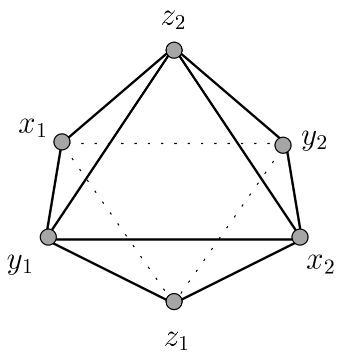
B Appendix: Solutions

Many of the solutions below were contributed by students of the 1999 COCOA Summer School in Turin, Italy, although their solutions have been edited for clarity and for technical reasons. When the printed solution has more than one author, the contributing author is labeled with an asterisk (*). When more than one group contributed solutions which were subsequently amalgamated by the authors of these lectures, each group has been parenthesized. All exercises which include computations refer to the program CoCoA [8].

B.1 Solutions for Day 1

Solution B.1.1 *By (Bahman Engheta, Leah Gold*, Ed Mosteig), (Kimberly Presser*), (Gerhard Quarg*), and (Carolyn Yackel*)*

In fact, these ideals appear in Lecture VI (Example 6.3). Here, we prefer to label the vertices of the octahedron as indicated in the picture:



In order to get the ideal I_Δ, write down the minimal nonfaces of the octahedron, which are the edges connecting variables on the same axis:

$$I_\Delta = \langle x_1x_2, y_1y_2, z_1z_2 \rangle .$$

For $I_{\Delta^\vee}$, just switch the roles of the minimal generators and prime components:

$$\begin{aligned} I_{\Delta^\vee} &= \langle x_1, y_1\rangle \cap \langle y_1, y_2\rangle \cap \langle z_1, z_2\rangle \\ &= \langle x_2y_2z_2, x_1y_2z_2, x_2y_1z_2, x_1y_1z_2, x_2y_2z_1, x_1y_2z_1, x_2y_1z_1, x_1y_1z_1\rangle. \end{aligned}$$

The Hilbert series for S/I_Δ can be calculated by hand from the polytopes as described in Lecture I or read off of the minimal free resolution calculated below:

$$\frac{1 - x_1x_2 - y_1y_2 - z_1z_2 + x_1x_2y_1y_2 + x_1x_2z_1z_2 + y_1y_2z_1z_2 - x_1x_2y_1y_2z_1z_2}{(1-x_1)(1-x_2)(1-y_1)(1-y_2)(1-z_1)(1-z_2)} .$$

We get the coarse Hilbert series by setting all variables equal to t:

$$\frac{1 - 3t^2 + 3t^4 - t^6}{(1-t)^6} = \frac{1 + 3t + 3t^2 + t^3}{(1-t)^3}.$$

Similarly for $I_{\Delta^\vee}$: the numerator of the Hilbert series of $S/I_{\Delta^\vee}$ is

$$\begin{aligned} 1 \ &- \ x_1y_1z_1 - x_2y_1z_1 - x_1y_2z_1 - x_2y_2z_1 - x_1y_1z_2 - x_2y_1z_2 - x_1y_2z_2 - x_2y_2z_2 \\ &+ \ x_2y_2z_1z_2 + x_1y_2z_1z_2 + x_2y_1z_1z_2 + x_1y_1z_1z_2 + x_2y_1y_2z_2 + x_1y_1y_2z_2 \\ &+ \ x_1x_2y_2z_2 + x_1x_2y_1z_2 + x_2y_1y_2z_1 + x_1y_1y_2z_1 + x_1x_2y_2z_1 + x_1x_2y_1z_1 \\ &- \ x_2y_1y_2z_1z_2 - x_1y_1y_2z_1z_2 - x_1x_2y_2z_1z_2 - x_1x_2y_1z_1z_2 - x_1x_2y_1y_2z_2 \\ &- \ x_1x_2y_1y_2z_1 + x_1x_2y_1y_2z_1z_2 \,, \end{aligned}$$

while the coarse Hilbert series of $I_{\Delta^\vee}$ is

$$\frac{1 - 8t^3 + 12t^4 - 6t^5 + t^6}{(1-t)^6} = \frac{1 + 2t + 3t^2 - 4t^3 + t^4}{(1-t)^4}.$$

