Equivariant Invariants and Linear Geometry

Robert MacPherson

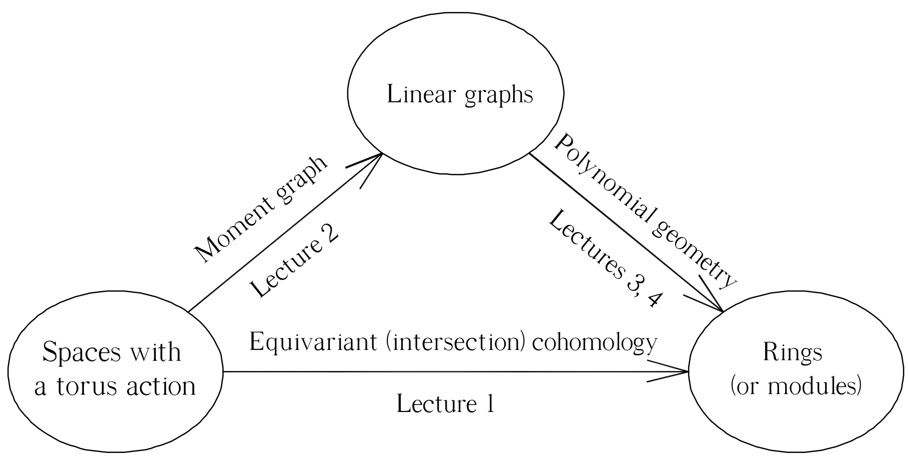
IAS/Park City Mathematics Series
Volume 14, 2004

Equivariant Invariants and Linear Geometry

Robert MacPherson

Introduction

⟨**0.1**⟩ This course will concern the following triangle of ideas.

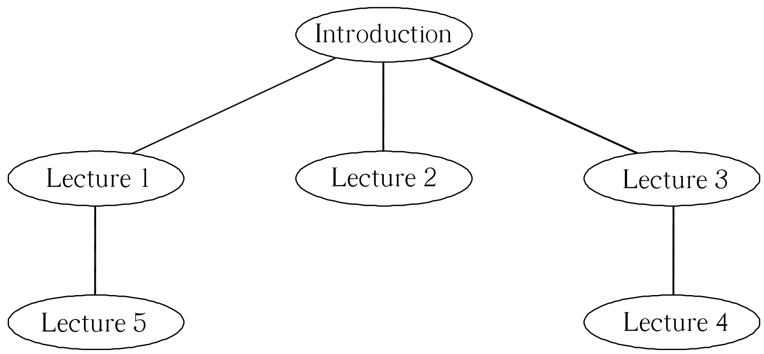


The vertices of this triangle represent mathematical objects. They will be defined in this introduction. The edges from one vertex to another represent mathematical constructions: given an object of the first type, we construct an object of the second type. These constructions will be the subject of the separate Lectures. The main theorem is that the diagram commutes: the construction on the bottom is the same as the composition of the two constructions on the top.

The constructions represented by the three edges all involve geometry, but they are of a completely different character from each other.

[1]Institute for Advanced Study, Princeton NJ 08540.
E-mail address: `rdm@ias.edu`.

⟨0.2⟩ Guide to reading. The Lectures have been made independent of each other as much as possible, so as to allow several different points of entry into the subject. The following is the diagram of dependencies:



The mathematical knowledge required in advance has been kept to a minimum. *Starred sections and exercises are exceptions to this rule. They have mathematical prerequisites that go beyond those of the other sections, and are not needed for the rest of what we will do. The reader is invited to skip the *starred sections on a first reading.

The exercises are designed to be an integral part of the exposition.

⟨0.3⟩ Credit and thanks. All of my work on this subject has been joint with Bob Kottwitz, Mark Goresky, and Tom Braden. A deep study of moment graphs has been carried out by Victor Guillemin, Tara Holm, and Catalin Zara; Lecture 3 may serve as an introduction to their papers.

I am grateful to Tom Braden and to many participants of PCMI for corrections and improvements to this exposition.

0.1. Spaces with a Torus Action

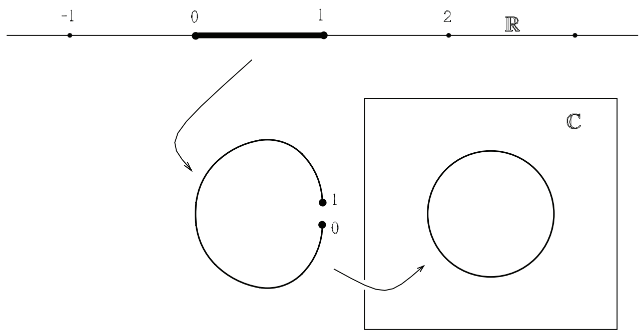
⟨1.1⟩ Definition. The *n-torus* T is the group

$$T = \mathbb{T}/L = (S^1)^n.$$

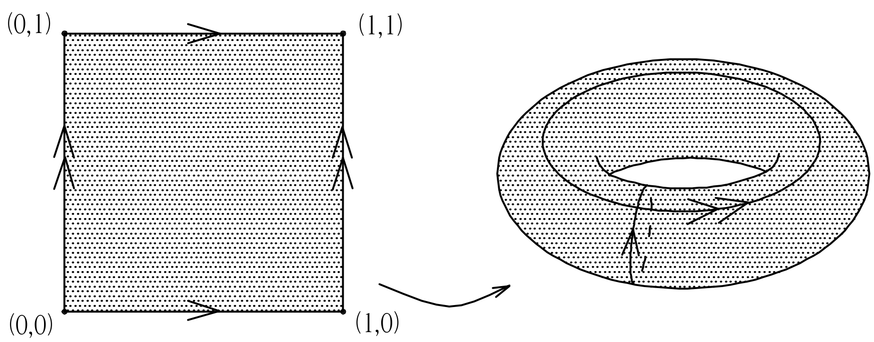
Here $\mathbb{T}$ is an n-dimensional real vector space, which we may take to be $\mathbb{R}^n$. The space $\mathbb{T}$ is a group under vector addition. The subgroup L is a lattice (i.e. a subgroup which is discrete as a topological space, with the property that $\mathbb{T}/L$ is compact). We may take L to be $\mathbb{Z}^n \subset \mathbb{R}^n$, the subgroup consisting of points whose coordinates are integers. The group S^1 is the unit circle group: the elements of norm 1 in the complex plane $\mathbb{C}$ considered as a group under multiplication. We may identify $\mathbb{R}/\mathbb{Z} \cong S^1$ by the map $\mathbb{R} \longrightarrow \mathbb{C}$ that sends x to $e^{2\pi ix}$, whose kernel is $\mathbb{Z}$. From this we get an identification

$$\mathbb{T}/L = \mathbb{R}^n/\mathbb{Z}^n = (\mathbb{R}/\mathbb{Z})^n = (S^1)^n.$$

⟨1.2⟩ We can visualize the n-torus as an n-cube $[0,1]^n$ with the opposite faces identified. For example, if $n = 1$, we have $S^1 = [0,1]/\sim$ where $\sim$ identifies 0 and 1.



Or, for example, the 2-torus is the square with the opposite edges identified,

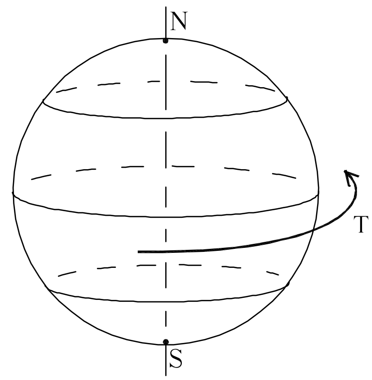


which shows why it's called a torus.

⟨**1.3**⟩ **Exercise.** Show in general that an n-torus as an n-cube $[0,1]^n$ with the opposite faces identified. Hint: Show every $\mathbb{Z}^n$ coset in $\mathbb{R}^n$ meets the unit cube $[0,1]^n \subset \mathbb{R}^n$, so $\mathbb{R}^n/\mathbb{Z}^n = [0,1]^n/\sim$ where $x \sim y$ when $x - y \in \mathbb{Z}^n$. Check that $\sim$ identifies opposite faces.

