = \beta_+$ for some composition $\beta \in M(v(\mathbf{r}))$. The skew shape λ/D_{Hom} afforded by (7.12) consists of a union of partition diagrams in which the partition λ_i is contained in the $r_{i-1} \times r_i$ rectangle comprising the i^{th} block row of $D(\Omega_0)/D_{Hom}$. In this situation, we say that the partition list $\underline{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ is obtained by **deleting** D_{Hom} **from** λ, and we write $\underline{\lambda} = \lambda - D_{Hom}$.

Proposition 7.9. *Suppose that* β *is a composition in* $M(v(\mathbf{r}))$*, that* $\lambda = \beta_+$*, and that* $\underline{\lambda}$ *is obtained from* λ *by deleting* D_{Hom}*. Then*

$$\kappa_\beta \;=\; \mathbf{x}^{D_{Hom}} \prod_{i=1}^{n} s_{\lambda_i}(\mathbf{x}^{i-1}).$$

Proof. Break D_{Hom} into a sequence $(\mu_0, \ldots, \mu_n)$ of shapes, where $\mu_i \in \mathbb{N}^{r_i}$ is rectangular (all of its parts are equal) and $\mu_n = (0^{r_n})$. Let w_0^i be the longest element of the symmetric group S_{r_i} acting on the i^{th} block of row indices. By (7.1) and (7.4) we have

$$\begin{aligned}
\kappa_\beta &= \pi_{\mathbf{w}_0}\mathbf{x}^\lambda \\
&= \prod_{i=0}^{n-1} \pi_{w_0^i}(\mathbf{x}^i)^{\lambda_{i+1}+\mu_i} \\
&= \prod_{i=0}^{n-1} (\mathbf{x}^i)^{\mu_i} s_{\lambda_{i+1}}(\mathbf{x}^i) \\
&= \mathbf{x}^{D_{Hom}} \prod_{i=1}^{n} s_{\lambda_i}(\mathbf{x}^{i-1}).
\end{aligned}$$

$\square$

7.3. Quiver constants are Stanley coefficients

Having an expansion of the double quiver function as a sum of products of Stanley symmetric functions automatically produces an expansion as a sum of products of Schur functions. Next, in Theorem 7.10, we shall see that the coefficients in this expansion are the Buch–Fulton quiver constants. Using this result, we show in Theorem 7.14 that quiver constants are special cases of Stanley coefficients.

For a list $\underline{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ of partitions and two sequences $\mathbf{x} = (\mathbf{x}^0, \ldots, \mathbf{x}^n)$ and $\mathring{\mathbf{y}} = (\mathbf{y}^n, \ldots, \mathbf{y}^0)$ of infinite alphabets, let

$$s_{\underline{\lambda}}(\mathbf{x} - \mathring{\mathbf{y}}) \;=\; \prod_{i=1}^{n} s_{\lambda_i}(\mathbf{x}^{i-1} - \mathbf{y}^i)$$

be the product of Schur functions in differences of alphabets. This notation parallels that with $\mathbf{y} = \mathbf{x}$ in (1.16), and for products of Stanley symmetric functions in Section 6.1. We follow the conventions after (6.2) for finite alphabets.

Theorem 7.10. *If $c_{\underline{\lambda}}(\mathbf{r})$ is the quiver constant from Theorem 1.20, then*

$$\mathcal{F}_{\mathbf{r}}(\mathbf{x} - \mathring{\mathbf{y}}) \;=\; \sum_{\underline{\lambda}} c_{\underline{\lambda}}(\mathbf{r}) s_{\underline{\lambda}}(\mathbf{x} - \mathring{\mathbf{y}}).$$

Moreover, using the Stanley coefficients α_w^λ from (6.1) *and writing $\alpha_{\mathbf{w}}^{\underline{\lambda}} = \prod_{i=1}^n \alpha_{w_i}^{\lambda_i}$, we get*

$$c_{\underline{\lambda}}(\mathbf{r}) \;=\; \sum_{\mathbf{w} \in W(\mathbf{r})} \alpha_{\mathbf{w}}^{\underline{\lambda}}. \tag{7.14}$$

Proof. Expanding the right-hand side of Theorem 6.20 into Schur functions yields

$$\mathcal{F}_{\mathbf{r}}(\mathbf{x} - \mathring{\mathbf{y}}) \;=\; \sum_{\mathbf{w} \in W(\mathbf{r})} \sum_{\underline{\lambda}} \alpha_{\mathbf{w}}^{\underline{\lambda}} s_{\underline{\lambda}}(\mathbf{x} - \mathring{\mathbf{y}}).$$

Specializing $\mathring{\mathbf{y}}$ to $\mathring{\mathbf{x}}$ yields an expression for $\mathcal{F}_{\mathbf{r}}(\mathbf{x} - \mathring{\mathbf{x}})$ as a sum of Schur function products $s_{\underline{\lambda}}(\mathbf{x} - \mathring{\mathbf{x}})$. By Corollary 6.23 and Theorem 1.20, uniqueness in the Main Theorem of [BF99] implies that the coefficient $\sum_{\mathbf{w} \in W(\mathbf{r})} \sum_{\underline{\lambda}} \alpha_{\mathbf{w}}^{\underline{\lambda}}$ on $s_{\underline{\lambda}}(\mathbf{x} - \mathring{\mathbf{x}})$ in this expression is $c_{\underline{\lambda}}(\mathbf{r})$. □

Remark 7.11. Theorem 7.10 and Corollary 6.23 are the only results in this paper that logically depend on the Main Theorem of [BF99], other than Theorem 1.20 (which needs the Main Theorem of [BF99] for its statement). In other words, starting from the multidegree characterization of quiver polynomials in Definition 1.18, the statements and proofs of all of our other combinatorial formulae—including all double and stable versions, as well as our combinatorial formula for the quiver constants to come in Theorem 7.21—are independent from [BF99]. It is only to identify the constants on the right side of (7.14) as the quiver constants appearing in the Conjecture from [BF99] that we apply Theorem 1.20, and hence the Main Theorem of [BF99].

Remark 7.12. The quiver constants $c_{\underline{\lambda}}(\mathbf{r})$ may be computed fairly efficiently using (7.14).

We need a proposition, in which our choice of notation (using $\mathbf{x} - \mathring{\mathbf{x}}$ in arguments of quiver polynomials) should finally become clear: we use $\mathcal{Q}_{\mathbf{r}}(\mathbf{x})$ and $\mathcal{F}_{\mathbf{r}}(\mathbf{x})$ to denote the $\mathbf{y} = \mathbf{0}$ specializations (as opposed to $\mathbf{y} = \mathbf{x}$ specializations) of the double quiver polynomial $\mathcal{Q}_{\mathbf{r}}(\mathbf{x} - \mathring{\mathbf{y}})$ and the double quiver function $\mathcal{F}_{\mathbf{r}}(\mathbf{x} - \mathring{\mathbf{y}})$. The "doubleness" of these expressions is crucial here, because in the ordinary single case it is impossible in the summation formulae to set just the second set of variables in each factor of every summand to zero.

