Combinatorics, geometry, algebra, and applications

Introduction

My research centers around combinatorial, computational, and cohomological problems originating in geometry and algebra, with ties to computer science. Currently, my participation at the Institute for Mathematics and its Applications (IMA) year on Mathematics of Molecular and Cellular Biology is stimulating potential interactions with geometric methods in biology and statistics (not to be confused with algebraic statistics and the related questions in biology, which were topics of last year's IMA program).

The unifying idea in my research has been to isolate or exploit combinatorial structures that govern or arise from continuous contexts. For example, if a continuous process carries underlying discrete data, then those data might be harnessed to produce algorithms for the continuous process. On the other hand, the goal could also be to understand the combinatorics rather than the geometry; the geometry then serves as a vehicle for interpolating between different interpretations of the combinatorics.

The aim of this exposition is to give a broad perspective on more specific areas in which I have worked and am currently interested. It is broken into four sections:

These sections do not cover my expository works. These include two graduate textbooks written: one recently completed with a number of coauthors on the muti-faceted topic of local cohomology [1], and another with Bernd Sturmfels summarizing a decade's worth of developments in combinatorial commutative algebra [4]. With Sturmfels and Vic Reiner, I edited the volume [2] of graduate courses from the PCMI Summer School on Geometric Combinatorics, which the three of us also organized; Vic Reiner and I wrote the overview [3]. In connection with the MSRI year on Commutative Algebra, I wrote an article on Hilbert schemes of points in the plane [5], as part of my duties as a TA for Mark Haiman's short course. Subsequent to a COCOA (Computation in Commutative Algebra) conference, I wrote a long article on monomial ideals with David Perkinson [6]. Finally, in connection with a lecture series I gave at CRM Montréal, I am in the process of writing an article with Laura Matusevich as well as two of the students there, Huilan Li and Craig Sloss, on connections between binomial ideals and hypergeometric systems.

1 Metric polyhedral geometry

This project is currently undergoing rapid development, with two papers in progress [7, 8]. The stage for this research is set by joint work of mine with Igor Pak [16]. Motivated by relations to a number of classical algorithmic problems in discrete and computational geometry, our fundamental observation there is that convexity and polyhedrality together impose rich combinatorial structures on the collection of shortest paths in a metrized sphere. The methods of metric combinatorics constitute a blend of differential geometry, classical polyhedral geometry, poset combinatorics, geometric topology, and complexity theory.

The first main result of [16] says that the boundary S of any $(d+1)$-polytope has a certain polyhedral nonoverlapping unfolding into $\mathbb{R}^d$. More precisely, the exponential map

$T_v \to S$ from the tangent space at any generic point $v \in S$ is surjective, and the set of tangent vectors exponentiating to shortest paths (length-minimizing geodesics) is a closed, star-shaped, polyhedral set in T_v, called the *source foldout*. Equivalently, the cut locus $\overline{K}_v \subseteq S$ for the source point v is a polyhedral complex of pure dimension $d-1$ (though not a subcomplex of S), and slicing S open along $\overline{K}_v$ results in a polyhedral ball that can be laid flat in $\mathbb{R}^d$ without overlap. This construction had been known in dimension $d = 2$ [VP71, SS86], but there seems to have been no known unfolding at all for $d \geq 3$.

The second main result of [16] is an effective algorithm for constructing the source foldout. This involves imposing—and subsequently navigating—certain order-theoretic combinatorial structures on the set of shortest paths, using iterated tangent data from an expanding wavefront. Currently, I am advising an undergradute (UROP) student, Nate Born, in a project to implement our algorithm; see Section 4.1 for the importance of doing this. The main mathematical open question from [16] is the complexity of the algorithm: although Pak and I were able to prove that it is polynomial in the size of the output, we could only conjecture that the output is polynomial in the number of facets of S. (It is exponential in the dimension d, of course.) I wish to stress that this is a natural, purely geometric question, and does not depend on the nature of our algorithm: given a point v on a polyhedral manifold S and a facet F of S, how many shortest paths start at v and end in F? The answer can be exponential in the number of facets if S is allowed to have negative curvature, even if S is a polyhedral 2-sphere, but nonetheless the number of shortest paths from v to F should be polynomial in the number of facets for convex polyhedral spheres.

Before describing my approach to the metric complexity of polyhedral spheres, it will help to review one more observation from [16]. In dimension $d = 2$, there is a second *Alexandrov unfolding* [Ale48] of a polyhedral sphere S, in a sense dual to the source foldout: fix any source point $v \in S$, and slice S open along the shortest paths from v to the vertices of S. That the resulting locally flat disk can be globally laid flat in $\mathbb{R}^2$ without overlap was proved by Aronov and O'Rourke [AO92]. However, the construction itself fails in dimensions $d \geq 3$: the union of the shortest paths from a fixed source point v to the $(d-2)$-skeleton is not contractible. Roughly speaking, the angles of the shortest paths to (say) an edge E make a quantum leap as they swing past any $(d-2)$-face between v and E.