⟨**1.4**⟩ **Exercise*.** Let T be the n-torus $\mathbb{R}^n/\mathbb{Z}^n$ and let T' be the k-torus $\mathbb{R}^k/\mathbb{Z}^k$. Every group homomorphism $h : \mathbb{Z}^n \longrightarrow \mathbb{Z}^k$ extends uniquely to a continuous group homomorphism $\tilde{h} : \mathbb{R}^n \longrightarrow \mathbb{R}^k$, and so it passes to a continuous group homomorphism $\bar{h} : T \longrightarrow T'$. Show that the map $\mathrm{Hom}(\mathbb{Z}^n, \mathbb{Z}^k) \longrightarrow \mathrm{Hom}(T, T')$ that sends h to $\bar{h}$ is an isomorphism. Here $\mathrm{Hom}(T, T')$ is the set of continuous homomorphisms from T to T'.

⟨**1.5**⟩ **Definition.** A *space with a torus action* is a (Hausdorff) topological space X together with a self map $X \xrightarrow{t} X$ for every $t \in T$, notated $x \mapsto tx$, such that composition of homeomorphisms corresponds to multiplication in the group $t_1(t_2x) = (t_1 \times t_2)x$, and $(t, x) \mapsto tx$ is jointly continuous in x and t. We symbolize this by $T \circlearrowright X$. A quintessential example will be the circle action on the 2-sphere, where the circle rotates the 2-sphere about an axis. (Think of the action of the 24 hour day on the surface of the Earth.)



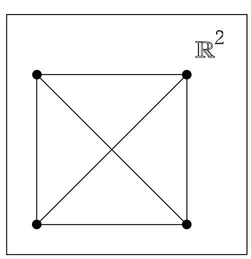
⟨**1.6**⟩ **Exercise.** Suppose that an n-torus T acts on a space X. Show that the orbit Tx of every point x in X is itself homeomorphic to a k-torus for some $k \leq n$. (Note the special case $k = 0$, which occurs at the North Pole N and the South Pole S of the example above.)

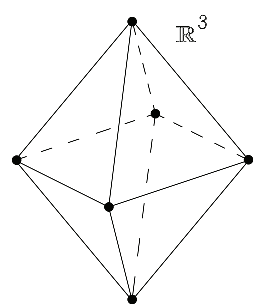
⟨**1.7**⟩ Why are we interested in torus actions $T \circlearrowright X$, rather than the actions of more general connected Lie groups $G \circlearrowright X$? In fact, computations for $G \circlearrowright X$ reduce to the computations for $T \circlearrowright X$, as explained in §3.8.11.

0.2. Linear Graphs

⟨**2.1**⟩ **Definition.** A *linear graph* is a finite set of points $\{v_i\}$ in a real vector space $\mathbb{V}$, called *vertices*, and a finite set of line segments $\{e_k\}$ in $\mathbb{V}$, called *edges* such that the two endpoints of each edge are both vertices.

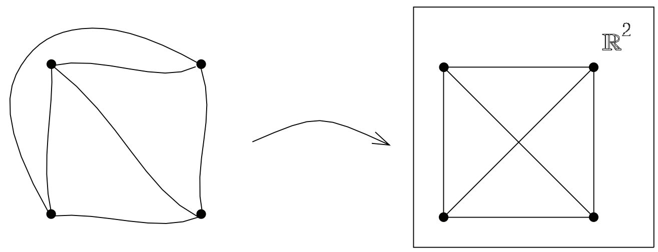
⟨**2.2**⟩ For example, the following are linear graphs:



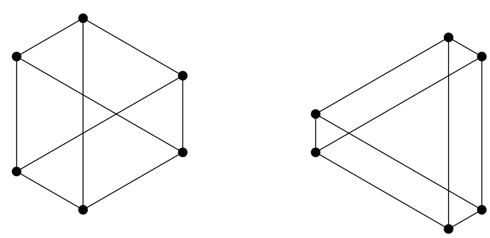


The first one, in $\mathbb{R}^2$, has four vertices and six edges. Note that the edges do not have to be disjoint: In this example, the two diagonals cross each other. The second one has six vertices and twelve edges. It is just the vertices and edges of an octahedron in $\mathbb{R}^3$. Any convex polyhedron gives rise to a linear graph by taking the vertices and the edges.

⟨**2.3**⟩ A topological graph is, of course, defined in a similar way, but without the embedding into a vector space. (For our purposes, a topological graph has at most one edge between a pair of vertices, and has no edge going from a vertex to itself.) So a linear graph is a graph together with a mapping into $\mathbb{V}$ in such a way that its edges are mapped into straight lines.



⟨2.4⟩ Equivalent linear graphs. We consider two linear graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ in $\mathbb{V}$ to be *equivalent* if they correspond to the same topological graph Γ, and for each edge of Γ, the corresponding line in $\mathcal{G}_1$ is parallel to the corresponding line in $\mathcal{G}_2$. For example, these two linear graphs are equivalent:



⟨2.5⟩ Directions and direction data. We define a *direction* in $\mathbb{V}$ to be a parallelism class of lines in $\mathbb{V}$, or equivalently, a line through the origin in $\mathbb{V}$. (To specify a direction, it suffices to give a nonzero vector in $D \in \mathbb{V}$. If $\lambda \in \mathbb{R}$ is nonzero, then λD and D determine the same direction, since they determine the same line through the origin.) To give an equivalence class of linear graphs of graphs in $\mathbb{V}$, it suffices to give a topological graph with *direction data*, i.e. for each edge of the graph, we give a direction D.

⟨2.6⟩ Exercise. What is the dimension of the space of linear graphs equivalent to the linear graphs pictured above?

⟨2.7⟩ Exercise. Suppose an abstract graph is embedded in the plane as a linear graph. Can you find a formula for the dimension of its equivalence class?

⟨2.8⟩ Exercise. Consider the triangle graph with the direction data that assigns to the three edges the following three directions D in $\mathbb{R}^3$: $(1, 0, 1)$, $(-1, 1, 1)$, and $(0, -1, 1)$. Show that there is no linear graph with this direction data.

0.3. Rings and Modules

⟨3.1⟩ Our rings R will all be graded algebras over the real numbers $\mathbb{R}$.

⟨3.2⟩ Definition. A *graded* $\mathbb{R}$-*algebra* is an $\mathbb{R}$-algebra with a direct sum decomposition

$$R = \bigoplus_{i \geq 0} R^i$$

into $\mathbb{R}$ vector spaces called the *graded pieces*, indexed by the non-negative integers, so that the multiplication is compatible with the grading: If $r \in R^i$ and $r' \in R^j$, then $rr' \in R^{i+j}$. Similarly, a *graded module* over R module M with a direct sum decomposition

$$M = \bigoplus_{i \geq 0} M^i$$

into $\mathbb{R}$ vector spaces, so that if $r \in R^i$ and $m \in M^j$, then $rm \in M^{i+j}$. Ring and module homomorphisms are required to respect the gradings.

All of our graded rings and modules will have the property that the odd numbered graded pieces are all zero, so $R = \bigoplus_{j \in \mathbb{Z}, j \geq 0} R^{2j}$. This perverse factor of 2 comes from the topological side of the story.

⟨3.3⟩ The polynomial ring $\mathcal{O}(\mathbb{T})$. We denote by $\mathcal{O}(\mathbb{T})$ the ring of real valued polynomial functions on the real vector space $\mathbb{T}$. It is the same as the ring of polynomials with real coefficients in n variables, where n is the dimension of $\mathbb{T}$. This is a graded ring. The $2j$-th graded piece is the space of polynomials of *homogeneous degree* j, i.e. the space spanned by monomials of degree j.

⟨3.4⟩ In all our graded rings and modules, the graded pieces are finite dimensional real vector spaces. Their dimensions are encoded in the Hilbert series.