Proposition 7.13. *If $\mathbf{r}$ is any rank array, then $\mathcal{Q}_{\mathbf{r}}(\mathbf{x}) = \mathcal{F}_{\mathbf{r}}(\mathbf{x}_{\mathbf{r}})$ is the finite $\mathbf{x}$ alphabet specialization of the double quiver function at $\mathbf{y} = \mathbf{0}$.*

Proof. Calculate $\mathcal{Q}_{m+\mathbf{r}}(\mathbf{x})$ and $\mathcal{Q}_{\mathbf{r}}(\mathbf{x})$ using Theorem 5.5. Consider a reduced pipe dream $D \in \mathcal{RP}(v(m+\mathbf{r}))$ that happens to lie entirely in rows indexed by $\mathbf{x}_{\mathbf{r}}$ (as opposed to $\mathbf{x}_{m+\mathbf{r}}$). Restricting attention to the $(j-1)^{\text{st}}$ block row of D for the moment, let w_j satisfy $w_j(D) = m + w_j$ using Corollary 5.16. No cross $+$ can be farther to the left of the vertical dividing line in that block row (from after Lemma 5.11) than the leftmost $+$ in the upside-down (rotated by 180°) top pipe dream for w_j. Therefore, the miniature pipe dream in block row $j-1$ of D has at least m blank columns on its left. By Proposition 5.15, horizontally shifting all crosses m cells to the left therefore induces a canonical bijection onto $\mathcal{RP}(v(\mathbf{r})$ from the set of pipe dreams in $\mathcal{RP}(v(m+\mathbf{r}))$ that happen to lie entirely in rows indexed by $\mathbf{x_r}$. Corresponding pipe dreams obviously yield the same monomial in $\mathbf{x}$. Therefore setting all variables in the set $\mathbf{x}_{m+\mathbf{r}} \smallsetminus \mathbf{x_r}$ to zero in $\mathcal{Q}_{m+\mathbf{r}}(\mathbf{x})$ yields $\mathcal{Q}_{\mathbf{r}}(\mathbf{x})$. Now use Corollary 6.13. □

Theorem 7.14. *The quiver constant $c_{\underline{\lambda}}(\mathbf{r})$ is the Stanley coefficient $\alpha^{\lambda}_{v(\mathbf{r})}$, where λ is the unique partition such that the partition list $\underline{\lambda}$ is obtained from λ by deleting D_{Hom}.*

Proof. Multiplying through by the implicit denominator in Proposition 7.13 yields

$$\mathfrak{S}_{v(\mathbf{r})}(\mathbf{x}) \quad = \quad \mathfrak{S}_{v(Hom)}(\mathbf{x}) \sum_{\underline{\lambda}} c_{\underline{\lambda}}(\mathbf{r}) s_{\underline{\lambda}}(\mathbf{x_r}), \tag{7.15}$$

where $c_{\underline{\lambda}}(\mathbf{r})$ is the quiver constant from Theorem 1.20 by Theorem 7.10. Proposition 7.9 shows that the Demazure character expansion of $\mathfrak{S}_{v(\mathbf{r})}(\mathbf{x})$ has the same form as the right side of (7.15). Linear independence of the polynomials $s_{\underline{\lambda}}(\mathbf{x_r})$ therefore implies that (7.15) is the Demazure character expansion. Equate coefficients on each Demazure character in Theorem 7.2 and (7.15) and apply Corollary 7.3. □

Remark 7.15. The comparison of (7.15) with Theorem 7.2 via Proposition 7.9 occurs at the level of ordinary Schubert polynomials (in a single set $\mathbf{x}$ of variables) rather than double Schubert polynomials (in $\mathbf{x}$ and $\mathbf{y}$). This restriction is forced upon us by the fact that the Demazure character theory behind Proposition 7.9 has not been sufficiently developed in the double setting. (See [Las03] for the beginnings of such a 'double' theory.) Since the comparison of (7.15) with Theorem 7.2 is the key point in the proof of Theorem 7.14, it is our single most important motivation for developing the double version of quiver polynomials via the ratio formula: we must be able to set $\mathbf{y} = \mathbf{0}$ for this comparison.

7.4. Peelable tableaux

Peelable tableaux [RS95a, RS98] arose from Stanley's problem of counting reduced decompositions of permutations [Sta84], the expansion by Lascoux and Schützenberger of the Schubert polynomial as a sum of Demazure characters [LS89], and from Magyar's character formula for the global sections of line bundles over Bott–Samelson varieties [Mag98]. Here we use them to give our first combinatorial formula for quiver constants. Although other combinatorial interpretations of Stanley coefficients would also produce formulae, it is the peelable tableaux that will connect to factor sequences in Section 8.

Let D be a **diagram**, meaning a finite set of pairs of positive integers. The diagram D is **northwest** if $(i_1, j_2) \in D$ and $(i_2, j_1) \in D$ implies $(i_1, j_1) \in D$ for $i_1 < i_2$ and $j_1 < j_2$; thus, if two cells lie in D, then so does the northwest corner of the smallest rectangle containing both. Permutation diagrams are always northwest. The diagram $D(\lambda)$ of a partition λ consists of λ_i left-justified cells in the i^{th} row for each i.

For tableaux Q and P, write $Q \supset P$ if Q contains P in its northwest corner. Write $Q - P$ for the skew tableau obtained by removing the subtableau P from Q.

A column of a diagram can be identified with the set of indices of rows in which it has a cell. This subset can also be identified with the decreasing word consisting of these row indices. As in Section 7.1, denote by $[u]$ the unique tableau Knuth-equivalent to a word u.

Now consider a northwest diagram D. A tableau Q is D-**peelable** provided that:

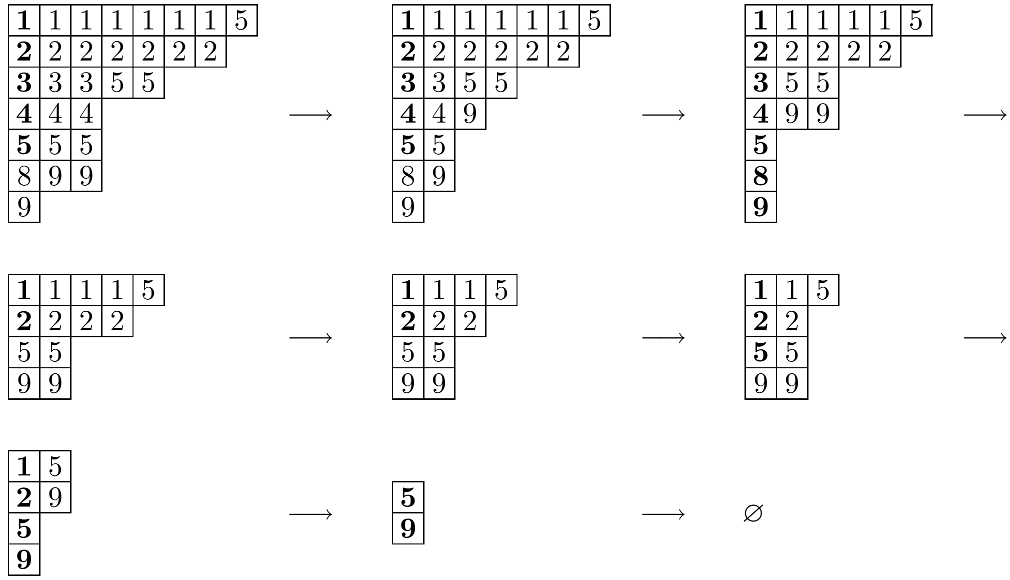


FIGURE 4. Peeling a peelable tableau

1. Q is the empty tableau when D is the empty diagram; or else
2. $Q \supset C$ and $[Q - C]$ is $(D - C)$-peelable when C is the first nonempty column of D.

We refer to the map $Q \mapsto [Q - C]$ as **peeling**, and denote by $\mathrm{Peel}(D)$ the set of D-peelable tableaux of partition shape.

Remark 7.16. The weight of any D-peelable tableau is the **code** of D, meaning the sequence $\mathrm{code}(D) = (c_1, c_2, \dots)$ in which c_i is the number of cells in row i of D.

Example 7.17. The tableau in Fig. 4 is $D_{\mathbf{r}}$-peelable as exhibited, where $\mathbf{r}$ is the rank array in Example 1.5, and the diagram $D_{\mathbf{r}}$ is in Example 1.9. The sequences of removed columns C (appearing in boldface below) are given by the columns of $D_{\mathbf{r}}$, after each cell has been labeled by its row index (this labeling of cells in $D_{\mathbf{r}}$ is depicted explicitly in Example 8.1, below).

Remark 7.18. The set of D-peelable tableaux may be constructed by "unpeeling" as follows. Let C be as in the definition of D-peelability, and p the number of cells in C. Suppose all of the $(D - C)$-peelable tableaux have been constructed. For each pair (T, V) where T is a $(D - C)$-peelable tableau and V is a vertical p-strip (a skew shape p cells, no two of which lie in the same row) whose union with the shape of T is a partition, use the jeu de taquin [Ful97, Section 1.2] to slide T into the topmost cell of V, then into the next topmost cell of V, and so on. This vacates p cells at the top of the first column. Place the single-column tableau C into the vacated cells. If the result is a (semistandard) tableau then it is D-peelable by construction. All D-peelable tableaux can be obtained in this manner.