The first part [7] of my current work aims to demonstrate that, nonetheless, Alexandrov unfoldings generalize naturally to dimensions $d \geq 3$. The idea is to consider the *gradient vector field* on S, which points in the direction of steepest distance increase from v. That there even exists a unique such vector at each point of S is already nontrivial, and fails immediately for any polyhedral manifold exhibiting negative curvature of any sort. More surprising still is the observation that this distinctly discontinuous vector field should determine a continuous flow on S away from v. For a generic source point v, the *Alexandrov slice set* is the union of the flow lines into the $(d-2)$-skeleton of S; in dimension $d = 2$, this is precisely the Alexandrov slice set from before, but in higher dimensions, it fills in the gaps left after joining v to the $(d-2)$-skeleton by shortest paths. In general, this slice set is contractible because the (continuous!) gradient flow pushes its complement onto a retract of the cut locus $\overline{K}_v$. Which retract? The one obtained by gradient flow on $\overline{K}_v$ itself.

Aside from generalizing a classical $d = 2$ construction to arbitrary dimension, why are Alexandrov unfoldings by gradient flow important? Simply put, the combinatorics of the Alexandrov slice set controls the metric complexity of the cut locus. Making this statement precise is the goal of the second half [8] of my current work. The intuition is as follows. Imagine that S is your universe, and you are standing at the source point v. As you look out, your horizon is studded with pieces of $(d-2)$-faces, where there is nontrivial curvature. These pieces don't subdivide your horizon into regions (this is why the union of shortest paths isn't a slice set); however, the union of the gradient flow lines into the boundary of each facet should.

In dimension $d = 3$, the picture is just small enough to be describable in everyday terms: your horizon is a 2-sphere, and the flow lines to the boundary of a single facet of S should pierce the sphere in a polygon. Taking all of the facets at once, you see a constellation of polygons. These subdivide your horizon into regions, and we need the number of these regions to be polynomial in the number of facets. The underlying structure, which I am developing for [8], is a certain generalization of oriented matroids, the motivation being that the number of regions in a hyperplane arrangement is polynomial in the number of hyperplanes (but exponential in the dimension d, of course).

An oriented matroid can be represented as a collection of pseudospheres, each of which divides the horizon into two pieces. Here, instead, we have a collection of *hedroids*, the flow-projections of the edges (in $d = 3$; for higher d, these will be faces of dimension $d-2$), each of which only locally divides the horizon into two pieces. The *polyhedroid* structure, which generalizes the notion of oriented matroid, amounts to control over how these hedroids are allowed to meet. In a matroid, the nonempty intersections of pseudospheres look locally like oriented matroids of smaller dimension; in a polyhedroid, these intersections are required to look like arbitrary polyhedral fans (or perhaps like general polyhedroids) of smaller dimension. The goal of [8], then, is twofold: first, develop a theory of polyhedroids enough to conclude polynomiality for the number of regions; and second, prove that the Alexandrov slice set is a polyhedroid. This latter part, if true, would encode fundamental geometric statements about the collections of shortest paths in a convex polyhedral sphere.

Where, in general, do I see the program of flows on polyhedral spaces going? In geometric topology, *entropy* is a measure of the exponential geodesic complexity of a manifold [Man79]. In that context, however, entropy is always zero for positively curved manifolds. Here, in contrast, the facial structure provides us with growth rate measures finer than the topological ones, allowing us to make meaningful statements about subexponential numbers of combinatorial types of geodesics. I wonder what can be said about metric complexity for geodesics that aren't shortest paths, or on general polyhedral manifolds; for example, is there anything between polynomial and exponential growth? Still in the topological realm, a number of geometrically defined flows in polyhedral geometry could potentially be viewed as flows on higher-dimensional polyhedral spaces; for example, consider the combinatorial Yamabe [Gli05], Ricci [CL03], or scalar curvature flows on polyhedral 3-manifolds (the latter having been used by Bobenko and Izmestiev [BI06] to give an effective algorithmic version of Alexandrov's embedding theorem for convex polyhedral 2-spheres). In another direction, it was noticed already by Volkov [Vol68] (or see [Ale50, Section 12.1] for a translation) that the intrinsic metric on a convex polyhedral sphere S can be intimately tied to the extrinsic geometry of its convex hull in dimension $d = 2$. It would be very interesting to see what can be said in higher dimensions. In any case, I find our limited understanding of the metric geometry of convex polyhedra in general to be an impediment to progress on the most basic of questions, even in $d = 2$: do polyhedra admit nonoverlapping foldouts by slicing along ridges—that is, faces of dimension $d-1$?