Definition. The *Hilbert series* of R is the power series whose coefficients are the dimensions of the graded pieces of R

$$\mathrm{Hilb}(R) = \sum_{i \geq 0} x^i \dim(R^i).$$

Since all of our graded rings are zero in odd degree, it is conventional to introduce the variable $q = x^2$.

$$\mathrm{Hilb}(R) = \sum_{j \geq 0} (x^2)^j \dim(R^{2j}) = \sum_{j \geq 0} q^j \dim(R^{2j}).$$

⟨3.5⟩ Proposition. The Hilbert series of the polynomial ring $\mathcal{O}(\mathbb{T})$ is

$$\mathrm{Hilb}(\mathcal{O}(\mathbb{R})) = \left(\frac{1}{1-q} \right)^n$$

where n is the dimension of the vector space $\mathbb{T}$.

⟨3.6⟩ Exercise. Prove this. Hint: here are two possible strategies:

1) Show directly that the number of monomials in n variables of degree j is the coefficient of q^j in $(q-1)^n$. For example, the number of monomials of degree j in 3 variables is the $(j+1)$-st triangular number: the number of points in a triangular array with $j+1$ points on a side. This is because the monomials of degree j can be arranged in a triangular array.

$$\begin{array}{ccccccc} & & & z^3 & & & \\ & & xz^2 & & yz^2 & & \\ & x^2z & & xyz & & y^2z & \\ x^3 & & x^2y & & xy^2 & & y^3 \end{array}$$

2) Or, calculate the Hilbert series of the polynomial ring $\mathcal{O}(\mathbb{R})$ of polynomials in one variable

$$\text{Hilb}(\mathcal{O}(\mathbb{R})) = 1 + q + q^2 + \cdots = \frac{1}{1-q}$$

then justify the following manipulations:

$$\text{Hilb}(\mathcal{O}(\mathbb{T})) = \text{Hilb}(\mathcal{O}(\underbrace{\mathbb{R} \times \cdots \times \mathbb{R}}_{n \text{ factors}}) = \text{Hilb}(\underbrace{\mathcal{O}(\mathbb{R}) \otimes \cdots \otimes \mathcal{O}(\mathbb{R})}_{n \text{ factors}}) =$$

$$= \underbrace{\text{Hilb}(\mathcal{O}(\mathbb{R})) \cdots \text{Hilb}(\mathcal{O}(\mathbb{R}))}_{n \text{ factors}} = \left(\frac{1}{1-q}\right)^n$$

⟨**3.7**⟩ **Exercise.** Let R be the ring of continuous functions on the real line $\mathbb{R}$, whose restriction to the positive reals $\mathbb{R}^{>0}$ and the negative reals $\mathbb{R}^{<0}$ are both polynomial functions. Show that R is a graded ring isomorphic to the polynomial ring in two variables x and y divided by the principal ideal generated by the polynomial xy, i.e. $R = \mathbb{R}[x, y]/(xy)$, and that its Hilbert series is $(1 + q)/(1 - q)$.

LECTURE 1
Equivariant Homology and Intersection Homology

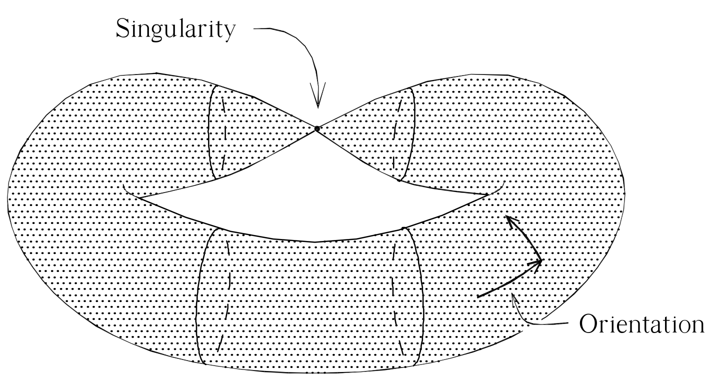
(Geometry of Pseudomanifolds)

1.1. Introduction

⟨**1.1**⟩ In this lecture, we will give a geometric way of defining equivariant homology and equivariant intersection homology. The standard definitions of these homology theories, as found in the literature, are good for proving properties, but are perhaps not so intuitive. In this lecture, we will consider $G \circlearrowright X$: an action of a general Lie group G on a space X, although in the other lectures we are interested mainly in the case that G is a torus T.

⟨**1.2**⟩ The definitions we present are based on the notion of a pseudomanifold. A k-dimensional manifold is a space that looks locally like k-dimensional Euclidean space near every point. A k-dimensional pseudomanifold P is allowed to have *singularities*, i.e. points where it doesn't locally look like Euclidean space. However, it must satisfy two properties:

(1) The part of P where it is a k-manifold is open and dense in P and it must be oriented.
(2) The set of singularities has dimension at most $k-2$ (i.e. codimension at least 2).



A pseudomanifold (the pinched torus)

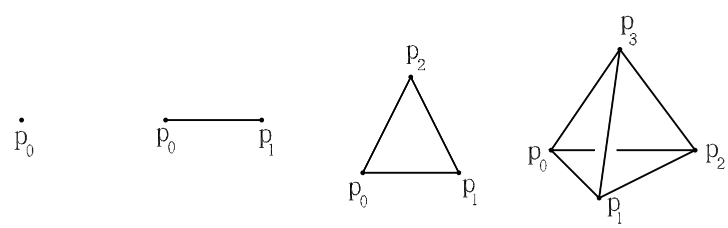
There are several ways to make this intuitive notion of a pseudomanifold rigorous. We will use simplicial complexes, because that is the one most in keeping with the spirit of these notes. Readers who are comfortable with pseudomanifolds can skip directly to §1.4

⟨**1.3**⟩ Equivariant homology theories are difficult to compute directly from the definitions as given in this Lecture. However, the methods of Lectures 3 to 5 provide effective computations in many interesting cases.

1.2. Simplicial Complexes

Readers familiar with simplicial complexes and orientations can skip this section.

⟨**2.1**⟩ A *k-simplex* Δ is the convex hull of $k+1$ points $p_0, \dots, p_k$ in general position in some Euclidean space. Here general position just means that the points don't all lie in any $(k-1)$-dimensional Euclidean subspace. The k-simplex is a polyhedron. Its faces are themselves simplices; they are the convex hulls of subsets of the points p_i. The points p_i are the vertices of Δ.



0-simplex 1-simplex 2-simplex 3-simplex

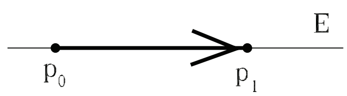
⟨**2.2**⟩ **Definition.** A *simplicial complex* is a set S of simplices in some Euclidean space with the properties

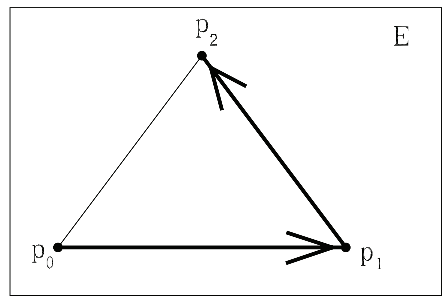
(1) Any two simplices in S are either disjoint or intersect in a set that is a face of each of them.
(2) Any face of a simplex in S is itself in S.

⟨**2.3**⟩ **Spaces of finite type.** We define a *space of finite type* to be a topological space homeomorphic to the difference $S - S'$ where S is a simplicial complex and S' is a sub simplicial complex. We will assume without further mention that all of our spaces are of finite type. Spaces generated by finite operations, such as real algebraic varieties, or their images under algebraic maps, are all of finite type (although proving it takes technology developed over many years). On the other hand, a Cantor set, or $\mathbb{Z}$ are not of finite type.

⟨**2.4**⟩ An *orientation* $\mathbb{O}$ of a simplex Δ is an ordering of the vertices of Δ, two orderings being considered equivalent if one is an even permutation of the other. (This definition doesn't work for a 0-simplex. An orientation of a 0-simplex is simply one of the symbols $+$ or $-$.) Any simplex has exactly two orientations, these two orientations are called opposite orientations of each other.