One may calculate the coarse Hilbert series with CoCoA as follows:

```
Use S ::= Q[x[1..2]y[1..2]z[1..2]];
I := Ideal(x[1]x[2],y[1]y[2],z[1]z[2]);
IDual := Intersection(Ideal(x[1],x[2]),Ideal(y[1],y[2]),
                      Ideal(z[1],z[2]));
IDual;
Ideal(x[2]y[2]z[2], x[1]y[2]z[2], x[2]y[1]z[2], x[1]y[1]z[2],
x[2]y[2]z[1], x[1]y[2]z[1], x[2]y[1]z[1], x[1]y[1]z[1])
-------------------------------
Poincare(S/I);
(1 + 3x[1] + 3x[1]^2 + x[1]^3) / (1-x[1])^3
-------------------------------
Poincare(S/IDual);
(1 + 2x[1] + 3x[1]^2 - 4x[1]^3 + x[1]^4) / (1-x[1])^4
-------------------------------
```

One may use CoCoA to calculate the minimal free resolutions. Continuing from above:

```
Res(S/I);
0 --> S(-6) --> S^3(-4) --> S^3(-2) --> S
-------------------------------
Describe It;

Mat[
  [z[1]z[2], y[1]y[2], x[1]x[2]]
]
Mat[
  [y[1]y[2], x[1]x[2], 0],
  [-z[1]z[2], 0, x[1]x[2]],
  [0, -z[1]z[2], -y[1]y[2]]
]
```

```
Mat[
  [x[1]x[2]],
  [-y[1]y[2]],
  [z[1]z[2]]
]
-------------------------------
```

Using monomial matrices the minimal free resolution of S/I_Δ is

$$0 \longleftarrow S \xleftarrow{\begin{array}{c|ccc} & z_1z_2 & y_1y_2 & x_1x_2 \\ \hline 1 & 1 & 1 & 1 \end{array}} S^3 \xleftarrow{\begin{array}{c|ccc} & y_1y_2z_1z_2 & x_1x_2z_1z_2 & x_1x_2y_1y_2 \\ \hline z_1z_2 & 1 & 1 & 0 \\ y_1y_2 & -1 & 0 & 1 \\ x_1x_2 & 0 & -1 & -1 \end{array}} S^3 \xleftarrow{\begin{array}{c|c} & x_1x_2y_1y_2z_1z_2 \\ \hline y_1y_2z_1z_2 & 1 \\ x_1x_2z_1z_2 & -1 \\ x_1x_2y_1y_2 & 1 \end{array}} S^1 \longleftarrow 0$$

The alternating sum of the column labels is the numerator of the Hilbert series of S/I_Δ.

The minimal free resolution of $S/I_{\Delta^\vee}$ is

$$0 \longleftarrow S \longleftarrow S^8 \longleftarrow S^{12} \longleftarrow S^6 \longleftarrow S^1 \longleftarrow 0\,.$$

The matrices are too big to fit comfortably on the page (either in monomial matrix form or usual form). However, we note that the monomial matrices are filled with the boundary maps of the cube (Proposition 5.4)—observe the reflection of this fact in the total Betti numbers $(1, 8, 12, 6, 1)$ of $S/I_{\Delta^\vee}$.