2 Combinatorial positivity in algebraic geometry

One of the major ways in which combinatorics interacts with algebraic geometry is through questions of positivity. In my work [22] with Knutson, we provide a proof by geometric degeneration that the coefficients of Schubert polynomials are positive. These polynomials represent the classes Poincaré dual to Schubert varieties in the cohomology ring of the flag variety. Their coefficients had been known since their inception [LS82] to be positive, but the connection of these particular polynomial representatives to geometry had been more tenuous. In contrast, the polynomials we prove geometrically to be positive in joint work additionally with Shimozono [17] were not known to be so—by combinatorics or by any other means. In particular, one of our four formulae proves the essence of the main Conjecture of Buch and Fulton in [BF99].

The first part of my work [13] with Knutson and Yong brings the combinatorics of decompositions of simplicial complexes to bear on the algebraic geometry of degenerations. In particular, we introduce *geometric vertex decomposition*, a procedure to squeeze algebraic varieties onto coordinate subspaces one by one. The rest is an embellishment on [22]: we use our new technology to show that degenerating by an opposite torus weight—equivalently, using a diagonal term order instead of an antidiagonal one—yields different combinatorial formulae for Schubert polynomials when the permutation is grassmannian. In another offshoot of [22], Kogan and I show that under the Gonciulea–Lakshmibai degeneration [GL96] of the flag variety to a toric variety, the Schubert varieties degenerate to reduced unions of toric faces [24] combinatorially identifiable as summands in Schubert polynomials.

In K-theory, as opposed to cohomology, positivity really means sign-alternation: for some fixed degree c, all of the coefficients in degree $c+i$ have sign $(-1)^i$. Knutson and I observed in 2000, while working on [22], that this phenomenon for K-classes of Schubert varieties arises from Cohen–Macaulay initial ideals by Alexander duality. A different sign alternation was conjectured by Buch [Buc02] for multiplication in the ordinary K-theory of flag varieties with its basis of Schubert structure sheaves, and it was proved by Brion [Bri02]. Brion's proof relied on a certain transversality theorem for K-theory on homogeneous spaces. A number of other results in Schubert calculus depend on related (but not equal) versions of this transversality. To avoid endlessly proving minor extensions of the known such Kleiman–Bertini type results, Speyer and I proved a rather general version of vanishing for sheaf Tor on homogeneous spaces [14], which holds even in positive characteristic. As general as Brion's sign alternation theorem is, it does not cover the alternation in Buch's conjecture [Buc02a] for the expansion of quiver K-polynomials in terms of Grothendieck polynomials for partitions. This required separate proofs, which Buch [Buc03] and I [21] independently produced.

In a current project with Stephen Griffeth, a postdoc I am mentoring at Minnesota (and former student at Wisconsin under Arun Ram), we are thinking about sign alternation in *equivariant* K-theory of homogeneous spaces. We intend to apply ideas from equivariant Chow theory combined with Brion's sign alternation theorem in ordinary K-theory. If this works, it will generalize Graham's positivity [Gra01] in the case of equivariant cohomology for flag varieties.

Sometimes algebraic geometry like that above can give rise to purely combinatorial problems—or solutions. In the course of our study of Schubert polynomials, Knuston and I found a positive combinatorial rule for generating Schubert polynomials, and I subsequently used it to give a quick positive construction of Schubert polynomials from scratch [28]. A student, Ning Jia, whom I mentored at Minnesota, gave an elegant simplification [11] of our proof [22] of duality between certain combinatorial objects associated to Schubert polynomials (my name is on [11] for a small contribution only because she insisted). Following up on our proof of the Cohen–Macaulay condition for certain simplicial complexes in [22], Knutson and I discovered a general topological way to view the exchange axiom in an arbitrary Coxeter group [26]: given a fixed word, the set of subwords with a given product forms a ball or sphere. These *subword complexes* include, as special cases, certain simplicial complexes whose facets are the semistandard Young tableaux of a given shape and whose interior faces are the semistandard set-valued tableaux [Buc02]. This suggested a different, purely combinatorial (poset-theoretic) construction of a family of nonisomorphic simplicial complexes with the same properties [15].

3 Commutative and homological algebra

Most of my ealiest work, starting in graduate school, concerned combinatorial methods in commutative and homological algebra. Many themes along these lines persist in my research to this day. To make the descriptions easier, I have (more or less arbitrarily) split the exposition into three subsections.

3.1 Multivariate hypergeometric systems

Hypergeometric mathematics, of the type arising in my research, combines algebraic geometry, lattice point combinatorics, binomial commutative algebra, homological and noncommutative algebra of differential operators, and complex analysis.

Hypergeometric series are power series whose adjacent coefficients are related by rational functions. In 1889, J. Horn introduced bivariate versions of the classical univariate hypergeometric series [Hor1889]. In the 1940s, Erdélyi discovered sporadic solutions to Horn's hypergeometric systems [Erd50]. In work with Alicia Dickenstein and Laura Matusevich [10], we explain these extra solutions as consequences of binomial primary decomposition, which we describe combinatorially in terms of lattice points. The specific form of our primary decompositions, which refine those of Eisenbud and Sturmfels [ES96], allow us to reduce—by filtration methods—to the toric hypergeometric techniques developed by Gelfand, Graev, Kapranov, Zelevinsky, and others, particularly Adolphson, in the 1980's and 1990's [GGZ87, GKZ89, Ado94].