⟨**2.5**⟩ An orientation of a Euclidean space is an ordered set of basis vectors, two being considered equivalent if one is a continuous deformation of the other. We can draw an orientation by representing the basis vectors as arrows, and signaling the ordering by placing the tail of each arrow at the head of the previous one. An orientation $\mathbb{O}$ of k-simplex Δ determines an orientation of the k-dimensional Euclidean space E containing Δ as follows: Suppose $\mathbb{O} = \{p_0 < p_1 < \cdots < p_k\}$. Then $\{p_1 - p_0, p_2 - p_1, \ldots, p_k - p_{k-1}\}$ is the ordered basis.





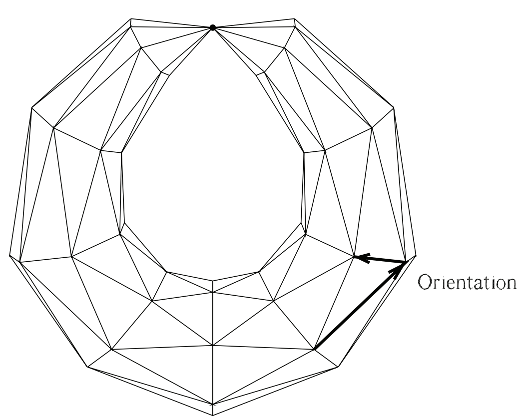
Exercise. Show that two orientations of Δ are equivalent if and only if they determine equivalent ordered bases of E.

⟨**2.6**⟩ If Δ is a k-simplex and Δ' is a $(k-1)$-simplex, an orientation $\mathbb{O}$ of Δ induces an orientation $\mathbb{O}'$ of Δ' as follows: Pick an equivalent ordering such that the unique vertex of Δ not in Δ' is the last one of the ordering. Then $\mathbb{O}'$ is the restriction of that ordering to Δ'. (This definition doesn't work if Δ is a 1-simplex. In this case, $\mathbb{O}'$ is $-$ if Δ' is the first vertex of the ordering, and it is $+$ if it is the second one.)

1.3. Pseudomanifolds

⟨**3.1**⟩ **Definition.** A *k-dimensional pseudomanifold* is a simplicial complex together with an orientation $\mathbb{O}(\Delta)$ of each of its k-simplices, with the following properties:

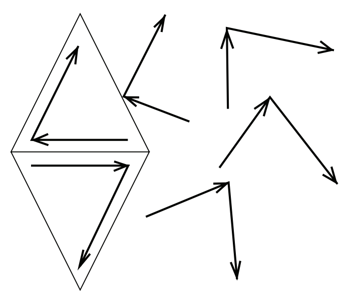
(1) Every simplex is a face of some k-simplex.
(2) Every $(k-1)$-simplex is the face of exactly two k-simplices.
(3) *(The continuity of orientation property)* If Δ' is a $(k-1)$-simplex and Δ and $\tilde{\Delta}$ are the two k-simplices that contain Δ' in their boundary, then the given orientations $\mathbb{O}(\Delta)$ and $\mathbb{O}(\tilde{\Delta})$ induce opposite orientations on Δ'.



A pseudomanifold (the simplicial pinched torus)

⟨**3.2**⟩ The following exercise shows why property 3 is called continuity of orientation.

Exercise. Suppose that Δ and $\tilde{\Delta}$ are two k-simplices in a Euclidean k-space, and that they intersect in a $(k-1)$-simplex Δ'. Show that the orientations $\mathbb{O}(\Delta)$ and $\mathbb{O}(\tilde{\Delta})$ induce opposite orientations on Δ' if and only if the ordered basis for E determined by Δ as in exercise 1.2.5 can be continuously deformed into the ordered basis for E determined by $\mathbb{O}(\tilde{\Delta})$.



A path in the space of ordered bases of the plane

⟨**3.3**⟩ **Definition.** A *k-dimensional pseudomanifold with boundary* is a simplicial complex S, an orientation $\mathbb{O}(\Delta)$ of each of its k-simplices, and a sub simplicial complex B called the *boundary*, with the following properties:

(1) The boundary B is a $(k-1)$-dimensional pseudomanifold
(2) Every simplex of S is a face of some k-simplex.
(3) Every $(k-1)$-simplex Δ' that not in B is the face of exactly two k-simplices, and the continuity of orientation property holds for Δ'.
(4) Every $(k-1)$-simplex Δ' in B is the face of exactly one k-simplex Δ in S. The orientation of Δ' induced from $\mathbb{O}(\Delta)$ coincides with the orientation $\mathbb{O}(\Delta')$ of Δ' from the pseudomanifold structure on B.

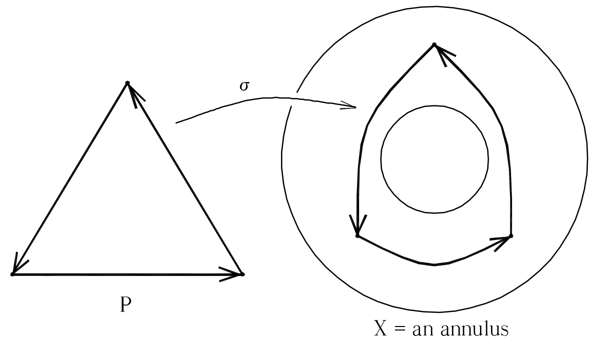
⟨**3.4**⟩ **Exercise.** Show that the continuity of orientation property for the boundary B of a pseudomanifold with boundary follows from the other properties in the definition.

1.4. Ordinary Homology Theory

As a warm up, we will give a definition of ordinary homology theory in the spirit of the definitions of more complicated theories to come. This definition of ordinary homology has roots going back to Poincaré and Veblen and the earliest days of homology theory.

⟨**4.1**⟩ **Definition.** Let X be a topological space. An *i-cycle* is an i-dimensional pseudomanifold P together with a map $\sigma : P \longrightarrow X$.

The idea is that an i-cycle captures the "holes" in a topological space by surrounding them. For example, the following 1-cycle surrounds the hole in the annulus:

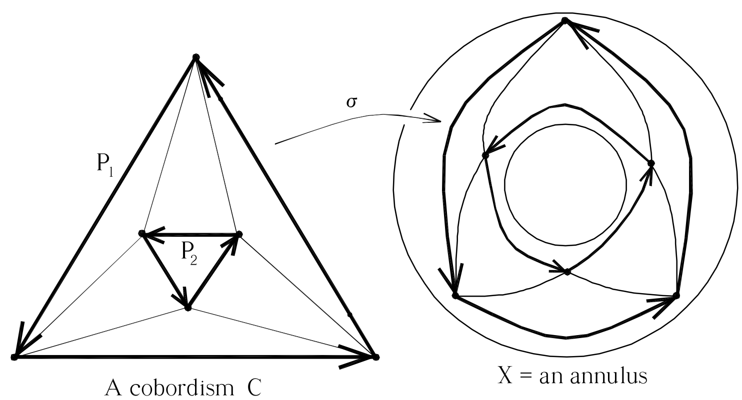


We will refer to the i-cycle $\sigma : P \longrightarrow X$ by the symbol P when there's no confusion about what σ is.

⟨4.2⟩ Definition. If $\sigma_1 : P_1 \longrightarrow X$ and $\sigma_2 : P_2 \longrightarrow X$ are two i-cycles, then the *sum* $P_1 + P_2$ is their union $\sigma : P_1 \cup P_2 \longrightarrow X$ where $\sigma|P_1 = \sigma_1$ and $\sigma|P_2 = \sigma_2$.

The *negative* $-P$ of an i-cycle is the same i-cycle P with the opposite orientation for every i-simplex. As usual, $P_1 - P_2$ is $P_1 + (-P_2)$.

⟨4.3⟩ Definition. A *cobordism* between two i-cycles $\sigma_1 : P_1 \longrightarrow X$ and $\sigma_2 : P_2 \longrightarrow X$ is a $(i+1)$-dimensional pseudomanifold with boundary C, and a map $\sigma : C \longrightarrow X$ such that the boundary B of C is $P_1 - P_2$ and the restriction of σ to B coincides with σ_1 and σ_2. Two i-cycles $\sigma_1 : P_1 \longrightarrow X$ and $\sigma_2 : P_2 \longrightarrow X$ are said to be *cobordant* if there is a cobordism between them.