Use Hochster's formula (Theorem 1.8), $\beta_{i,\sigma}(S/I_\Delta) = \dim_k \widetilde{H}^{|\sigma|-i-1}(\Delta|_\sigma; k)$, to relate the Betti numbers with simplicial homology. For each incidence vector σ—which we identify with a subset of the vertices of the octahedron—the Betti number $\beta_{i,\sigma}(S/I_\Delta)$ is 1 precisely when $\mathbf{x}^\sigma$ is a column label for the i-th monomial matrix appearing in the minimal free resolution of S/I_Δ; otherwise it is 0. Thus, for instance, from the column label $x_1x_2z_1z_2$ appearing in the middle matrix of the resolution displayed earlier we see that

$$\beta_{2,\sigma} = \dim_k \widetilde{H}^{|\sigma|-3}(\Delta|_\sigma; k) = \dim_k \widetilde{H}^1(\Delta|_\sigma; k) = \dim_k \widetilde{H}_1\Delta|_\sigma; k) = 1.$$

Here, σ corresponds to the four vertices of the octahedron labeled x_1, x_2, z_1, and z_2. Hence, $\Delta|_\sigma$ is the boundary of a square. Since the column label $x_1x_2z_1z_2$ only appears once in the resolution, Hochster's formula shows that the reduced homology of $\Delta|_\sigma$ is 0 in every other homological degree, as expected.

When $\sigma = \mathbf{1}$ is defined by $\mathbf{x}^\sigma = x_1x_2y_1y_2z_1z_2$, we get $\Delta|_\sigma = \Delta$. Using Hochster's formula, we see that $\widetilde{H}_2(\Delta; k) \cong k$ and the other homology is zero.

Solution B.1.2

Although CoCoA doesn't readily calculate $\mathbb{Z}^n$-graded Betti numbers, the only Betti numbers we need to find the homology of a simplicial complex Δ are $\beta_{i,(1,\ldots,1)}(S/I_\Delta)$, and these are the only Betti numbers in total ($\mathbb{Z}$-graded) degree $n = \#$variables.

Therefore, we can calculate $\dim_k \widetilde{H}_i(\Delta; k) = \beta_{n-i-1,(1,\ldots,1)}(S/I_\Delta)$ (Theorem 1.8) simply by reading off the number of summands of the form $S(-n)$ at the i^{th} step in the minimal free resolution of S/I_Δ.

Solution B.1.3

First create the 5×5 chessboard complex and calculate its Hilbert series:

```
N := 5;
Use S ::= Q[x[1..N,1..N]];

G := []; -- generators for the ideal
For I := 1 To N Do
  For J := 1 To N Do
    For L := 1 To N Do
      If J <> L Then
G := Concat(G,[x[I,J]x[I,L],x[J,I]x[L,I]]);
      End;
    End;
  End;
End;
Len(Set(G)); -- just to make sure we get 100 monomials
100
-------------------------------
IDelta := Ideal(G); -- create the ideal
Poincare(S/IDelta);
(1 + 20x[1,1] + 110x[1,1]^2 + 140x[1,1]^3 -
95x[1,1]^4 - 56x[1,1]^5) / (1-x[1,1])^5
-------------------------------
```

Now compute the f-vector:

```
-- function for computing the f-vector
Define FVector(I)
  L := HVector(CurrentRing()/I);
  D := Dim(CurrentRing()/I);
  Using Qt Do
    H := Sum([L[N]t^(N-1) | N In 1..Len(L)]);
    E := Subst(H,t,1);
    F := [E];
    For N := 0 To D-1 Do
      H := (H-E*t^(D-N))/(1-t);
      E := Subst(H,t,1);
      F := Concat([E],F);
    End;
  End;
```

```
    Return [LC(X) | X In F];  -- LC used to suppress printing of 'Qt'
End;

FVector(IDelta);
[1, 25, 200, 600, 600, 120]
-------------------------------
```

Hence, there are 25 vertices, 200 edges, 600 two-dimensional faces, etc.