Our work on Horn systems relied in an absolutely essential way on the homological *Euler–Koszul* technology introduced with Matusevich and Walther [20]. The main purpose of [20] was to prove a conjecture by Sturmfels on the equivalence of the Cohen–Macaulay condition with the lack of rank-jumping parameters. We accomplished this using sheaf-theoretic techniques on the space of parameters. We identified the rank-jump locus as an algebraic variety, defined via sheaf cohomology of the associated toric variety, whose nonemptiness witnesses the failure of the Cohen–Macaulay condition. In the case where the toric variety is simplicial, Matusevich and I had proved this connection between local cohomology and rank jumps by a direct polyhedral argument [18].

Currently, I am guiding one of my graduate students, Robert Edman, in his chosen project to understand Kapranov's analogues of hypergeometric systems for reductive groups (instead of tori). This research involves the same list as ordinary hypergeometric mathematics, except that lattice point combinatorics (a.k.a. representation theory of a torus) is replaced with deeper topics in representation theory for reductive groups.

3.2 Homological combinatorics: free and injective resolutions

Much of the work in my dissertation [31] concerned combinatorial aspects of homological algebra in situations that are *finely graded*: the ambient ring is graded in such a way that its components have vector space dimension 1. Over polynomial rings, I made Alexander duality, an a priori combinatorial topological operation, into a functor [32] that interchanges free and injective resolutions. I realized later [40] that my functorial Alexander duality for resolutions, when considered in the derived category, is a special case of finely graded Greenlees–May duality [GM92].

Duality for resolutions allowed me to homologically interpret the Eagon–Reiner duality [ER98] between the Cohen–Macaulay and linear resolution conditions for squarefree monomial ideals, and thereby to generalize it to affine semigroup rings [29]. The appearance of injective resolutions—and their finely graded truncations, the *irreducible resolutions*, which I introduced—indicated their utility in computations; hence David Helm and I produced an algorithm to calculate them [23]. Part of this algorithm relies on a bound from [27], obtained in an abstract homological setting, on how the finely graded shifts of the indecomposable summands in an injective resolution vary from prime to prime. The purpose of [27], however, was to prove the finiteness of Bass numbers for local cohomology over simplicial semigroup rings, and the failure in the nonsimplicial case. This provided a general toric framework for Hartshorne's counterexample [Har70] to Grothendieck's conjecture on this finiteness issue.

My most recent research on resolutions is joint work [12] with a bright graduate student, Shin-Yao Jow, of Rob Lazarsfeld's at Michigan. Shin-Yao's idea was to provide a geometric, sheaf-theoretic (as opposed to characteristic p algebraic) proof of a formula for the multiplier ideals of a sum of ideal sheaves. We reduced the problem to an analytically local one on a resolution of singularities, where it became a mixture of combinatorial commutative algebra, particularly cellular resolutions of monomial ideals, and simplicial topology, particularly homology-manifolds-with-boundary.

3.3 Monomial ideals and related combinatorics

Special classes of monomial ideals give rise to particularly beautiful and useful combinatorics. This is the case, for example, with *generic* monomial ideals ([BPS98] and [33]), whose exponent vectors are, in effect, random. These ideals are characterized in numerous ways [33], particularly by the invariance of their minimal free and injective resolutions (which are explicit convex-geometric simplicial complexes) under deformations of the exponents of the generators. Similarly, three-dimensional monomial ideals give rise to pretty staircase pictures, in which the minimal resolutions can be drawn explicitly as planar graphs [30, 42].

Generalizations of monomial ideals can also carry simplicial or polyhedral combinatorics. This well-known phenomenon is illustrated by my joint work with Reiner [19, 25]. The first of those concerns ideals of simplicial posets, which are certain flexible weakenings of simplicial complexes; we simplify Masuda's proof [Mas03] of Stanley's conjecture [Sta91] concerning the h-vectors of these objects. The second concerns certain radical monomial ideals in semigroup rings; we generalize and place in a (local co)homological framework one of Stanley's reciprocity theorems [Sta74].

I have recently begun working with Raman Sanyal, a graduate student under Günter Ziegler at Technische Universität Berlin, on "combinatorial moduli spaces" for resolutions of monomial ideals. The basic idea is to classify commutative polynomial ideals generated by n monomials in d variables according to their minimal free or injective resolutions. In slightly more detail, such an ideal is determined by choosing a $d \times n$ matrix of nonnegative integers, the columns being the exponent vectors of the monomial generators. However, the homological properties of the ideal depend only on order-theoretic combinatorial data extracted from the integer matrix, such as the permutation of the columns required to make a given row weakly increasing. The goal is to describe the subdivision of the space of matrices into the equivalence classes consisting of the monomial ideals with identical homological properties. For example, the generic monomial ideals ([BPS98] and [33]) constitute the maximal regions in the subdivision. The eventual goal would be to attack the (apparently much harder) analogue for lattice ideals; see [PS98], where the generic ideals are identified. For the time being, we are starting with the case $d = 3$, where planar graphs help immensely [30].