The 1-cycles P_1 and P_2 are cobordant

The idea behind this definition is that if P_1 and P_2 are cobordant, they surround the same holes in the same way. For example, if σ has appropriate differentiability assumptions so that it makes sense, any closed differential i-form will have the same integral on P_1 and on P_2, by Stokes' Theorem.

⟨4.4⟩ Proposition. Cobordism is an equivalence relation among i-cycles.

Exercise Prove this. For example, if S_1 is a cobordism between P_1 and P_2, and S_2 is a cobordism between P_2 and P_3, then S_1 and S_2 can be glued together to provide a cobordism between P_1 and P_3.

⟨**4.5**⟩ **Definition – Proposition.** The i-th homology, notated $H_i(X)$, is the set of cobordism classes of i-cycles. The operations $+$ and $-$ induce the structure of an Abelian group on this set. The identity element is represented by the empty pseudomanifold.

⟨**4.6**⟩ For example, if X is the annulus, $H_0(X)$ is $\mathbb{Z}$ generated by a point, and $H_1(X)$ is $\mathbb{Z}$ generated by the cycle P_1 or P_2 as in the pictures above.

⟨**4.7**⟩ **Exercise.** Show for any X that $H_0(X)$ is $\mathbb{Z}^k$ where k is the number of path connected components of X.

⟨**4.8**⟩ **Convention.** We write $H_*(X)$ for $\bigoplus_i H_i(X)$. It's a summation convention: wherever a star appears, it means a direct sum over the possible indices i that might appear there.

1.5. Basic Definitions of Equivariant Topology

⟨**5.1**⟩ A *topological group* G is a set that is simultaneously a group and a topological space, with the property that the multiplication operation $G \times G \longrightarrow G$ and the inverse operation $G \longrightarrow G$ are both continuous. G is a Lie group if it is one of our spaces of finite type §1.2.3 (or, what turns out to be the same thing for topological groups, if it's a topological manifold with finitely many connected components.)

⟨**5.2**⟩ A *space with a group action* $G \circlearrowright X$ is a topological space X (which for us will always be of finite type), and an a map $G \times X \longrightarrow X$ that is continuous, such that $(g \cdot g')x = g(g'(x))$ (§0.1.5).

⟨**5.3**⟩ **The equivariant category.** Suppose $G \circlearrowright X$ and $G' \circlearrowright X'$ are two topological spaces with a group action. A morphism $G \circlearrowright X \Longrightarrow G' \circlearrowright X'$ is a continuous group homomorphism $\phi : G \longrightarrow G'$ together with a continuous map $\psi : X \longrightarrow X'$ such that $\psi(gx) = \phi(g)\psi(x)$.

For example, for any $G \circlearrowright X$ there is a canonical morphism $G \circlearrowright X \Longrightarrow 1 \circlearrowright X/G$. Here 1 is the one element group; X/G is the quotient space $X/\sim$ where $\sim$ is the equivalence relation $x \sim x'$ if there is a $g \in G$ such that $gx = x'$; $\phi : G \longrightarrow 1$ is the only thing it could be; and $\phi : X \longrightarrow X/G$ is the quotient map.

⟨**5.4**⟩ G **equivariant maps.** If G is a fixed group, then *the category of G-spaces* is the sub category of the equivariant category where the map ϕ on G is the identity. The maps in this category are called *G equivariant maps.* In other words, if X and X' are both G-spaces, then an equivariant map from X to X' is a continuous map $\psi : X \longrightarrow X'$ such that $\psi(gx) = g\psi(x)$ for all $g \in G$ and all $x \in X$. We consider two G spaces *equivalent* if they are isomorphic in this category. This means that the map ψ is a homeomorphism.

⟨**5.5**⟩ **Free actions.** An action of G on X is *free* if no element of G except the identity fixes any point in X, i.e. $gx = x$ implies g is the identity.

Another commonly used terminology for the same thing is this: The map $\pi : X \longrightarrow X/G$ is called a *principal G-bundle* if and only if the action $G \circlearrowright X$ is free. In this terminology, X/G is called the *base* of the principal bundle; X is called the total space; and π is called the projection.

Yet another popular terminology is to say that X is a *G-torsor* over X/G.

⟨**5.6**⟩ **Exercise.** Show that every orbit Gx of a free action $G \circlearrowright X$ is homeomorphic to G.

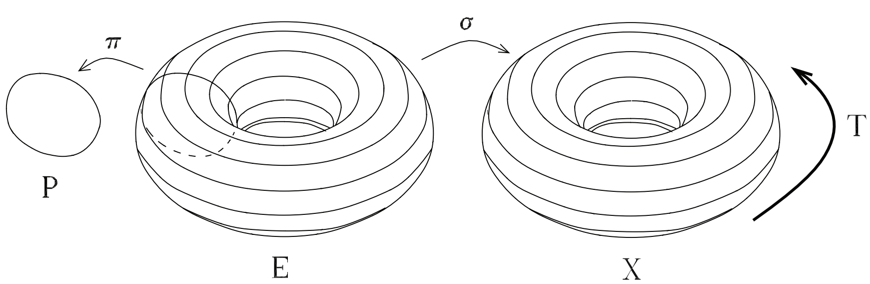
1.6. Equivariant Homology

⟨**6.1**⟩ **Definition.** Let $G \circlearrowright X$ be a G-space. An *equivariant i-cycle* is a diagram

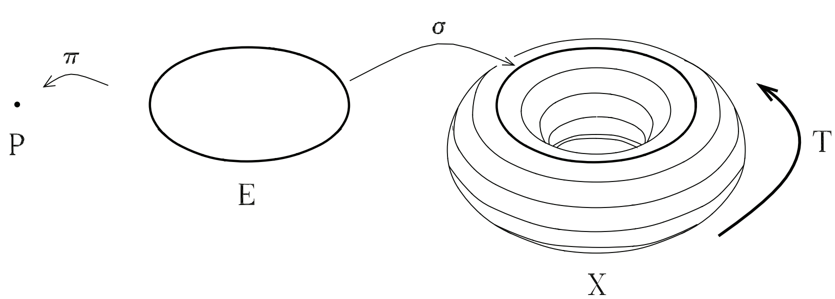
$$P \xleftarrow{\pi} E \xrightarrow{\sigma} X$$

where P is an i-dimensional pseudomanifold, E is a G-space with a free G action, $\pi : E \longrightarrow P$ is the projection on the quotient by the G action (so $P = E/G$), and σ is a G equivariant map.

Another way of describing the same thing (see §1.5.5) is this: A i-cycle is an equivariant map into X of the total space of a G-principal bundle E, whose base space is an i-dimensional pseudomanifold.



An equivariant 1-cycle for the circle T acting on the torus X



An equivariant 0-cycle for the circle T acting on the torus X

⟨**6.2**⟩ When we want to refer to an i-cycle, we use the symbol P for the pseudomanifold, even though it is really a 4-tuple of data. The *sum* $P_1 + P_2$ of two equivariant i-cycles P_1 and P_2 is $P_1 \cup P_2 \longleftarrow E_1 \cup E_2 \longrightarrow X$. The **negative** $-P$ of an equivariant i-cycle is the same i-cycle with the opposite orientation on P.

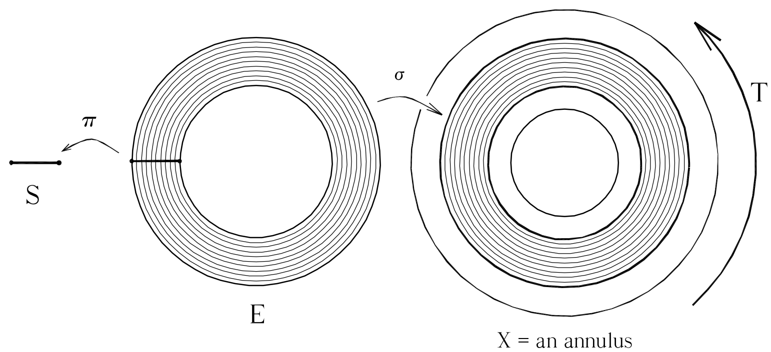
⟨6.3⟩ Definition. A *cobordism* between two equivariant i-cycles P_1 and P_2 is a commutative diagram

$$\begin{array}{ccccc} C & \xleftarrow{\ \pi\ } & E & \xrightarrow{\ \sigma\ } & X \\ {\scriptstyle \text{inclusion as boundary B}} \uparrow & & {\scriptstyle \text{inclusion}} \uparrow & & \uparrow = \\ P_1 - P_2 & \xleftarrow{\pi_1, \pi_2} & E_1 \cup E_2 & \xrightarrow{\sigma_1, \sigma_2} & X \end{array}$$

where C is a pseudomanifold with boundary $B = P_1 - P_2$, E is a G principal bundle over C, σ is an equivariant map, and $E_1 \cup E_2$ is $\pi^{-1}(P_1 \cup P_2)$.