In theory we should be able to calculate the homology using the CoCoA command `Res(S/IDelta)` and looking for the number of summands of the form $S(-25)$ in each degree. Unfortunately, the calculation does not terminate successfully in CoCoA. For the 3×3 case, go back and set `N := 3`. We get

```
FVector(IDelta);
[1, 9, 18, 6]
-------------------------------
Res(S/IDelta);
0 --> S^4(-9) --> S^27(-8) --> S^72(-7) --> S^9(-5)(+)S^90(-6) -->
S^45(-4)(+)S^45(-5) --> S^48(-3)(+)S^9(-4) --> S^18(-2) --> S
-------------------------------
```

which according to Solution B.1.2 means that $\dim_k \widetilde{H}_1(\Delta; k) = \beta_7(S/I_\Delta) = 4$ and the other homology groups are zero.

Solution B.1.4

Here is some CoCoA code to accomplish the task. Unfortunately, this code will probably not finish in a reasonable amount of time, except for the reverse lex gin.

```
S ::= Q[x,y,z];
T ::= Q[x[1..10]];  --Default order Reverse Lex
U ::= Q[x[1..10]], Lex;
Use T;

Using S Do                           --Create a matrix whose columns
  M := Ideal(Indets())^3;            --are the exponent vectors of the
  L := [Log(J) | J In MinGens(M)];   --10 degree 3 monomials
  L := Transposed(L);
End;

I := Toric(L);  --Form the toric ideal for this matrix

Define Gin(I)
  RandLin(N) := [Sum([Rand(-100,100) J | J In Indets()]) | M In 1..N];
  J := Eval(I,RandLin(NumIndets()));
  Return(LT(J));
End;
```

The following is the result of taking the reverse lex initial ideal:

```
Use T;
Gin(I);
Ideal(x[1]^2, x[1]x[2], x[2]^2, x[1]x[3], x[2]x[3], x[3]^2, x[1]x[4],
x[2]x[4], x[3]x[4], x[4]^2, x[1]x[5], x[2]x[5], x[3]x[5], x[4]x[5],
x[5]^2, x[1]x[6], x[2]x[6], x[3]x[6], x[4]x[6], x[5]x[6], x[6]^2,
x[1]x[7], x[2]x[7], x[3]x[7], x[4]x[7], x[5]x[7], x[6]x[7], x[7]^3)
```

Solution B.1.5 *By Carolyn Yackel*

Any nonradical ideal I whose radical is $\sqrt{I} = \mathfrak{m} = \langle x_1, \ldots, x_n \rangle$ cannot be an initial ideal of a homogeneous prime. Indeed, any homogeneous ideal J having initial ideal I is artinian, and therefore has support on $\mathfrak{m}$. The Hilbert function of such a J gives away the fact that J is not radical. In particular, I can be Borel-fixed—for instance, I can be a power of $\mathfrak{m}$. In general, Borel-fixed ideals which are primary to ideals generated by initial subsets of the variables will have the same problems.

Solution B.1.6 *By Mircea Mustaţă and Greg Smith**

Every initial ideal can is the result of taking an initial with respect to a *weight order* [12, Chapter 15]. We start by finding a list of weight vectors which give different initial ideals. Without loss of generality, it is enough to consider vectors of the form $(A, 2, 1)$: because the ideal is homogeneous the initial ideal remains the same if one adds or subtracts $(1, 1, 1)$ and if one multiplies by a scalar. We may assume the first entry of $(A, 2, 1)$ is larger than the second and that the second is larger than the third, as long as we remember to apply all permutations of the variables when we're done.

```
LL := [];
For A := 3 To 20 Do
    W := Mat[[A,2,1]];
    S ::= Q[abc], Weights(W);
    Using S Do
        L := GenericPoints(11);
        L := Tail(Tail(Tail(Tail(L))));
        I := IdealOfProjectivePoints(L);
        J := [Log(X) | X In Gens(LT(I))];
        If Not (J IsIn LL) Then
            Append(LL, J);
            Print([A,2,1]);
        End;
     End;
End;
Print(LL);
```

From this we obtain four vectors: $(3, 2, 1)$, $(4, 2, 1)$, $(6, 2, 1)$ and $(8, 2, 1)$. Now we determine the interval associated to each vector, using the fact that if two different