4 Interdisciplinary projects

A good deal of my previous work has overt connections to computer science; see Section 4.1. More recently, a triad of independent and nearly simultaneous circumstances have placed me in an ideal position to foster immediate collaborations with scientists in disparate areas of mathematics as well as theoretical and more traditional experimental sciences. The triad of events consists of (i) Minnesota's recent hire of Gilad Lerman, an applied mathematician, with whom I have been having tantalizing conversations; (ii) my introduction to R. Dennis Cook, an established statistician at Minnesota specializing in diagnostics (finding anomalies in data); and (iii) the current year-long program on Mathematics of Molecular and Cellular Biology at the Institute for Mathematics and its Applications (IMA). It is becoming increasingly clear to me that my work in geometry—polyhedral as well as algebraic—has great potential for these collaborations.

The projects in this section are at varying developmental stages; because of the timing, many have not yet coalesced into concrete lines of investigation.

4.1 Computer science

A number of times in the past, I have found myself working in areas directly influeced by explicit algorithms for various kinds of computations: in algebra (of resolutions [39, 42] and of hypergeometric series [36]), in combinatorics (of planar graphs [30, 42]; see also the works of Felsner [Fel01, Fel03]), and in geometry (of polyhedra [16]). In particular, my work [23, 39] with Helm was the first algorithm for computing local cohomology over any class of singular rings, and my work with Pak was the first algorithm to unfold convex polyhedra in any dimension $d \geq 3$. It is the latter where I see important advances in the near future. Looking back at Section 1, once the Alexandrov unfolding is defined for dimensions $d \geq 3$, it will be important to have an algorithm for producing it. Indeed, it is natural to conjecture that the unfolding will be nonoverlapping, as it is for $d = 2$ [AO92], and computers are the best way to produce copious evidence (or a counterexample). This brings me back to the undergraduate research project I am conducting with Nate Born: if I understand things correctly, then it should be quite easy to rearrange pieces of the source foldout, which Nate is computing, to get the Alexandrov unfolding. This will truly be a computational geometry experiment, and I am hoping to get people like O'Rourke and Joe Mitchell [MMP87] interested in it.

4.2 Statistics

At a meeting to discuss future programs for the IMA, I began conversations with R. Dennis Cook, a statistician here at Minnesota, on fitting data with nonlinear geometric structures. These structure could be real algebraic varieties, or polyhedral complexes, or something else—we'll have to see what's possible abstractly, what's desirable, and what's computationally feasible. (But first, we'll have to teach each other our respective subjects!) In contrast with the field that has come to be known as "algebraic statistics", in which algebra or algebraic geometry describes statistical models (allowed sets of probabilities for a fixed set of random variables), Cook and I are talking about the completely different issue of methods to make geometric sense of data sets.

4.3 Geometric measure theory

In conversations with Gilad Lerman and his student, J. Tyler Whitehouse, we have reduced certain questions in geometric measure theory to various kinds of high-dimensional metric polyhedral geometry of the sort that I have been thinking about recently. A typical problem in this area is how to place a hyperplane so as to make it pass within ε of a given set of points. Gilad and I have plans to pursue further conversations to mine each other's expertise.

4.4 Mathematical biology

During high-school, I lived in the vicinity of Bethesda, MD, and I was lucky enough to spend four summers working at the National Institutes of Health [NIH] there, working in laboratories on questions in biochemistry and molecular biology. My career goals turned later to mathematics, but I have retained and nurtured my interest in the biological sciences. For some time, now, I have been looking for an opportunity to integrate these interests of mine, and this year's program at the IMA is it. It is still early in the year, but already I am seeing connections to my work on metric geometry. For example, flows on polyhedral spaces and differential inclusions are useful in modeling self-regulating network models; see [CdG06] for an instance of this.

References

Publications and preprints by Ezra Miller (numbered as in CV)

Books and expository articles

2007 [1] (book with Srikanth Iyengar, Graham Leuschke, Anton Leykin, Claudia Miller, Anurag Singh, and Uli Walther) *Twenty-four hours of local cohomology*, Graduate Studies in Mathematics, American Mathematical Society, Providence, RI. (xvi+282 pages)

[2] (book edited with Vic Reiner and Bernd Sturmfels) *Geometric Combinatorics.* Lecture notes from the Graduate Summer School of the Institute for Advanced Study/Park City Mathematics Institute held in Park City, UT, July 11–31, 2004. IAS/Park City Math. Series, American Math. Society, Providence, RI; Institute for Advanced Study (IAS), Princeton, NJ. (xvi+691 pages)

[3] (with Vic Reiner) *What is geometric combinatorics?* In Geometric combinatorics (Park City, UT, 2004), IAS/Park City Math. Series, American Math. Society, Providence, RI, pp. 1–17.