Two equivariant i-cycles P_1 and P_2 are said to be *cobordant* if there is a cobordism between them.

For example, the following picture shows a cobordism between two equivariant 0-cycles. Here $T \circlearrowright X$ is a circle rotating the annulus.



⟨6.4⟩ Proposition. Cobordism is an equivalence relation on equivariant i-cycles.

⟨6.5⟩ Definition – Proposition. The i-th equivariant homology of $G \circlearrowright X$, which we will notate $H_i(G \circlearrowright X)$, is the set of cobordism classes of equivariant i-cycles. The operations $+$ and $-$ induce the structure of an Abelian group on this set.

The usual notation for the ith equivariant homology group is $H_i^G(X)$, and Borel's original notation was $H_i(X_G)$.

⟨6.6⟩ The topological juice of this definition comes in the requirement that the G action on the cycles and cobordisms be free.

Exercise. Show that if we dropped the requirement that the G action be free from the definitions of equivariant cycles and cobordisms, then we would just get the homology of the quotient $H_i(X/G)$.

⟨6.7⟩ Exercise. Show that if G acts freely on X, then $H_i(G \circlearrowright X)$ is, in fact, the homology of the quotient $H_i(X/G)$.

For example, if T is the 1-torus, X is the 2-torus, and T rotates X as in the pictures above, then the T action is free and $H_i(T \circlearrowright X) = H_i(X/T)$. Since X/T is a circle, we have

$$H_i(T \circlearrowright X) = \begin{cases} \mathbb{R} & \text{if} \quad i = 0 \\ \mathbb{R} & \text{if} \quad i = 1 \\ 0 & \text{otherwise} \end{cases}$$

The two nonzero equivariant homology groups are generated by the equivariant cycles shown in the pictures above.

Similarly, the equivariant homology of the annulus is nonzero only in degree 0, where it is generated by the 0-cycle shown in the picture.

In both of these examples, the equivariant homology is smaller than the ordinary homology. For the cases we are going to study later, the reverse is true.

⟨**6.8**⟩ **Exercise.** Show that $H_0(G \circlearrowright X) = H_0(X)$, the free group generated by the connected components of X.

⟨**6.9**⟩ **Exercise.** Show that $H_i(1 \circlearrowright X) = H_i(X)$, where 1 is the one element group.

1.7. Formal Properties of Equivariant Homology

⟨**7.1**⟩ **Functoriality.** Suppose that we have an equivariant map

$$G \circlearrowright X \Longrightarrow G' \circlearrowright X'.$$

Then there is an induced map on equivariant homology

$$H_i(G \circlearrowright X) \longrightarrow H_i(G' \circlearrowright X').$$

If $G = G'$ and the map $G \longrightarrow G'$ is the identity, then this map can be defined simply by composition of G-equivariant maps. The general case is a little complicated: If

$$P \xleftarrow{\;\pi\;} E \xrightarrow{\;\sigma\;} X$$

is a cycle for $H_i(G \circlearrowright X)$, then

$$P \xleftarrow{\;\pi\;} G' \times_G E \xrightarrow{\;\sigma'\;} X'$$

is the corresponding cycle for $H_i(G' \circlearrowright X')$, where $G' \times_G E$ is the "associated bundle" defined as the quotient space $(G' \times E)/G$ where G acts on $G' \times E$ by $g(g', e) = (g' \cdot \phi g^{-1}, ge)$. The group G' acts freely on the associated bundle $G' \times_G E$ by $g'(g'', e) = (g \cdot g'', e)$. The quotient of this G' action is again P. The map σ' : $G' \times_G E \longrightarrow X'$ sends (g', e) to $g'(\psi(\sigma(e)))$. One needs to check that this is well defined.

⟨**7.2**⟩ **Coefficients.** Suppose that R is a ring containing the integers $\mathbb{Z}$. We can define the equivariant homology with coefficients in R by

$$H_i(G \circlearrowright X; R) = H_i(G \circlearrowright X) \otimes_{\mathbb{Z}} R$$

Assumption. *We will assume that our coefficient ring is the real numbers* $\mathbb{R}$. (Actually, any other field of characteristic zero would work as well for everything we will do.) With these coefficients, the equivariant homology is a real vector space; we notate it simply by $H_i(G \circlearrowright X)$. There are torsion phenomena that are killed by taking these coefficients, but the most interesting phenomena survive. Many of our theorems are false, or are at least much more complicated to state, for coefficients that are not fields of characteristic zero.

⟨**7.3**⟩ **The Kunneth theorem.** Given two spaces with group actions $G_1 \circlearrowright X_1$ and $G_2 \circlearrowright X_2$, we have the *Kunneth map*

$$\begin{array}{ccccc} H_i(G_1 \circlearrowright X_1) & \times & H_j(G_2 \circlearrowright X_2) & \longrightarrow & H_{i+j}((G_1 \times G_2) \circlearrowright (X_1 \times X_2)) \\ & & & & \\ [P_1] & \times & [P_2] & \mapsto & [P_1 \times P_2] \end{array}$$

where $P_1 \times P_2$ is defined to be the equivariant $(i+j)$-cycle

$$P_1 \times P_2 \xleftarrow{\pi_1 \times \pi_2} E_1 \times E_2 \xrightarrow{\sigma_1 \times \sigma_2} X_1 \times X_2$$

Proposition. The Kunneth map induces an isomorphism

$$H_i(G_1 \circlearrowright X_1) \otimes H_j(G_2 \circlearrowright X_2) \xrightarrow{\cong} H_{i+j}((G_1 \times G_2) \circlearrowright (X_1 \times X_2)).$$

⟨**7.4**⟩ **Note on proofs.** From here until the end of the lecture, we will state hard

theorems without proofs. What is important is to understand the statements, and to understand why the various objects and maps are well defined. If you invent proofs of any of these statements without knowing a lot of topology, that will be quite an accomplishment.

⟨**7.5**⟩ **The cohomology ring.** The cohomology is the vector space dual of the homology

$$H^i(G \circlearrowright X) = (H_i(G \circlearrowright X))^* = \operatorname{Hom}(H_i(G \circlearrowright X), \mathbb{R}).$$

Consider the diagram

$$H_i(G \circlearrowright X) \otimes H_j(G \circlearrowright X) \xrightarrow{\cong} H_{i+j}(G \times G \circlearrowright X \times X) \xleftarrow{\Delta} H_{i+j}(G \circlearrowright X)$$

where $\Delta : G \circlearrowright X \Longrightarrow (G \times G) \circlearrowright (X \times X)$ is the diagonal map that sends g to (g, g) and sends x to (x, x).

Now take the vector space dual of the whole diagram

$$H^i(G \circlearrowright X) \otimes H^j(G \circlearrowright X) \xleftarrow{\cong} H^{i+j}(G \times G \circlearrowright X \times X) \xrightarrow{\Delta^*} H^{i+j}(G \circlearrowright X)$$

Proposition. The composed map from the left to the right in this diagram is the multiplication rule of a ring structure on equivariant cohomology.

⟨**7.6**⟩ **Signs.** The product in cohomology satisfies the following sign rule: If $x \in H^i(G \circlearrowright X)$ and $y \in H^j(G \circlearrowright X)$, then $xy = (-1)^{ij} yx$. The reason for this it that if P_1 and P_2 are pseudomanifolds of dimensions i and j, then $P_1 \times P_2 = (-1)^{ij} P_2 \times P_1$. A nice exercise is to figure out how to define precisely the product of two pseudomanifolds, and to show this commutation rule.