Theorem 7.19. [RS95a, Theorem 29] *For any (partial) permutation w,*

$$\mathfrak{S}_w(\mathbf{x}) \quad = \sum_{Q \in \mathrm{Peel}(D(w))} \kappa_{\mathrm{wt}(K_-(Q))}.$$

Remark 7.20. Theorem 7.19 gives a combinatorial description of the multiset $M(w)$ in Theorem 7.2: it is the multiset of compositions given by $\mathrm{wt}(K_-(Q))$ for $Q \in \mathrm{Peel}(D(w))$.

Theorem 7.21. *The quiver constant $c_{\underline{\lambda}}(\mathbf{r})$ is the number of $D_{\mathbf{r}}$-peelable tableaux of shape λ, where $\underline{\lambda}$ is obtained from λ by deleting D_{Hom}.*

Proof. This follows from Theorems 7.14 and 7.19 along with Corollary 7.3, since for any tableau Q, the partition $\mathrm{wt}(K_-(Q))_+$ is just the shape of Q. □

Section 8. Factor sequences from peelable tableaux

In [BF99] it was conjectured that the quiver constant $c_{\underline{\lambda}}(\mathbf{r})$ counts certain lists of tableaux called '$\mathbf{r}$-factor sequences of shape $\underline{\lambda}$' (see the end of Section 8.1 for a precise statement). We already know by Theorem 7.21 that the quiver constants count peelable tableaux. We shall verify a variant (Corollary 8.23) of the Buch–Fulton conjecture by establishing a bijection from peelable tableaux to factor sequences.

8.1. Factor sequences

The **tableau array** $T(\mathbf{r}) = (T_{ij}(\mathbf{r}))$ for the rank array $\mathbf{r}$ is defined as follows. View the diagram $D_{\mathbf{r}} = D(v(\mathbf{r}))$ of the Zelevinsky permutation as being filled with integers, where every cell (either $*$ or $\square$) in row k is filled with $k \in \mathbb{Z}$. Define $T_{ij}(\mathbf{r})$ to be the rectangular tableau given by the i^{th} block column (from the right) and the j^{th} block row (from the top), defined for blocks on or below the superantidiagonal. The tableau T_{ij} has shape $R_{i-1,j+1}$ by Lemma 1.8.

For later use, we define tableaux Y_i and K_i as follows. Let Y be the tableau given by restricting the filling of $D_{\mathbf{r}}$ to the blocks above the superantidiagonal, and set Y_i equal to the part of Y in the i^{th} block column. Let K_i be the tableau obtained by stacking (left-justified) the tableaux in the i^{th} column of the tableau array, with $T_{i,i-1}$ on top, $T_{i,i}$ below it, and so on. Equivalently,

$$K_i \quad = \quad [T_{i,n-1}T_{i,n-2}\cdots T_{i,i}T_{i,i-1}]. \tag{8.1}$$

Let A_i be the interval of row indices occurring in the i^{th} block row, and then set $B_i = A_i \cup A_{i+1} \cup \cdots$. Given a set A, let $A^\bullet$ denote the set of words in the alphabet A. Then

$$Y_i \in (A_0 \cup \cdots \cup A_{i-2})^\bullet. \tag{8.2}$$

Example 8.1. For the ranks $\mathbf{r}$ in Example 1.5, with diagram $D_{\mathbf{r}}$ in Example 1.9, Fig. 5 depicts the filling of $D_{\mathbf{r}}$, the tableau array $T = T(\mathbf{r})$, and the tableaux Y_i and K_i. □

We now recall the recursive structure underlying the definition of an $\mathbf{r}$-factor sequence. Let $\widehat{\mathbf{r}}$ be the rank array obtained by removing the entries r_{ii} for $0 \le i \le n$. Using notation from Definition 1.10, observe that $D^{\square}_{\widehat{\mathbf{r}}}$ is obtained from $D^{\square}_{\mathbf{r}}$ by removing the superantidiagonal cells and some empty rows and columns. Hence we may identify $D^{\square}_{\widehat{\mathbf{r}}}$ with a subdiagram of $D^{\square}_{\mathbf{r}}$ by reindexing the nonempty rows. Under this identification the tableau array $T(\widehat{\mathbf{r}})$ is obtained from $T(\mathbf{r})$ by removing the superantidiagonal tableaux.

Example 8.2. Continuing with Example 8.1, we get $\widehat{\mathbf{r}}$, $\widehat{\mathbf{R}}$, $D_{\widehat{\mathbf{r}}}$, and $T(\widehat{\mathbf{r}})$ as in Fig. 6. □

Returning to the general case, let $T_i = T_{i,i-1}$ for $1 \le i \le n$.

Definition 8.3. The notion of **r-factor sequence** is defined recursively, as follows.

1. If $n = 1$ then there is a unique $\mathbf{r}$-factor sequence, namely (T_1).

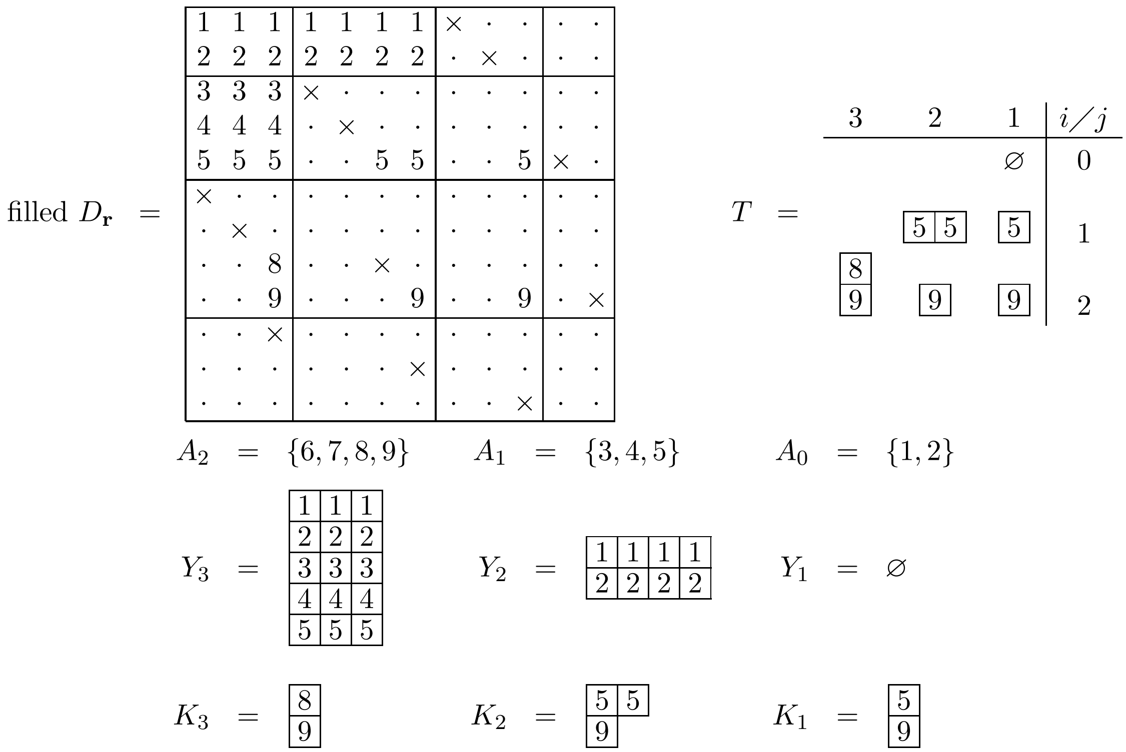


FIGURE 5. Tableau array and other data from Example 8.1

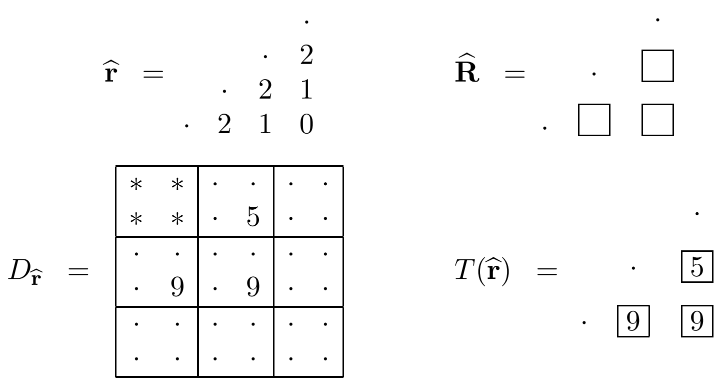


FIGURE 6. Data obtained by deleting the antidiagonal of $\mathbf{r}$ from Fig. 5

2. For $n \geq 2$, an $\mathbf{r}$-factor sequence is a tableau list $(W_1, \ldots, W_n)$ such that there exists an $\widehat{\mathbf{r}}$-factor sequence $(U_1, \ldots, U_{n-1})$ (for the tableau array $T(\widehat{\mathbf{r}})$ as defined above) and factorizations