2004 [4] (book with Bernd Sturmfels) *Combinatorial Commutative Algebra*, Graduate Texts in Mathematics Vol. 227, Springer–Verlag, New York. (xiv+417 pages)

[5] *Hilbert schemes of points in the plane*, Appendix to *Commutative algebra of N points in the plane*, by Mark Haiman, in Luchezar Avramov et al., (eds.), *Trends in Commutative Algebra*, MSRI Publications Vol. 51, Cambridge University Press, New York, pp. 153–180.

2001 [6] (with David Perkinson) *Eight lectures on monomial ideals*, in Queen's Papers in Pure and Applied Mathematics, no. 120, 3–105.

Articles in progress

[7] *Metric combinatorics of convex polyhedra, II: gradient flow and Alexandrov unfolding.*

[8] *Metric complexity of convex polyhedra.*

[9] (with Huilan Li, Laura Matusevich, and Craig Sloss) *Multivariate hypergeometric functions and binomial ideals*.

Submitted journal articles

[10] (with Alicia Dickenstein and Laura Matusevich) *Binomial D-modules*, 47 pages. arXiv:math.AG/0610353

[11] (with Ning Jia) *Duality of antidiagonals and pipe dreams*, 5 pages. arXiv:math.CO/0706.3031

Peer-reviewed journal articles

to appear [12] (with Shin-Yao Jow) *Multiplier ideals of sums via cellular resolutions*, Mathematical Research Letters, 15 pages. arXiv:math.AG/ 0703299

[13] (with Allen Knutson and Alex Yong) *Gröbner geometry of vertex decompositions and of flagged tableaux*, Journal für die reine und angewandte Mathematik, 23 pages. arXiv:math.CO/0502144

[14] (with David Speyer) *A Kleiman–Bertini theorem for sheaf tensor products*, Journal of Algebraic Geometry, 5 pages. arXiv:math.AG/0601202

[15] (with Allen Knutson and Alex Yong) *Tableau complexes*, Israel Journal of Math., 18 pages. arXiv:math.CO/0510487

2006 [16] (with Igor Pak) *Metric combinatorics of convex polyhedra: cut loci and nonoverlapping unfoldings*, Discrete and Computational Geometry (electronic), DOI: 10.1007/s00454-006-1249-0, pages OF1-OF50. arXiv:math.MG/0312253

[17] (with Allen Knutson and Mark Shimozono) *Four positive formulae for type A quiver polynomials*, Inventiones Mathematicae **166** no. 2, 229–325. arXiv:math.AG/0308142

[18] (with Laura Matusevich) *Combinatorics of rank jumps in simplicial hypergeometric systems*, Proceedings of the American Mathematical Society **134**, 1375–1381. arXiv:math.AC/0402071

[19] (with Vic Reiner) *Stanley's simplicial poset conjecture, after M. Masuda*, Communications in Algebra **34** (2006), no. 3, 1049–1053.

2005 [20] (with Laura Matusevich and Uli Walther) *Homological methods for hypergeometric families*, Journal of the American Mathematical Society **18**, no. 4, 919–941. arXiv:math.AG/0406383

[21] *Alternating formulas for K-theoretic quiver polynomials*, Duke Mathematical Journal **128**, 1–17. arXiv:math.CO/0312250

[22] (with Allen Knutson) *Gröbner geometry of Schubert polynomials*, Annals of Mathematics **161**, 1245–1318. arXiv:math.AG/0110058

[23] (with David Helm) *Algorithms for graded injective resolutions and local cohomology over semigroup rings*, Journal of Symbolic Computation **39**, 373–395. arXiv:math.AC/0309256

[24] (with Mikhail Kogan) *Toric degeneration of Schubert varieties and Gelfand–Tsetlin polytopes*, Advances in Mathematics **193**, no. 1, 1–17. arXiv:math.AG/0303208

[25] (with Vic Reiner) *Reciprocal domains and Cohen–Macaulay d-complexes in* $\mathbb{R}^d$, The Electronic Journal of Combinatorics **11(2)**, #N1 (9 pages). arXiv:math.CO/0408169

2004 [26] (with Allen Knutson) *Subword complexes in Coxeter groups*, Advances in Mathematics **184**, 161–176. arXiv:math.CO/0309259

2003 [27] (with David Helm) *Bass numbers of semigroup-graded local cohomology*, Pacific Journal of Mathematics **209**, no. 1, 41–66. arXiv:math.AG/0010003

[28] *Mitosis recursion for coefficients of Schubert polynomials*, Journal of Combinatorial Theory, Series A **103**, 223–235. arXiv:math.CO/0212131

2002 [29] *Cohen–Macaulay quotients of normal semigroup rings via irreducible resolutions*, Mathematical Research Letters **9**, no. 1, 117–128. arXiv:math.AC/0110096

[30] *Planar graphs as minimal resolutions of trivariate monomial ideals*, Documenta Mathematica **7**, 43–90. (electronically published: http://www.math.uiuc.edu/documenta/vol-07/03.html)

2000 [31] *Resolutions and duality for monomial ideals*, Ph.D. thesis, University of California at Berkeley.