⟨**7.7**⟩ **Exercise.** Use the map $1 \circlearrowright X \Longrightarrow G \circlearrowright X$ to show that there is a canonical map $H_i(X) \longrightarrow H_i(G \circlearrowright X)$, or dually $H^i(G \circlearrowright X) \longrightarrow H^i(X)$.

⟨**7.8**⟩ **Exercise.** Let A be the ring $H^*(G \circlearrowright \mathrm{pt})$, where pt is a point. Use the map $G \circlearrowright X \Longrightarrow G \circlearrowright \mathrm{pt}$ to show that there is a map $A \longrightarrow H^*(G \circlearrowright X)$, so $H^*(G \circlearrowright X)$ is an A-module. We will be interested mainly in actions where this map is an injection, unlike the illustrative examples considered in the last section.

⟨**7.9**⟩ **Homology vs. cohomology.** There is a psychological dilemma. An equivariant cohomology class is hard to imagine. It is an element in a dual space — it eats a homology class and gives you a number. But the cohomology is an algebra, which most people find to be an intuitive structure. An equivariant homology class is easy to visualize, but the homology forms a co-algebra, which is hard to think about.

Choosing the demons of dual spaces over the demons of co-algebras, our computations will be in equivariant cohomology. Of course, the computation of equivariant cohomology is mathematically equivalent to the computation of equivariant homology, so the information is the same in the end.

1.8. Torus Equivariant Cohomology of a Point

The ordinary homology of a point p is uninteresting: It is simply a 1-dimensional vector space in homology degree 0, and zero in every other degree. However, for Lie groups of positive dimension, the equivariant homology (or cohomology) or cohomology of a point is quite interesting: It is infinite dimensional.

⟨8.1⟩ Generators of the circle equivariant homology of a point. Suppose T^1 is a 1-torus and pt is a point. Then for every integer $k \geq 0$, we have the following equivariant $2k$-cycle:

$$\mathbb{CP}^k \xleftarrow{\ \pi\ } S^{2k+1} \xrightarrow{\ \sigma\ } \mathrm{pt}$$

Here S^{2k+1} is the real unit real sphere in complex $(k+1)$-space, given by the equation $|z_0|^2 + |z_1|^2 + \cdots + |z_k|^2 = 1$. The circle $T^1 = S^1 \subset \mathbb{C}$ acts on it freely by scalar multiplication. The quotient space $\mathbb{CP}^k$ is the complex projective k-space (see §2.3.1), a pseudomanifold (indeed a manifold) of real dimension $2k$.

⟨8.2⟩ It is useful to think about why this $2k$-cycle is nonzero. 1. The $2k+1$ sphere bounds the $2k+2$ ball $|z_0|^2 + |z_1|^2 + \cdots + |z_k|^2 \leq 1$. The S^1 action extends to the ball. If $k > 0$, the quotient is a pseudomanifold with boundary $\mathbb{CP}^k$. Why isn't this a cobordism to zero? 2. The fibers of the map $S^{2k+1} \longrightarrow \mathbb{CP}^k$ are all circles S^1. There is a "trivial" example of a map to $\mathbb{CP}^k$ whose fibers are all circles: if we had a homeomorphism $S^1 \times \mathbb{CP}^k \longrightarrow \mathbb{CP}^k$ which is cobordant to zero, since we can take $S^1 \times C(\mathbb{CP}^k) \longrightarrow C(\mathbb{CP}^k)$ where $C(\mathbb{CP}^k)$ is the cone on $\mathbb{CP}^k$. So, since our $2k$-cycle is not cobordant to zero, it must not be equivalent to the"trivial" example.

⟨8.3⟩ Exercise*. Prove that the $2k$-cycle above is not 0 in $H_{2k}(T^1 \circlearrowright \mathrm{pt})$. Hint: Use characteristic classes.

⟨8.4⟩ Proposition. The equivariant homology of $T^1 \circlearrowright \mathrm{pt}$, where T^1 is the 1-torus and pt is a point, is given by

$$H_i(T^1 \circlearrowright \mathrm{pt}) = \begin{cases} \mathbb{R} \quad \text{generated by } \mathbb{CP}^k & \text{if} \quad i = 2k \\ \qquad 0 & \text{if} \quad i \quad \text{is odd} \end{cases}$$

Dually, the equivariant cohomology ring is

$$H^*(T^1 \circlearrowright \mathrm{pt}) = \{\text{polynomial functions on } \mathbb{T}^1 = \mathbb{R}\} = \mathcal{O}(\mathbb{T}^1)$$

The basis $\{1, t, t^2, \ldots\}$ of $H^*(T^1 \circlearrowright \mathrm{pt})$ is dual to the basis $\{\mathbb{CP}^0, \mathbb{CP}^1, \mathbb{CP}^2, \ldots\}$ of $H_*(T^1 \circlearrowright \mathrm{pt})$.

⟨8.5⟩ The torus equivariant homology of a point. A general n-torus $T = \mathbb{T}/L$ is a product of n copies of the circle $T^1 = \mathbb{T}^1/L^1$. Therefore, we have the product of spaces with group action

$$T \circlearrowright \mathrm{pt} = \underbrace{T^1 \circlearrowright \mathrm{pt} \times T^1 \circlearrowright \mathrm{pt} \times \cdots \times T^1 \circlearrowright \mathrm{pt}}_{n \text{ factors}}$$

Applying the Kunneth theorem for cohomology, we have

$$\begin{aligned} H^*(T \circlearrowright \text{pt}) &= \underbrace{H^*(T^1 \circlearrowright \text{pt}) \otimes \cdots \otimes H^*(T^1 \circlearrowright \text{pt})}_{n \text{ factors}} \\ &= \left\{ \text{polynomial functions on } \underbrace{\mathbb{T}^1 \times \cdots \times \mathbb{T}^1}_{n \text{ factors}} \right\} \\ &= \{\text{polynomial functions on } \mathbb{T}\} = \mathcal{O}(\mathbb{T}) \end{aligned}$$

where $\mathbb{T} = \mathbb{R}^n$ is the product of n copies of $\mathbb{T}^1$.

1.9. The Equivariant Cohomology of a 2-Sphere

⟨9.1⟩ Homology of the fixed point set $N \cup S$. Suppose that the circle T^1 acts on the 2-sphere $X = S^2$ by rotation as in §0.1.4. There are two fixed points, the North pole N and the South pole S. The space $N \cup S$ is just two points, so its T^1 equivariant homology is just two copies of the equivariant homology of a point:

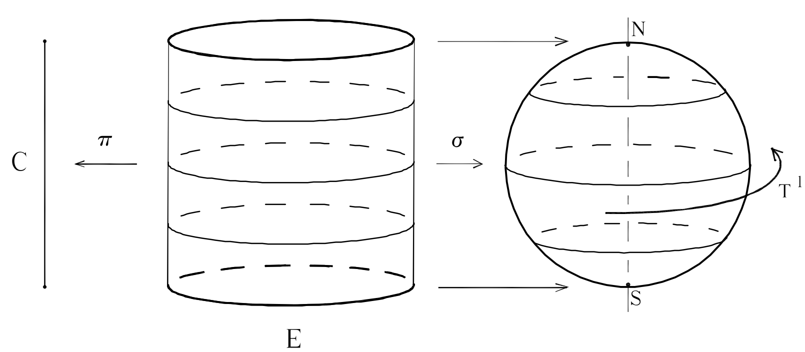
$$H_i(T^1 \circlearrowright (N \cup S)) = \begin{cases} \mathbb{R}[(\mathbb{CP}^k)_N] \oplus \mathbb{R}[(\mathbb{CP}^k)_S] = \mathbb{R}^2 & \text{if } i = 2k \\ 0 & \text{if } i \text{ is odd} \end{cases}$$

The equivariant map $N \cup S \hookrightarrow X$ is T^1 equivariant, so it induces a map on equivariant homology $H_i(T^1 \circlearrowright (N \cup S)) \longrightarrow H_i(T^1 \circlearrowright X)$.