$$U_i = [P_i Q_i] \qquad \text{for } 1 \leq i \leq n-1 \tag{8.3}$$

such that

$$W_i = [Q_i T_i P_{i-1}] \qquad \text{for } 1 \leq i \leq n, \tag{8.4}$$

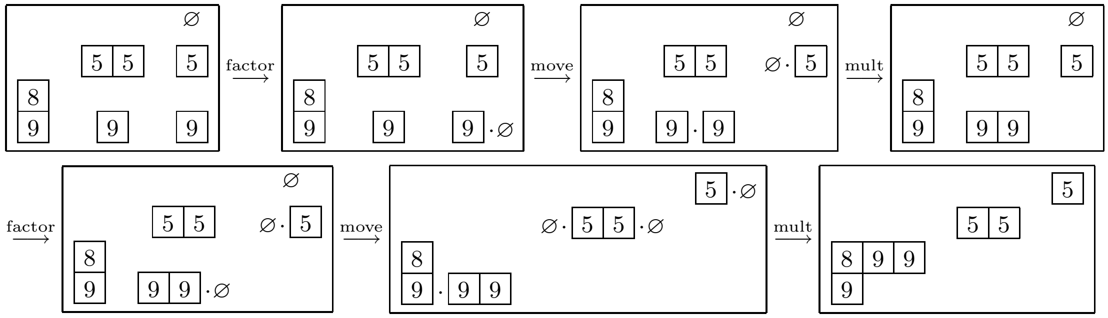


FIGURE 7. Computation of a factor sequence

where by convention

$$Q_n = P_0 = \varnothing. \tag{8.5}$$

The global computation of a factor sequence proceeds as follows. Index the block antidiagonals by their distance (going northwest) to the main superantidiagonal.

1. Initialize $W_{ij} := T_{ij}$ for all $n \geq j+1 \geq i \geq 1$.
2. Set $k := n-1$.
3. At this point the k^{th} antidiagonal of W is an $\mathbf{r}^{(k)}$-factor sequence, where $\mathbf{r}^{(k)}$ is obtained from $\mathbf{r}$ by removing the 0^{th} through $(k-1)^{\text{st}}$ antidiagonals. Factor each tableau $W_{ij} = [P_{ij}Q_{ij}]$ on the k^{th} antidiagonal. Now move the left factor P_{ij} to the west and the right factor Q_{ij} to the north.
4. At this point $W_{ij} = T_{ij}$ on the $(k-1)^{\text{st}}$ antidiagonal. For every position on the $(k-1)^{\text{st}}$ antidiagonal, multiply the current entry T_{ij} on the left with the tableau $Q_{i,j+1}$ coming from the south and on the right with the tableau $P_{i-1,j}$ coming from the east. That is, set $W_{ij} := [Q_{i,j+1}T_{ij}P_{i-1,j}]$.
5. Set $k := k-1$. If $k = 0$, stop. Otherwise go to step 3.

Example 8.4. In Fig. 7 we compute a factor sequence as above, except that after a tableau is factored and its factors moved, its position is vacated. Compare Example 8.14. □

Remark 8.5. It is clear from the global computation that for any $\mathbf{r}$-factor sequence $(W_1, \ldots, W_n)$, the letters of W_i are a submultiset of the letters from the tableaux $T_{k\ell}$ to the south and east of the i^{th} block column and $(i-1)^{\text{st}}$ block row (that is, for $\ell + 1 \geq i \geq k$).

We now recall the original definition of factor sequence and the associated conjecture in [BF99]. The tableau we index by T_{ij} is indexed as $T_{n-1-j,n+1-i}$ in [BF99].

Let (T_{ij}) be any array of semistandard tableaux with T_{ij} of shape $R_{i-1,j+1}$ (which in turn is defined in terms of $\mathbf{r}$ in Section 1.2). Say that a (T_{ij})-factor sequence is any sequence of tableaux obtained from the factor-move-multiply algorithm described above, but starting from the array (T_{ij}) rather than our canonical array $(T_{ij}(\mathbf{r}))$ of semistandard tableaux. Say that (T_{ij}) satisfies the factor sequence condition if the number of (T_{ij})-factor sequences of shape $\underline{\lambda}$ is equal to the quiver constant $c_{\underline{\lambda}}(\mathbf{r})$. The Buch–Fulton factor sequence conjecture asserts that the factor sequence condition holds for tableau arrays (T_{ij}) such that the tableaux collectively have no repeated entries and that each entry of T_{ij} is larger than each entry of $T_{i,j-1}$ and of $T_{i+1,j}$.

Corollary 8.23 states that the factor sequence condition holds for the canonical semistandard tableau array $(T_{ij}(\mathbf{r}))$. This result is not a special case of the Buch–Fulton conjecture, nor does the former immediately imply the latter, although it does rather easily imply that the factor sequence condition holds for a number of tableau arrays encompassed by the Buch–Fulton conjecture. It is an interesting and purely combinatorial problem to find the largest class of tableau arrays (T_{ij}) which satisfy the factor sequence condition.

8.2. Zelevinsky peelable tableaux

This section contains some scattered but essential results regarding peelable tableaux. Our discussion culminates in Definition 8.13, which says how to construct sequences of tableaux from peelable tableaux associated to the diagram of a Zelevinsky permutation.

The following lemma generalizes the dominance bounds for $D(w)$-peelable tableaux implied by Proposition 7.4 to the case of peelables with respect to any northwest diagram. This added generality will be used in the proof of Proposition 8.17. Recall the definitions of $\trianglelefteq$, $D\!\uparrow$, and $D\!\leftarrow$ given just before Proposition 7.4.

Lemma 8.6. *If a D-peelable tableau for a northwest diagram D has shape λ, then*

$$D\!\leftarrow\!\uparrow \;\trianglelefteq\; \lambda \;\trianglelefteq\; D\!\uparrow\!\leftarrow .$$

Proof. If a tableau has shape λ and weight γ, then $\gamma_+ \trianglelefteq \lambda$ [Ful97, Section 2.2]. Noting that $D\!\leftarrow\!\uparrow = \mathrm{code}(D)_+$, the first dominance condition holds by Remark 7.16. The second condition is a consequence of the first, using the following facts.

1. The transposing of shapes gives an anti-isomorphism of the poset of partitions under the dominance partial order.
2. There is a shape-transposing bijection $\mathrm{Peel}(D) \to \mathrm{Peel}(D^t)$, where D^t is the transpose diagram of D [RS98, Definition–Proposition 31]. □

There are a number of operations $D \to D'$ one may perform on diagrams that induce maps $\mathrm{Peel}(D) \to \mathrm{Peel}(D')$ on the corresponding sets of peelable tableaux. For notation, let s_rD be the diagram obtained by exchanging the r^{th} and $(r+1)^{\text{st}}$ rows of D, and similarly for Ds_r exchanging columns. Given a diagram D and a set I of row indices, let $D|_I$ be the diagram obtained by taking only the rows of D indexed by I. If u is a word or a tableau, let $u|_I$ be the word obtained from u by erasing all letters not in I, and set $[u]_I = [u|_I]$.