A return the same initial ideal then so do all intermediate A. For instance, let's see what happens when we increase A from 3 to 5.

```
W := [3,2,1];
S ::= Z/(32003)[abc], Weights(W);
Define TestWeight()
  Using S Do
    L := GenericPoints(11);
    L := Tail(Tail(Tail(Tail(L))));
    I := IdealOfProjectivePoints(L);
    G := ReducedGBasis(S/I);
  End;
  Return(LT(G));
End;

TestWeight();
[S :: ab^2, S :: a^2b, S :: a^3, S :: b^4]
-------------------------------
W := [301,200,100];
S ::= Z/(32003)[abc], Weights(W);
TestWeight();
[S :: a^2c, S :: a^2b, S :: a^3, S :: ab^2c, S :: ab^3, S :: b^5]
-------------------------------
W := [5,2,1];
S ::= Z/(32003)[abc], Weights(W);
TestWeight();
[S :: a^2c, S :: a^2b, S :: a^3, S :: ab^2c, S :: ab^3, S :: b^5]
-------------------------------
W := [501,200,100];
S ::= Z/(32003)[abc], Weights(W);
TestWeight();
[S :: a^2c, S :: a^2b, S :: a^3, S :: ab^2c, S :: ab^3, S :: abc^3,
 S :: b^6]
-------------------------------
```

Note that we had to use [301,200,100] in place of [3.01,2,1] since CoCoA requires integers for weights. The above CoCoA session shows that the initial ideal is constant for $3 < \mathtt{A} \leq 5$, but changes when $\mathtt{A} = 3$ or $\mathtt{A} > 5$. Doing this for the four critical values of A found above, the intervals are

$$2 \leq \mathtt{A} \leq 3, \quad 3 < \mathtt{A} \leq 5, \quad 5 < \mathtt{A} \leq 7, \quad 7 < \mathtt{A}\,.$$

One checks easily that applying the permutations of the variables to the initial ideals for $\mathtt{A} = 3, 5, 7, 8$ doesn't produce any ideal twice (look at what happens to the powers of variables in these ideas). Therefore, there are $6 \cdot 4 = 24$ distinct initial ideals.

Solution B.1.7

Distractions are covered to some extent in the other lectures in this volume, so we'll just give some sketches and references. A finite generating set is obtained from the minimal generators of M by replacing each occurrence of $x_i^{b_i}$ by $x_i(x_i-1)(x_i-2)\cdots(x_i-b_i+1)$. Thus D_M is generated by the *distractions* $D(m)$ of the monomials m generating M, where, for example,

$$D(x^3y^4z) = x(x-1)(x-2)y(y-1)(y-2)(y-3)z\,.$$

Every scalar in k is less than every variable in a term order, so the initial term of a distraction $D(m)$ is m itself. This implies that M is contained in any initial ideal of D_M. If M is artinian then M is the initial ideal of D_M with respect to all term orders because S/M and S/D_M have the same length (k-vector space dimension). In particular, the distraction of the minimal generating set of M is a Gröbner basis if M is artinian. For general M, introduce powers of the variables so high that they cannot interfere with Buchberger's algorithm on the distractions of the minimal generators of M. Since we know for artinian M that the distractions form a Gröbner basis, it must be that Buchberger's criterion applies without the presence of the distractions of the introduced high powers of variables. Therefore, $\{D(m) \mid m \text{ is a minimal generator of } M\}$ is a reduced Gröbner basis for every term order.

The prime decomposition of $D(M)$ is given by the set of *standard pairs* [35], which refine the information in irreducible decomposition. However, our CoCoA routine to calculate the arithmetic degree will be based simply on irreducible decomposition and some counting, without listing the standard pairs.