[32] *The Alexander duality functors and local duality with monomial support*, Journal of Algebra **231**, 180–234.

[33] (with Bernd Sturmfels and Kohji Yanagawa) *Generic and cogeneric monomial ideals*, Journal of Symbolic Computation **29**, 691–708.

[34] *Icosahedra constructed from congruent triangles*, Discrete and Computational Geometry **24**, no. 2–3, 437–451.

1998 [35] *Multiplicities of ideals in noetherian rings.* Beiträge zur Algebra und Geometrie **39**(1) 47–51.

Conference publications (peer-reviewed and/or invited)

2007 [36] (with Alicia Dickenstein and Laura Matusevich) *Extended abstract: Binomial D-modules*, Proceedings MEGA (Effective Methods in Algebraic Geometry), Strobl, Austria, 2007, 13 pages.

[37] (with Shin-Yao Jow) *Extended abstract: Cellular resolutions of multiplier ideals of sums*, in *Topological and geometric combinatorics*, abstracts from the workshop held January 28–February 3, 2007, organized by Anders Björner, Gil Kalai, and Günter Ziegler, Oberwolfach reports, 3 pages.

2006 [38] (with Laura Matusevich and Uli Walther) *Extended abstract: Homological methods for hypergeometric families*, in *Convex and algebraic geometry*, abstracts from the workshop held January 29–February 4, 2006, organized by Klaus Altmann, Victor Batyrev, and Bernard Teissier, Oberwolfach reports, 3 pages.

2003 [39] (with David Helm) *Extended abstract: Algorithms for graded injective resolutions and local cohomology over semigroup rings*, Proceedings MEGA (Effective Methods in Algebraic Geometry), Kaiserslautern, Germany, 2003, 5 pages.

2002 [40] *Graded Greenlees–May duality and the Čech hull*, Local cohomology and its applications (Guanajuato, 1999), Lecture Notes in Pure and Appl. Math., vol. 226, Dekker, New York, 233–253.

[41] (with Allen Knutson) *Extended abstract: Gröbner geometry of Schubert polynomials*, Proceedings FPSAC (Formal Power Series and Algebraic Combinatorics), Melbourne 2002, 10 pages.

1999 [42] (with Bernd Sturmfels) *Monomial ideals and planar graphs*, in Applied Algebra, Algebraic Algorithms and Error-Correcting Codes, [M. Fossorier, H. Imai, S. Lin and A. Poli, eds.], Proceedings of AAECC-13 (Honolulu, Nov. 1999), *Springer Lecture Notes in Computer Science* **1719**, 19–28.

Cited publications by other authors

[Ado94] Alan Adolphson, *Hypergeometric functions and rings generated by monomials*, Duke Math. J. **73** (1994), no. 2, 269–290.

[Ale48] A. D. Alexandrov, *Vnutrennyaya geometriya vypuklykh poverkhnostey* (in Russian), M.–L.: Gostekhizdat, 1948; English translation: *Selected works. Intrinsic geometry of convex surfaces*, Vol. 2, Chapman & Hall/CRC, Boca Raton, FL, 2005.

[Ale50] A. D. Alexandrov, *Convex polyhedra*, Springer Monographs in Mathematics. Springer–Verlag, Berlin, 2005. Translated from the 1950 Russian edition by N. S. Dairbekov, S. S. Kutateladze and A. B. Sossinsky, with comments and bibliography by V. A. Zalgaller and appendices by L. A. Shor and Yu. A. Volkov.

[AO92] Boris Aronov and Joseph O'Rourke, *Nonoverlap of the star unfolding*, Discrete Comput. Geom. **8** (1992), no. 3, 219–250.

[BPS98] Dave Bayer, Irena Peeva, and Bernd Sturmfels, *Monomial resolutions*, Math. Res. Lett. **5** (1998), no. 1-2, 31–46.

[BI06] Alexander I. Bobenko and Ivan Izmestiev, *Alexandrov's theorem, weighted Delaunay triangulations, and mixed volumes*, preprint. arXiv:math.DG/0609447

[Bri02] M. Brion, *Positivity in the Grothendieck group of complex flag varieties*, Journal of Algebra **258** (2002), no. 1, 137–159.

[Buc02] A. S. Buch, *A Littlewood-Richardson rule for the K-theory ring of Grassmannians,* Acta Math., **189** (2002), no. 1, 137–159. math.AG/0004137

[Buc02a] Anders S. Buch, *Grothendieck classes of quiver varieties*, Duke Math. J. **115** (2002), no. 1, 75–103.