Proposition 8.7. *Let D be a northwest diagram.*

1. [RS98, Definition–Proposition 13] *If s_rD is also northwest, then there is a shape-preserving bijection $s_r : \mathrm{Peel}(D) \to \mathrm{Peel}(s_rD)$. (It is given by the action of the "r^{th} type A crystal reflection operator"; see* [RS98, Section 2] *for the definition of s_r.)*
2. [RS98, Lemma 51] *If both D and Ds_r are northwest then $\mathrm{Peel}(Ds_r) = \mathrm{Peel}(D)$.*
3. [RS98, Lemmata 54 and 56] *There is a surjective map $\mathrm{Peel}(D) \to \mathrm{Peel}(D|_I)$ given by $Q \mapsto [Q]_I$.*

Lemma 8.8. *Let D be a northwest diagram such that $D \supset D(\mu)$ for a partition μ. Then $Q \supset \mathrm{key}(\mu)$ for all tableaux $Q \in \mathrm{Peel}(D)$.*

Proof. Suppose μ has i nonempty rows. By Proposition 8.7.3 with $I = \{1, 2, \ldots, i\}$, we may assume that D has i rows. Since $D \supset D(\mu)$, the first column C of D is $C = i \cdots 321$. Let $\nu = (\mu_1 - 1, \mu_2 - 1, \ldots, \mu_i - 1)$. The first column of Q contains C and contains only letters in I, so it must be equal to C. Then $[Q - C]$ is merely Q with its first column erased. By definition $[Q - C]$ is $(D - C)$-peelable. By induction $Q - C \supset \mathrm{key}(\nu)$. But then $Q \supset C\,\mathrm{key}(\nu) = \mathrm{key}(\mu)$. □

We say that a diagram D is **λ-partitionlike** for the partition λ if D is northwest and $D(\lambda)$ can be obtained from D by repeatedly exchanging adjacent rows or columns, staying within the family of northwest shapes.

Lemma 8.9. Peel(D) *consists of one tableau, namely* key(code(D))*, if D is partitionlike.*

Proof. Using parts 1 and 2 of Proposition 8.7 to pass from D to $D(\lambda)$, there is a shape-preserving bijection from Peel(D) to Peel$(D(\lambda))$. Lemma 8.8 and Remark 7.16 imply that key(λ) is the unique $D(\lambda)$-peelable tableau. Thus Peel(D) consists of a single tableau of shape λ (since key(λ) has shape λ) and weight code(D) (by Remark 7.16). But key(code(D)) is the unique such tableau. □

Lemma 8.10. *Suppose D is a northwest diagram such that its r^{th} row (viewed as a subset of integers given by the column indices of its cells) either contains or is contained in its $(r+1)^{\text{st}}$ row. Suppose that rows r and $r+1$ have a and b cells, respectively. If Q is a D-peelable tableau, then $[Q]_{\{r,r+1\}} = [(r+1)^b r^a]$.*

Proof. Proposition 8.7.3 with $I = \{r, r+1\}$ implies $[Q]_I$ is $D|_I$-peelable. By Lemma 8.9 there is a unique $D|_I$-peelable tableau, namely $[(r+1)^b r^a]$. □

In light of Remark 7.20, the following two results are restatements of Lemma 7.5 and Proposition 7.7.

Lemma 8.11. *Let w be a permutation such that $w(i) < w(i+1)$. If $Q \in \text{Peel}(D(w))$ and $\beta = \text{wt}(K_-(Q))$, then $\beta_i \leq \beta_{i+1}$.*

Proposition 8.12. *For a fixed dimension vector $(r_0, r_1, \ldots, r_n)$, identify D_{Hom} and $D(\Omega_0)$ with partitions of those shapes. Let $\mathbf{r}$ be a rank array with the above dimension vector, $Q \in \text{Peel}(D_{\mathbf{r}})$, and $\lambda = \text{shape}(Q)$. Then $D_{Hom} \subset \lambda \subset D(\Omega_0)$.*

Let $Q \in \text{Peel}(D_{\mathbf{r}})$ and $\lambda = \text{shape}(Q)$. By Lemma 8.8 $Q \supset \text{key}(D_{Hom})$. The skew tableau $Q - \text{key}(Hom)$ has shape $\lambda/D_{Hom} \subset D(\Omega_0)/D_{Hom}$. The skew shape $D(\Omega_0)/D_{Hom}$ is a disjoint union of $r_{i-1} \times r_i$ rectangles for $1 \leq i \leq n$ in which the $r_{i-1} \times r_i$ rectangle resides in the $(i-1)^{\text{st}}$ block row and i^{th} block column, where the indexing of block rows and columns is the same as in Section 1.2. Proposition 8.12 and Lemma 8.8 imply that the following map $\Psi_{\mathbf{r}}$ from Peel$(D_{\mathbf{r}})$ to length n sequences of tableaux is well-defined. It is the peelable tableau analogue of Definition 7.8.

Definition 8.13. Let Q be a $D_{\mathbf{r}}$-peelable tableau. The sequence $\Psi_{\mathbf{r}}(Q) = (W_1, W_2, \ldots, W_n)$ of tableaux is obtained by **deleting D_{Hom} from Q** if W_i is the subtableau of the skew tableau $Q - \text{key}(D_{Hom})$ sitting inside the $r_{i-1} \times r_i$ rectangle.

Example 8.14. The image under $\Psi_{\mathbf{r}}$ of the $D_{\mathbf{r}}$-peelable tableau in Example 7.17 is the sequence (W_1, W_2, W_3) depicted in Fig. 8. □

8.3. Bijection to factor sequences

Next we give a condition for a tableau list to be obtained by deleting D_{Hom} from a $D_{\mathbf{r}}$-peelable tableau. This will be indispensable for proving the bijection from peelable tableaux to factor sequences, the aim of this section. The idea is to perform several peelings together, so as to remove one block column at a time. The 'code of D' is defined in Remark 7.16.

Lemma 8.15. *Let D be a northwest diagram, divided by a vertical line into diagrams D^w ("D-west") and D^e ("D-east"), where D^w is the first several columns of D. Assume D^w is λ-partitionlike, and set $Y = \text{key}(\text{code}(D^w))$.*

$$W_3 \;=\; \begin{array}{ccc} 8 & 9 & 9 \\ 9 & & \end{array} \qquad W_2 \;=\; \begin{array}{cc} 5 & 5 \end{array} \qquad W_1 \;=\; \begin{array}{c} 5 \end{array}$$

$$\begin{array}{cccccccc} 1&1&1&1&1&1&1&5 \\ 2&2&2&2&2&2&2& \\ 3&3&3&5&5&&& \\ 4&4&4&&&&& \\ 5&5&5&&&&& \\ 8&9&9&&&&& \\ 9&&&&&&& \end{array} \quad \longmapsto \quad \begin{array}{cccccccc} *&*&*&*&*&*&*&5 \\ *&*&*&*&*&*&*& \\ *&*&*&5&5&&& \\ *&*&*&&&&& \\ *&*&*&&&&& \\ 8&9&9&&&&& \\ 9&&&&&&& \end{array}$$

FIGURE 8. Peelable tableau to factor sequence under $\Psi_{\mathbf{r}}$

1. *If* $Q \in \mathrm{Peel}(D)$ *then* $Q \supset Y$.
2. *If* $Q \supset Y$, *then* $Q \in \mathrm{Peel}(D)$ *if and only if* $[Q - Y] \in \mathrm{Peel}(D^e)$.

Proof. By Proposition 8.7.2 we assume that the columns of D^w are decreasing with respect to containment. Suppose the rows of D^w are not decreasing with respect to containment. Then some row r of D^w is a proper subset of row $r+1$. Since D is northwest, row r of D^e is empty. This given, apply Proposition 8.7.1. It is straightforward to check that the above conditions for D and Q are equivalent to those for $s_r D$ and $s_r Q$. Hence we may assume that $D^w = D(\lambda)$ for a partition λ.