The ideal `M` used as the test case below is the Permutohedron ideal of Example 6.12, whose irreducible components have varying dimensions. It is also a good idea to test its Alexander dual,

```
Ideal(xyz,x^2y^2,y^2z^2,x^2z^2,x^3,y^3,z^3)},
```

which has only one primary component that splits into lots of irreducible components.

```
M := Ideal(xy^2z^3,x^2y^3z,x^3yz^2,x^2yz^3,x^3y^2z,xy^3z^2);
A:= Log(LCM(Gens(M)));
```

The next function `Pos` is a CoCoA command for supp(L), when applied to vectors $\mathtt{L} \in \mathbb{N}^n$. The command `Artinian` throws in the high coordinate of `A` to the exponent vector `B` whenever `B` has a zero there. We'll need this to compute the irreducible decomposition via Alexander duality, and to reduce everything to finite-dimensional k-vector spaces, later on.

```
Pos(L):= [Min(Concat(X,[1])) | X In List(Transposed(Mat(L)))];

Define Artinian(A,B)
  C := [];
  For N := 1 To Len(A) Do
```

```
    If B[N] = 0 Then Output := A[N]+1
      Else Output := B[N];
    End;
    Append(C,Output);
  End;
  Return(C);
End;
```

The following `AlexDual` command is one of those submitted by Robert Forkel for Solution B.3.3.

```
Define AlexDual(...)
  N:=NumIndets();
  I:=Minimalized(ARGV[1]);
  If Len(ARGV)=2 Then A:=ARGV[2]
  Else A:=Log(LCM(Gens(I)))
  End;
  I:=Mat([[1|K In 1..N]-Log(M)+A|M In Gens(ARGV[1])]);
  If Len(ARGV) > 2 Then Error('Too many arguments in AlexDual') End;
  L:=[];
  For K:=1 To Len(I) Do
  Append(L,[(Indet(M))^I[K,M]|M In 1..N And I[K,M]<=A[M]]);
  End;
  Return IntersectionList([Ideal(M)|M In L])
End;
```

Getting the irreducible decomposition from the Alexander dual is easy.

```
BToTheA(A,B) := A + [1 | X In Indets()] - Artinian(A,B);
Define IrrDecomp(I)
  J := AlexDual(I);
  A := Log(LCM(Gens(J)));
  Return([BToTheA(A,Log(B)) | B In Gens(J)]);
End;

Irr := IrrDecomp(M); Irr;
[[0, 0, 0, 1], [0, 3, 3, 3], [0, 0, 1, 0], [0, 1, 0, 0],
               [0, 0, 2, 2], [0, 2, 0, 2], [0, 2, 2, 0]]
-------------------------------
```

We have now fixed an ordering of the irreducible components. For each Q primary to P on the list we write down (i) the other components on the list whose associated primes are strictly contained in P, plus (ii) those associated to P which appear after Q.

```
Divides(Y,X):= Mod(LogToTerm(Y),LogToTerm(X)) = 0;
                                             --True if X divides Y
```

```
PartOrder(Irr):=
[[Irr[N] | N In 1..Len(Irr) And Divides(Pos([Irr[M]]),Pos([Irr[N]]))
  And (Pos([Irr[M]])<>Pos([Irr[N]]) Or N >= M)] | M In 1..Len(Irr)];

PO := PartOrder(Irr); PO;
[[[0, 0, 0, 1]],
[[0, 0, 0, 1], [0, 3, 3, 3], [0, 0, 1, 0], [0, 1, 0, 0],
               [0, 0, 2, 2], [0, 2, 0, 2], [0, 2, 2, 0]],
[[0, 0, 1, 0]],
[[0, 1, 0, 0]],
[[0, 0, 0, 1], [0, 0, 1, 0], [0, 0, 2, 2]],
[[0, 0, 0, 1], [0, 1, 0, 0], [0, 2, 0, 2]],
[[0, 0, 1, 0], [0, 1, 0, 0], [0, 2, 2, 0]]]
-------------------------------
```