[Buc03] Anders S. Buch, *Alternating signs of quiver coefficients*, J. Amer. Math. Soc. **18** (2005), no. 1, 217–237 (electronic).

[BF99] Anders Skovsted Buch and William Fulton, *Chern class formulas for quiver varieties*, Invent. Math. **135** (1999), no. 3, 665–687.

[CdG06] Richard Casey, Hidde de Jong, and Jean-Luc Gouzé, *Piecewise-linear models of genetic regulatory networks: equilibria and their stability*, J. Math. Biol. **52** (2006), no. 1, 27–56.

[CL03] B. Chow and F. Luo, *Combinatorial Ricci flows on surfaces*, J. Differential Geom. **63** (2003), 97–129.

[ER98] John A. Eagon and Victor Reiner, *Resolutions of Stanley–Reisner rings and Alexander duality*, J. Pure Appl. Algebra **130** (1998), no. 3, 265–275.

[ES96] David Eisenbud and Bernd Sturmfels, *Binomial ideals*, Duke Math. J. **84** (1996), no. 1, 1–45.

[Erd50] Arthur Erdélyi, *Hypergeometric functions of two variables*, Acta Math. **83** (1950), 131–164.

[Fel01] Stefan Felsner, *Convex drawings of planar graphs and the order dimension of 3-polytopes*, Order **18** (2001), no. 1, 19–37.

[Fel03] Stefan Felsner, *Geodesic embeddings and planar graphs*, Order **20** (2003), no. 2, 135–150.

[GGZ87] I. M. Gel′fand, M. I. Graev, and A. V. Zelevinskiĭ, *Holonomic systems of equations and series of hypergeometric type*, Dokl. Akad. Nauk SSSR **295** (1987), no. 1, 14–19.

[GKZ89] I. M. Gel′fand, A. V. Zelevinskiĭ, and M. M. Kapranov, *Hypergeometric functions and toric varieties*, Funktional. Anal. i Prilozhen. **23** (1989), no. 2, 12–26. Correction in ibid, **27** (1993), no. 4, 91.

[Gli05] David Glickenstein, *A combinatorial Yamabe flow in three dimensions*, Topology **44** (2005), no. 4, 791–808.

[GL96] N. Gonciulea and V. Lakshmibai, *Degenerations of flag and Schubert varieties to toric varieties*, Transform. Groups **1** (1996), no. 3, 215–248.

[Gra01] William Graham, *Positivity in equivariant Schubert calculus*, Duke Math. J. **109** (2001), no. 3, 599–614.

[GM92] John P. C. Greenlees and J. Peter May, *Derived functors of I-adic completion and local homology*, J. Algebra **149** (1992), no. 2, 438–453.

[Har70] Robin Hartshorne, *Affine duality and cofiniteness*, Invent. Math. **9** (1969/1970), 145–164.

[Hor1889] J. Horn, *Über die konvergenz der hypergeometrischen Reihen zweier und dreier Veränderlichen*, Math. Ann. **34** (1889), 544–600.

[LS82] Alain Lascoux and Marcel-Paul Schützenberger, *Polynômes de Schubert*, C. R. Acad. Sci. Paris Sér. I Math. **294** (1982), no. 13, 447–450.

[Man79] Anthony Manning, *Topological entropy for geodesic flows*, Ann. of Math. (2) **110** (1979), no. 3, 567–573.

[Mas03] Mikiya Masuda, *h-vectors of Gorenstein* simplicial posets*, 2003. arXiv:math.CO/0305203

[MMP87] J. S. B. Mitchell, D. M. Mount, and C. H. Papadimitriou, The discrete geodesic problem, *SIAM J. Comp.* **16** (1987), no. 4, 647–668.

[PS98] Irena Peeva and Bernd Sturmfels, *Generic lattice ideals*, J. Amer. Math. Soc. **11** (1998), no. 2, 363–373.

[SS86] M. Sharir and A. Schorr, On shortest paths in polyhedral spaces, *SIAM J. Comp.* **15** (1986), no. 1, 193–215.

[Sta74] R. P. Stanley, *Combinatorial reciprocity theorems*, Advances in Math. **14** (1974), 194–253.

[Sta91] Richard P. Stanley, *f-vectors and h-vectors of simplicial posets*, J. Pure Appl. Algebra **71** (1991), no. 2-3, 319–331.

[Vol68] Y. A. Volkov, *An estimate for the deformation of a convex surface in dependence on the variation of its intrinsic metric*, Ukrain. Geometr. Sb. **5-6** (1968), 44–69.

[VP71] Ju. A. Volkov and E. G. Podgornova, The cut locus of a polyhedral surface of positive curvature (in Russian), *Ukrainian Geometric Sbornik* **11** (1971), 15–25.