Now item 1 follows from Lemma 8.8. For item 2 we use the jeu de taquin [Ful97, Section 1.2]. Let C_j be the column word that gives the (decreasing) set of row indices for the elements in column j of D. The result $[Q - C_1]$ of one peeling step can be computed by sliding the skew tableau $Q - C_1$ into the cells occupied by C_1. Repeating this process for the columns of D^w, the tableau resulting from applying the corresponding peeling steps to Q is also obtained by removing the subtableau Y (which must be present in Q by item 1) and sliding the skew tableau $Q - Y$ into the cells that were occupied by Y. $\square$

Lemma 8.16. *If* $(W_1, \ldots, W_n) = \Psi_{\mathbf{r}}(Q)$ *for* $Q \in \mathrm{Peel}(D_{\mathbf{r}})$, *then*

$$W_i \;\in\; B^{\bullet}_{i-1} \tag{8.6}$$

$$\mathrm{height}(W_i) \;\leq\; r_{i-1} \tag{8.7}$$

$$\mathrm{width}(W_i) \;\leq\; r_i, \tag{8.8}$$

and Q *has the block column factorization*

$$Q \;=\; (W_n Y_n) \cdots (W_2 Y_2)(W_1 Y_1). \tag{8.9}$$

Proof. Statement (8.6) follows from (8.2) and the fact that W_i sits below Y_i in a block column of the tableau Q. The rest follows by definition, using Proposition 8.12. $\square$

Proposition 8.17. *The list* $(W_1, \ldots, W_n)$ *of tableaux is obtained from some* $D_{\mathbf{r}}$*-peelable tableau by deleting* D_{Hom} *if and only if there exist tableaux* $X_1, \ldots, X_n$ *such that*

$$X_n \;=\; W_n, \tag{8.10}$$

$$X_i \;\supset\; K_i \quad \textit{for all } i, \tag{8.11}$$

$$X_i \;=\; [(X_{i+1} - K_{i+1}) W_i] \quad \textit{for } i = 1, \ldots, n-1, \tag{8.12}$$

$$\textit{and} \quad X_1 \;=\; K_1; \tag{8.13}$$

and if

$$Z_i = [X_i W_{i-1} \cdots W_2 W_1]_{B_{i-1}} \tag{8.14}$$

then

$$\text{height}(Z_i) \leq r_{i-1}, \tag{8.15}$$
$$\text{width}(Z_i) \leq r_i, \tag{8.16}$$
$$\textit{and} \quad \text{wt}(Z_i) = \text{wt}(T_{\leq i, \geq i-1}), \tag{8.17}$$

where $\text{wt}(T_{\leq i, \geq j}) = \sum_{\substack{k \geq i \\ \ell \leq j}} \text{wt}(T_{k,\ell})$.

The proof will come shortly.

Lemma 8.18. *The conditions in Proposition 8.17 imply the following:*

$$X_i \in B^{\bullet}_{i-1} \tag{8.18}$$
$$\text{height}(X_i) \leq r_{i-1} \tag{8.19}$$
$$\text{width}(X_i) \leq r_i \tag{8.20}$$
$$\text{wt}(X_i) \leq \text{wt}(T_{\leq i, \geq i-1}) \tag{8.21}$$

Proof. (8.18) can be proved by induction using (8.6). X_i is a factor of Z_i by (8.18) and (8.14). The last three assertions then follow from (8.15), (8.16), and (8.17). □

Lemma 8.19. *If* (8.10) *through* (8.13) *hold, then the conditions* (8.18) *through* (8.20) *imply* (8.6) *through* (8.9)*, plus*

$$\text{wt}(W_i) \leq \text{wt}(T_{\leq i, \geq i-1}).$$

This last inequality is the same condition on $(W_1, \ldots, W_n)$ appearing in Remark 8.5.

Example 8.20. The tableaux X_i and Z_i associated with the above tableaux W_i are given by

$$X_3 = \begin{array}{|c|c|c|}\hline 8 & 9 & 9 \\ \hline 9 \\ \cline{1-1}\end{array} \qquad X_2 = \begin{array}{|c|c|}\hline 5 & 5 \\ \hline 9 & 9 \\ \hline\end{array} \qquad X_1 = \begin{array}{|c|}\hline 5 \\ \hline 9 \\ \hline\end{array} \tag{8.22}$$

$$Z_3 = \begin{array}{|c|c|c|}\hline 8 & 9 & 9 \\ \hline 9 \\ \cline{1-1}\end{array} \qquad Z_2 = \begin{array}{|c|c|c|}\hline 5 & 5 & 5 \\ \hline 9 & 9 \\ \cline{1-2}\end{array} \qquad Z_1 = \begin{array}{|c|}\hline 5 \\ \hline 9 \\ \hline\end{array} \tag{8.23}$$

Note that the X tableaux appear at the bottom of the leftmost block columns in the first, fourth, and eighth tableaux in Example 7.17. □

Proof of Proposition 8.17. Set $D = D_{\mathbf{r}}$, and let D_i be the diagram obtained from D by removing the n^{th} through $(i+1)^{\text{st}}$ block columns (or equivalently the leftmost $r_n + \cdots + r_{i+1}$ columns). Let $D_{i,j}$ be obtained from D_i by removing the 0^{th} through $(j-1)^{\text{st}}$ block rows. In other words, $D_{i,j}$ consists of the block at the intersection of the i^{th} block column and j^{th} block row, along with all blocks to its south and east.

Suppose $(W_1, \ldots, W_n) = \Psi_{\mathbf{r}}(Q)$ for a tableau $Q \in \text{Peel}(D)$. Let $Q_i \in \text{Peel}(D_i)$ be the tableau obtained from Q by $r_n + \cdots + r_{i+1}$ peelings. We shall proceed by induction, with hypothesis (for descending i) being that Q_j has the block column factorization

$$Q_j = (X_j Y_j)(W_{j-1} Y_{j-1}) \cdots (W_2 Y_2)(W_1 Y_1) \tag{8.24}$$

and that X_j, Z_j have been defined and have the desired properties for $n \geq j \geq i$.

Suppose $0 \le i \le n$ and the induction hypothesis holds for indices greater than i. It must be shown that the induction hypothesis holds for i. The tableau X_i is defined by (8.10) when $i = n$, or by (8.12) when $i < n$ (note that (8.11) holds for $i + 1$).

Apply Lemma 8.15 to $Q_{i+1} \in \mathrm{Peel}(D_{i+1})$, with D^w_{i+1} the west block column (the leftmost r_{i+1} columns) of D_{i+1}. By (8.24) for $i+1$, the first block column of Q_{i+1} is $X_{i+1}Y_{i+1}$. Also $\mathrm{key}(\mathrm{code}(D^w_{i+1})) = K_{i+1}Y_{i+1}$. Recalling that Q_i is defined by iterated peelings from Q, we have

$$\begin{aligned} Q_i &= [(X_{i+1}Y_{i+1} - K_{i+1}Y_{i+1})(W_iY_i)\cdots(W_1Y_1)] \\ &= [(X_{i+1} - K_{i+1})W_iY_i\cdots(W_1Y_1)] \\ &= [(X_iY_i)(W_{i-1}Y_{i-1})\cdots(W_1Y_1)], \end{aligned} \tag{8.25}$$

the last equality by definition of X_i. Apply Proposition 8.7.3 to $Q_i \in \mathrm{Peel}(D_i)$ and the interval B_{i-1}. Note that $(D_i)|_{B_j} = D_{i,j}$, so in particular $[Q_i]_{B_{i-1}} \in \mathrm{Peel}(D_{i,i-1})$. Thus

$$[Q_i]_{B_{i-1}} \;=\; [X_iW_{i-1}\cdots W_1]_{B_{i-1}} \;=\; Z_i,$$

by (8.25), (8.2), and (8.14). We have proved that

$$Q \in \mathrm{Peel}(D) \quad\Rightarrow\quad Z_i \in \mathrm{Peel}(D_{i,i-1}). \tag{8.26}$$

The sum of the heights of the rectangles in the i^{th} block column of $D_{i,i-1} \subset D$ is at most r_{i-1} by (1.3). The sum of the widths of the rectangles in the $(i-1)^{\text{st}}$ block row of $D_{i,i-1} \subset D$ is at most r_i by (1.4). Z_i fits inside a $r_{i-1} \times r_i$ rectangle, by Lemma 8.6, (1.7), and (1.8). This proves (8.15) and (8.16). Equation (8.17) follows from Remark 7.16. By Lemma 8.18, the corresponding properties (8.18) through (8.21) hold for the X tableaux. The block column factorization (8.24) for $j = i$ then follows from (8.25), (8.19), (8.20), and (8.18).