For each irreducible component in the list corresponding to Q, we add artinian generators coming from `A`, except that we ignore the variables not in P.

```
A(A,Q):= Artinian(A,[1|X In Indets()]-Pos([Q]))-[1 | X In Indets()];

Art(A,Ir,PO):=[[Artinian(A(A,Ir[N]),Q)|Q In PO[N]]|N In 1..Len(PO)];
APO := Art(A,Irr,PO);
```

The intersection of these "artinianized" components has finite codimension as a k-vector subspace of the polynomial ring. Omitting the artinianized version of Q itself from the intersection still leaves us with an ideal of finite codimension, less than the codimension with Q included. The difference of these codimensions is the contribution from the irreducible component Q.

```
M(B):= Ideal([Indet(N)^B[N] | N In 1..NumIndets()]);

WithQ(APO):=[IntersectionList([M(B)|B In X]) | X In APO]; WithQ(APO);
[Ideal(t, x, y, z), Ideal(t, z^4, y^4, x^4, xy^2z^3, x^2yz^3, xy^3z^2,
 x^3yz^2, x^2y^3z, x^3y^2z), Ideal(t, x, y, z), Ideal(t, x, y, z),
 Ideal(x, t, yz^2, y^2z, z^4, y^4), Ideal(y, t, xz^2, x^2z, z^4, x^4),
 Ideal(z, t, xy^2, x^2y, y^4, x^4)]
-------------------------------

WithoutQ(A,APO,Irr):=[IntersectionList([M(B) | B In APO[N] And
                      B <> Artinian(A(A,Irr[N]),Irr[N])])
                                         | N In 1..Len(APO)];
WithoutQ:= WithoutQ(A,APO,Irr); WithoutQ;
[[ ], Ideal(t, z^4, y^4, x^4, xy^2z^2, x^2yz^2, x^2y^2z), [ ], [ ],
 Ideal(x, t, yz, z^4, y^4), Ideal(z^4, x^4, y, t, xz),
 Ideal(xy, y^4, x^4, z, t)]
-------------------------------
```

```
Define Mult(I)
  If Type(I) = IDEAL Then Mult := Multiplicity(CurrentRing()/I)
    Else Mult := 0;
  End;
  Return(Mult);
End;

DegWithQ:= [Mult(K) | K In WithQ(APO)]; DegWithQ;
[1, 48, 1, 1, 8, 8, 8]
-------------------------------
DegWithoutQ:= [Mult(K) | K In WithoutQ]; DegWithoutQ;
[0, 44, 0, 0, 7, 7, 7]
-------------------------------
Degrees:= DegWithQ - DegWithoutQ; Degrees;
[1, 4, 1, 1, 1, 1, 1]
-------------------------------
ArithDeg:= Sum(Degrees); ArithDeg;
10
-------------------------------
```

The contribution of a single irreducible component to the arithmetic degree is only well-defined after choosing an ordering of the irreducible components. However, the total contribution from a given *associated* prime is well defined.

B.2 Solutions for Day 2

Solution B.2.1 *By (Bahman Engheta, Leah Gold*, Ed Mosteig), (Kimberly Presser*), (Gerhard Quarg*), and (Carolyn Yackel*)*

The following CoCoA code uses the function `FVector`, defined as in Solution B.1.3. We assume this function has already been entered into CoCoA.

```
Use S ::= Q[x[1..4]y[1..4]z[1..4]];
-- the polarization appears next:
IDelta := Ideal(x[1]x[2]x[3]x[4],y[1]y[2]y[3]y[4],z[1]z[2]z[3]z[4],
x[1]x[2]x[3]y[1]y[2]z[1],x[1]y[1]y[2]y[3]z[1]z[2],
x[1]x[2]y[1]z[1]z[2]z[3]);
FVector(IDelta);
[1, 12, 66, 220, 492, 768, 837, 609, 264, 51]
-------------------------------
```

Thus, Δ has 12 vertices, 66 edges, etc.

Gerhard Quarg found another way to calculate the f-vector. First calculate the Hilbert series.