COCOA Summer School 1999

Eight Lectures on Monomial Ideals

EZRA MILLER DAVID PERKINSON

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Preface

The lectures below are an expanded form of the notes from a course given by Bernd Sturmfels in May, 1999 at the COCOA VI Summer School in Turin, Italy. They are not meant to be a complete overview of the latest research on monomial ideals. Rather, a few representative topics are presented in what we hope is their most accessible form, with lots of examples and pictures. Many proofs are omitted or only sketched, and often we avoid the most general form of a theorem. The reader who peruses these notes should acquire a feel for some of the recent developments in monomial ideals and surrounding areas, and be able to apply the theorems in practical cases, such as those in the exercises of Appendix A. At the very least, these lectures should provide a guide to the references, for those who wish to explore this exciting and branching field for themselves.

The lectures can be broken down roughly as follows. After the Basics (definitions and notation), Lectures I and II give what can be considered as a mostly historical overview. The standard classes of monomial ideals are introduced, including squarefree ("Stanley-Reisner") and Borel-fixed ideals, along with the ways of getting structural and numerical information about their resolutions and Hilbert series. Lecture III accomplishes the transition to recent advances by emphasizing the geometric ways of thinking about monomial ideals and their resolutions which have been so influential. Each of Lectures IV through VIII then treats in greater detail some recent research topic.

Lecture IV introduces a new paradigm for monomial ideals, contrasting with those of Lectures I and II. This is a notion of *genericity* which rests on randomness of exponents on monomials rather than of coefficients in polynomials. It is shown how the infinite poset of monomials in the ideal can be whittled down in this case to just a few absolutely necessary elements, and how this reduction reflects the geometry and combinatorics introduced in Lecture III. In particular, geometric free resolutions of generic monomial ideals (*algebraic Scarf complexes*) are constructed using just the combinatorial data. By deforming exponents on generators, any monomial ideal is approximated by a generic one, and bounds for Betti numbers of ideals are thus attained.

Lecture V abstracts the construction of Scarf complexes to more general *cellular resolutions*. In particular, the geometry of the Scarf complex is elucidated as a special case of the *hull resolution* defined by convexity. This lecture shows how combinatorial topology interacts with monomial ideals via cellular complexes which are not necessarily simplicial.

Lecture VI then covers Alexander duality, illustrating how this topological duality is manifested in monomial ideals and their homological algebra. First the classical form for squarefree monomial ideals is reviewed, along with some recent consequences for resolutions of these. Then, Alexander duality is defined for arbitrary monomial ideals. The interactions with free resolutions are explored, including what happens to the numerical information in general, and in particular how the geometry of cellular resolutions demonstrates the duality. Cogeneric monomial ideals, which are generic with respect to their irreducible components rather than their generators, are presented as running examples.

Lecture VII connects monomial and binomial ideals via monomial modules. Certain kinds of binomial ideals—the *lattice ideals*, which include toric ideals—are viewed as coming from "infinite periodic" versions of monomial ideals. The cellular methods of Lecture V apply here as well, with the cell complexes having infinitely many cells but finite dimension. Taking the quotient by periodicity, the cell complex becomes a torus, and resolves the lattice ideal. Examples of the theory are provided by the classes of *unimodular Lawrence ideals* and *generic lattice ideals*.

Finally, Lecture VIII relates recent advances in the study of local cohomology for monomial ideals to the foundational theorem of Hochster on the subject. The standard notion of local cohomology with supports is reviewed, and calculations of Hilbert series as well as module structure are given. Using the *Čech hull*, it is then shown how these calculations are equivalent to the classical ones.

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