To prove (8.11), apply Lemma 8.15 with $Q_i \in \mathrm{Peel}(D_i)$ and D^w_i given by the westmost block column of D_i. This yields that $Q_i \supset \mathrm{key}(\mathrm{code}(D^w_i)) = K_iY_i$. By (8.24) the first block column of Q_i is X_iY_i, from which (8.11) follows.

For the converse (i.e. that (8.10)–(8.17) imply $(W_1, \ldots, W_n)$ lies in the image of $\Psi_{\mathbf{r}}$), suppose the X_i and Z_i exist and have the desired properties. By Lemmas 8.18 and 8.19, conditions (8.18)–(8.21) and (8.6)–(8.8) all hold. Let Q_i be defined by the block column factorization (8.24). The aforementioned properties guarantee that Q_i is a tableau. It will be shown by induction on increasing i that $Q_i \in \mathrm{Peel}(D_i)$.

For $i = 0$, Q_0 is empty by (8.24), and D_0 is the empty diagram so $Q_0 \in \mathrm{Peel}(D_0)$. Suppose $i > 0$. Apply Lemma 8.15 to D_i and Q_i with D^w_i the west block column of D_i. Then $Q_i \in \mathrm{Peel}(D_i)$ if and only if $Q_i \supset \mathrm{key}(\mathrm{code}(D^w_i)) = K_iY_i$ and $[Q_i - (K_iY_i)] \in \mathrm{Peel}(D_{i-1})$. But Q_i has first block column X_iY_i so by (8.11) it contains K_iY_i. Also

$$\begin{aligned} [Q_i - (K_iY_i)] &= [(X_i - K_i)Y_i(W_{i-1}Y_{i-1})\cdots(W_1Y_1)] \\ &= [X_{i-1}Y_{i-1}(W_{i-2}Y_{i-2}\cdots(W_1Y_1)] \\ &= Q_{i-1} \end{aligned}$$

by (8.24) for i and $i-1$, and (8.12) for $i-1$. Since by induction $Q_{i-1} \in \mathrm{Peel}(D_{i-1})$, we conclude that $Q_i \in \mathrm{Peel}(D_i)$. Taking $i = n$ we have $Q_n \in \mathrm{Peel}(D_n)$.

The proof is complete because $Q_n = Q$ and $D_n = D$. □

For the record, the next result is "rank stability for peelable tableaux", which is analogous to the rank stability statements we proved for components of quiver degenerations (Proposition 4.13) and Zelevinsky pipe dreams (Proposition 5.15).

Corollary 8.21. *There is a bijection* $\mathrm{Peel}(D_{\mathbf{r}}) \to \mathrm{Peel}(D_{m+\mathbf{r}})$ *inducing a shape-preserving bijection* $\mathrm{Im}(\Psi_{\mathbf{r}}) \to \mathrm{Im}(\Psi_{m+\mathbf{r}})$.

Proof. With notation as in Definition 1.10, the diagrams $D^{\square}_{\mathbf{r}}$ and $D^{\square}_{m+\mathbf{r}}$ are the same up to removing some empty columns and rows. By Proposition 8.17 the images of $\Psi_{\mathbf{r}}$ and $\Psi_{m+\mathbf{r}}$ differ by a trivial relabeling of their entries. □

Corollary 8.21 could be used instead of Proposition 7.13 as the conduit through which Corollary 6.13 enters into the proof of Theorem 7.21.

8.4. The Buch–Fulton conjecture

Theorem 8.22. *The map* $\Psi_{\mathbf{r}}$ *taking each* $D_{\mathbf{r}}$*-peelable tableau* $Q \in \mathrm{Peel}(D_{\mathbf{r}})$ *to the list* $\Psi_{\mathbf{r}}(Q)$ *of tableaux obtained by deleting* D_{Hom} *from* Q *is a bijection from the set of* $D_{\mathbf{r}}$*-peelable tableaux to* **r***-factor sequences.*

Proof. It suffices to show that the conditions on a tableau list $(W_1, \ldots, W_n)$ in Proposition 8.17 for membership in the image of $\Psi_{\mathbf{r}}$ are equivalent to the conditions defining an $\mathbf{r}$-factor sequence. The case $n = 1$ is trivial, so suppose $n \geq 2$. Write $\widehat{K}_i$ to denote the analogue of the tableau K_i for $\widehat{\mathbf{r}}$, for $1 \leq i \leq n-1$. Identifying the tableau array of $\widehat{\mathbf{r}}$ with that of $\mathbf{r}$ but with $T_i = T_{i,i-1}$ removed for all i, we have

$$K_i \;=\; [\widehat{K}_i T_i] \quad \text{for } 1 \leq i \leq n, \tag{8.27}$$

where by convention

$$\widehat{K}_n \;=\; \varnothing. \tag{8.28}$$

Suppose first that $(W_1, \ldots, W_n)$ is an $\mathbf{r}$-factor sequence. Let $(U_1, \ldots, U_{n-1})$ be an $\widehat{\mathbf{r}}$-factor sequence, with factorizations given by (8.3) and such that (8.4) and (8.5) hold. By induction the $\widehat{\mathbf{r}}$-factor sequence $(U_1, \ldots, U_{n-1})$ satisfies the conditions of Proposition 8.17 with associated tableaux $\widehat{X}_i$ for $1 \leq i \leq n-1$. For convenience define

$$\widehat{X}_n \;=\; \varnothing \tag{8.29}$$

The $\widehat{X}_i$ satisfy the following analogues of (8.12) through (8.21):

$$\widehat{X}_i \;=\; [(\widehat{X}_{i+1} - \widehat{K}_{i+1})U_i] \tag{8.30}$$

$$\widehat{X}_i \;\supset\; \widehat{K}_i \qquad \text{for } 1 \leq i \leq n-1 \tag{8.31}$$

$$\widehat{X}_1 \;=\; \widehat{K}_1. \tag{8.32}$$

Defining the tableau $\widehat{Z}_i$ by

$$\widehat{Z}_i \;=\; [\widehat{X}_i U_{i-1} \cdots U_1]_{B_i} \tag{8.33}$$

we have

$$\mathrm{height}(\widehat{Z}_i) \;\leq\; r_{i-1,i} \tag{8.34}$$

$$\mathrm{width}(\widehat{Z}_i) \;\leq\; r_{i,i+1} \tag{8.35}$$

$$\mathrm{wt}(\widehat{Z}_i) \;=\; \mathrm{wt}(T_{\leq i, \geq i}). \tag{8.36}$$

Note that the analogue $\widehat{X}_{n-1} = U_{n-1}$ of (8.10) follows from (8.28), (8.29) and (8.30).

Define X_i by

$$X_i \;=\; [\widehat{X}_i\, T_i\, P_{i-1}]. \tag{8.37}$$

By (8.31) and (8.27) we have (8.11) and

$$[X_i - K_i] \quad = \quad [(\widehat{X}_i - \widehat{K}_i)P_{i-1}]. \tag{8.38}$$

We shall show by descending induction on i that X_i and Z_i (the latter being defined by (8.14)) have the properties specified by Proposition 8.17.

We have $X_n = [\varnothing T_n P_{n-1}] = [Q_n T_n P_{n-1}] = W_n$ by (8.29), (8.5), and (8.4), proving (8.10). Let $n > i \geq 1$. We have

$$\begin{aligned} X_i &= [\widehat{X}_i T_i P_{i-1}] \\ &= [(\widehat{X}_{i+1} - \widehat{K}_{i+1})U_i T_i P_{i-1}] \\ &= [(\widehat{X}_{i+1} - \widehat{K}_{i+1})P_i Q_i T_i P_{i-1}] \\ &= [(\widehat{X}_{i+1} - \widehat{K}_{i+1})P_i W_i] \\ &= [(X_{i+1} - K_{i+1})W_i] \end{aligned}$$

by (8.37), (8.30), (8.3), (8.4), and (8.38), proving (8.12). We have $X_1 = [\widehat{X}_1 T_1 P_0] = [\widehat{K}_1 T_1] = K_1$ by (8.37), (8.32), (8.5), and (8.27), proving (8.13).

By construction Z_i satisfies (8.17). To see this, observe first that $\sum_{i=1}^n \mathrm{wt}(W_i)$ is the weight of the entire tableau array. Then by (8.10) and (8.12), $\mathrm{wt}(X_i W_{i-1} \cdots W_1)$ is the weight of the entire tableau array minus $\sum_{j=i+1}^m \mathrm{wt}(K_j)$, which is the part of the tableau array strictly west of the i^{th} block column. Finally, the restriction of $X_i W_{i-1} \cdots W_1$ to B_{i-1} restricts to the part of the tableau array weakly below the $(i-1)^{\text{st}}$ block row.

In particular, Z_i has at most r_i distinct letters appearing in it since the corresponding part of the tableau array has that property (see the proof of Proposition 8.17). Since Z_i is a tableau, (8.15) follows.

For (8.16), we have

$$\begin{aligned} Z_i &= [(\widehat{X}_i T_i P_{i-1})(Q_{i-1} T_{i-1} P_{i-2}) \cdots (Q_1 T_1 P_0)]_{B_{i-1}} \\ &= [\widehat{X}_i T_i U_{i-1} U_{i-2} \cdots U_1]_{B_{i-1}} \end{aligned} \tag{8.39}$$

by (8.14), (8.37), (8.4), (8.3), (8.5), and the fact that $T_j \in A^{\bullet}_{j-1}$. We get

$$\begin{aligned} \widehat{Z}_{i-1} &= [\widehat{X}_{i-1} U_{i-2} \cdots U_1]_{B_{i-1}} \\ &= [(\widehat{X}_i - \widehat{K}_i) U_{i-1} \cdots U_1]_{B_{i-1}} \\ &= [(\widehat{X}_i T_i - \widehat{K}_i T_i) U_{i-1} \cdots U_1]_{B_{i-1}} \\ &= [(\widehat{X}_i T_i - K_i) U_{i-1} \cdots U_1]_{B_{i-1}} \\ &= [Z_i - K_i] \end{aligned} \tag{8.40}$$

by (8.33), (8.30), (8.27), and (8.39). Here we have used $\widehat{X}_i \supset \widehat{K}_i$ so that $[\widehat{X}_i T_i] \supset [\widehat{K}_i T_i] = K_i$. By (8.40) we have

$$\mathrm{width}(\widehat{Z}_{i-1}) \quad \geq \quad \mathrm{width}(Z_i) - \mathrm{width}(K_i).$$

But $\mathrm{width}(K_i) = \mathrm{width}(T_i) = \mathrm{width}(R_{i-1,i}) = r_i - r_{i-1,i}$; together with (8.35) (applied to $i-1$ instead of i), this implies (8.16).

Conversely, suppose $(W_1, \ldots, W_n)$ satisfies Proposition 8.17 with tableaux X_i. We must show that it is an **r**-factor sequence. By Lemma 8.18, conditions (8.18)–(8.21) hold. We claim that

$$W_i \supset T_i \qquad \text{for all } 1 \leq i \leq n. \tag{8.41}$$

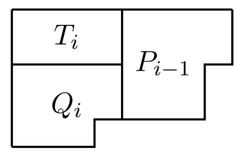
We have

$$X_i \supset T_i \qquad \text{for all } 1 \le i \le n \tag{8.42}$$

by (8.11) and (8.1). For $i = n$, (8.10) and (8.42) gives (8.41). For $n > i \ge 1$, (8.18) for $i+1$, (8.12), and (8.42) imply (8.41).

Consider the factorization of W_i pictured below.

$W_i \quad =$



By construction (8.4) holds.

Since T_n has the largest letters, $Q_n = \varnothing$. To obtain $P_0 = \varnothing$ (and finish up (8.5)), note that $[\widehat{K}_1 T_1] = K_1 = X_1 = [(X_2 - K_2)W_1]$ by (8.27), (8.13), and (8.12). But K_1 has the same width as T_1. It follows that W_1 does also, and hence that $P_0 = \varnothing$.

Define U_i by (8.3). By induction it suffices to prove that there exist tableaux $\widehat{X}_i$ such that $(U_1, \dots, U_{n-1})$ satisfies the conditions of Proposition 8.17. Define $\widehat{X}_n$ by (8.29) and

$$\widehat{X}_i \quad = \quad [(X_{i+1} - K_{i+1})Q_i] \qquad \text{for } 1 \le i \le n-1. \tag{8.43}$$

We have

$$[\widehat{X}_i T_i P_{i-1}] \;=\; [(X_{i+1} - K_{i+1})Q_i T_i P_{i-1}] \;=\; [(X_{i+1} - K_{i+1})W_i] \;=\; X_i$$

by (8.43), (8.4), and (8.12), proving (8.37). This implies (8.31). We get

$$\begin{aligned} \widehat{X}_i &= [(X_{i+1} - K_{i+1})Q_i] \\ &= [(\widehat{X}_{i+1} T_{i+1} P_i - \widehat{K}_{i+1} T_{i+1})Q_i] \\ &= [(\widehat{X}_{i+1} - \widehat{K}_{i+1}) P_i Q_i] \\ &= [(\widehat{X}_{i+1} - \widehat{K}_{i+1}) U_i] \end{aligned}$$

by (8.43), by (8.37), (8.27), and (8.31) for $i+1$ (which hold by induction), and (8.3), proving (8.30). We have $[\widehat{K}_1 T_1] = K_1 = X_1 = [\widehat{X}_1 T_1 P_0] = [\widehat{X}_1 T_1]$ by (8.27), (8.13), (8.37), and (8.5), from which (8.32) follows.

Define Z_i by (8.14) and $\widehat{Z}_i$ by (8.33). Note that (8.36) holds by construction. This implies (8.34) because $\widehat{Z}_i$ is a tableau in an alphabet with at most $r_{i,i+1}$ distinct letters. To show that (8.35) holds, note that (8.40) still holds. Let $D_{i,j}$ be as in the proof of Proposition 8.17. Recall that also that $Z_i \in \mathrm{Peel}(D_{i,i-1})$ by (8.26). Apply Lemma 8.15 with $D^w_{i,i-1}$ given by the west block column of $D_{i,i-1}$. Note that $K_i = \mathrm{key}(\mathrm{code}(D^w_{i,i-1}))$. It follows that the r_i-fold peeling of Z_i (which is none other than $\widehat{Z}_i$ by (8.40)) is $D_{i,i}$-peelable. The inequality (8.35) follows. Since the conditions of Proposition 8.17 have been satisfied, $(U_1, \dots, U_{n-1})$ is in the image of $\Psi_{\hat{\mathbf{r}}}$. By induction $(U_1, \dots, U_{n-1})$ is an $\hat{\mathbf{r}}$-factor sequence. Thus $(W_1, \dots, W_n)$ is an $\mathbf{r}$-factor sequence. □

Our final result is an alternate combinatorial version of the formula in Theorem 7.21. The list of shapes in a tableau list is called the **shape** of the tableau list.

Corollary 8.23 (Buch–Fulton conjecture [BF99]). *The quiver constant $c_{\underline{\lambda}}(\mathbf{r})$ in the expression $\mathcal{Q}_{\mathbf{r}}(\mathbf{x} - \mathring{\mathbf{x}}) = \sum_{\underline{\lambda}} c_{\underline{\lambda}}(\mathbf{r}) s_{\underline{\lambda}}(\mathbf{x_r} - \mathring{\mathbf{x}}_{\mathbf{r}})$ of the quiver polynomial in terms of Schur functions (or by Theorem 7.10, in the expression of double quiver functions in terms of double Schur functions) equals the number of $\mathbf{r}$-factor sequences of shape $\underline{\lambda}$.*

Proof. Theorem 8.22 and Theorem 7.21. □

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