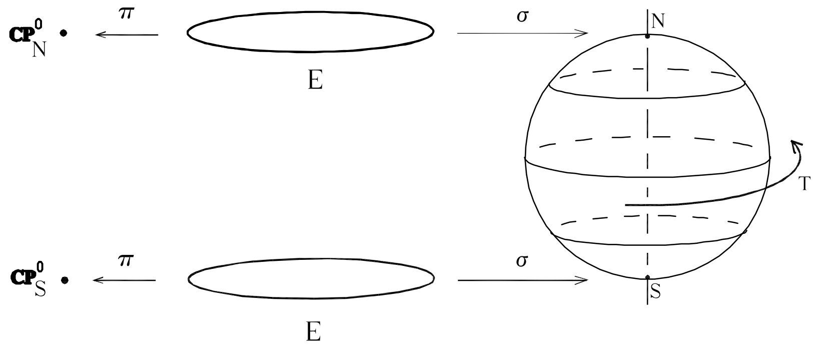
⟨9.2⟩ Circle equivariant homology of the 2-sphere. Proposition. The map

$$H_i(T^1 \circlearrowright (N \cup S)) \longrightarrow H_i(T^1 \circlearrowright X)$$

is an isomorphism for all $i > 0$. For $i = 0$, there is one relation $[\mathbb{CP}^0_N] = [\mathbb{CP}^0_S]$ given by the following cobordism whose boundary is $\mathbb{CP}^0_N - \mathbb{CP}^0_S$.



Cobordism giving the relation in $H_0(T^1 \circlearrowright X)$



The boundary of this cobordism

⟨9.3⟩ Exercise*. This result says that every equivariant cycle for $T \circlearrowright X$ is cobordant to one that maps into just the two fixed points N and S. The corresponding statement in ordinary homology (i.e. $1 \circlearrowright X$) is false. Can you see geometrically why this is true?

⟨9.4⟩ Translating this calculation to equivariant cohomology. In summary, the equivariant homology of X is a quotient of the equivariant homology of $N \cup S$; i.e. we have the exact sequence of graded vector spaces

$$0 \longrightarrow \mathbb{R} \xrightarrow{\;q\;} H_*(T^1 \circlearrowright (N \cup S)) \longrightarrow H_*(T^1 \circlearrowright X) \longrightarrow 0$$

where the map q sends 1 to $[\mathbb{CP}^0_N] - [\mathbb{CP}^0_S]$. Dualizing, the equivariant cohomology of X is a sub of the equivariant cohomology of $N \cup S$; i.e. we have the exact sequence of rings:

$$0 \longleftarrow \mathbb{R} \xleftarrow{\;q^*\;} H^*(T^1 \circlearrowright (N \cup S)) \longleftarrow H^*(T^1 \circlearrowright X) \longleftarrow 0.$$

Here $H^*(T^1 \circlearrowright (N \cup S)) = H^*(T^1 \circlearrowright N) \oplus H^*(T^1 \circlearrowright S)$ which is two copies of the ring $\mathcal{O}(\mathbb{T}^1)$ of polynomials on $\mathbb{T}^1$. The map q^* sends the difference of the identity elements of the two copies of the polynomial ring $1_N - 1_S$ to $1 \in \mathbb{R}$.

In other words, $H^*(T^1 \circlearrowright (N \cup S))$ is the ring of pairs (f_N, f_S) of polynomial functions on $\mathbb{T}^1 = \mathbb{R}$. The ring $H^*(T^1 \circlearrowright X)$ is pairs (f_N, f_S) such that $f_N(0) = f_S(0)$.

⟨9.5⟩ The torus equivariant cohomology of a sphere. Now suppose that an n torus T acts on the sphere $X = S^2$ by rotating it. By changing coordinates in the torus, we can arrange things so that $T = T^1 \times T^{n-1}$ where the circle T^1 acts on X as in the discussion above and the torus T^{n-1} acts trivially. Therefore, we have the product of spaces with group action

$$T \circlearrowright X = T^1 \circlearrowright X \times T^{n-1} \circlearrowright p.$$

Applying the Kunneth theorem, we get

$$\begin{aligned} H^*(T \circlearrowright X) &= H^*(T^1 \circlearrowright X) \otimes H^*(T^{n-1} \circlearrowright \mathrm{pt}) \\ &= \left\{(f_N, f_S) \in \mathcal{O}(T^1) \oplus \mathcal{O}(T^1) \text{ such that } f_N|0 = f_S|0\right\} \otimes \mathcal{O}(\mathbb{T}^{n-1}) \\ &= \left\{(f_N, f_S) \in \mathcal{O}(T^1 \times \mathbb{T}^{n-1}) \text{ such that } f_N|\mathbb{T}^{n-1} = f_S|\mathbb{T}^{n-1}\right\} \end{aligned}$$

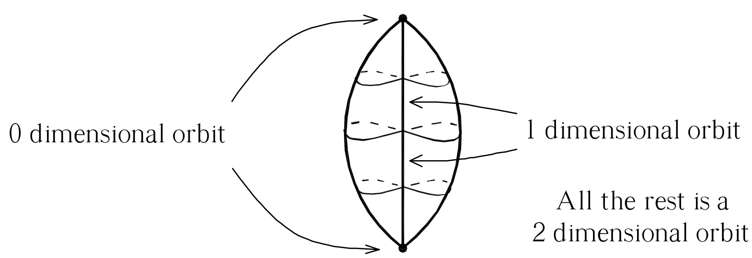
In Lecture 3, this simple calculation will lie at the root of all of our calculations of equivariant cohomology (and ordinary cohomology) of many complicated spaces.

1.10. Equivariant Intersection Cohomology

In this section, we will sketch the construction of intersection cohomology, so the reader can get the flavor.

Intersection cohomology is an invariant of pseudomanifolds. If the pseudomanifold is a manifold, then the equivariant intersection cohomology is the same as the ordinary cohomology. If the pseudomanifold is singular, then often it is the intersection cohomology (equivariant or otherwise) that is important for applications, rather than the ordinary cohomology.

⟨**10.1**⟩ Suppose X is a k-dimensional pseudomanifold of finite type. Consider the group $\mathbf{H} \circlearrowleft X$ of self-homeomorphisms of X. The group $\mathbf{H}$ is infinite dimensional, but its orbits in X are of finite type because X is of finite type. The space X will be "uniformly singular" along the orbits of $\mathbf{H} \circlearrowleft X$. For example, if X is "the suspension of ∞", we get this picture:



Orbits of the group of homeomorphisms, $\mathbf{H} \circlearrowleft X$

Let X_c be the union of all the orbits of codimension c, i.e of dimension $k - c$. The largest orbit X_0 is an open dense k-manifold in X. X_1 is empty because X is a pseudomanifold. *We assume that X_c is empty unless c is even.* This assumption holds for many spaces of interest — particularly for complex algebraic varieties, where X_c is a complex manifold, and therefore of even real dimension.

⟨**10.2**⟩ **Allowable cycles and cobordisms.** Now suppose $G \circlearrowleft X$ is a group of finite type acting on X. The group G will necessarily preserve the decomposition $X = \bigcup_c X_c$ into $\mathbf{H} \circlearrowleft X$ orbits, since G is a subgroup of $\mathbf{H}$. An *allowable* i-cycle is a diagram

$$P \xleftarrow{\ \pi\ } E \xrightarrow{\ \sigma\ } X$$

as in the definition of an equivariant i-cycle §1.6.1, that satisfies the *allowability condition*

$$\operatorname{codim} \sigma^{-1}(X_c) < \frac{c}{2}$$

where $\operatorname{codim} \sigma^{-1}(X_c)$ is the codimension of $\sigma^{-1}(X_c)$ in E .

Similarly, an *allowable* cobordism between two allowable equivariant i-cycles P_1 and P_2 is a diagram

$$\begin{array}{ccccc} C & \xleftarrow{\ \pi\ } & E & \xrightarrow{\ \sigma\ } & X \\ \text{inclusion as boundary B} \uparrow & & \text{inclusion} \uparrow & & \uparrow = \\ P_1 - P_2 & \xleftarrow{\ \pi_1, \pi_2\ } & E_1 \cup E_2 & \xrightarrow{\ \sigma_1, \sigma_2\ } & X \end{array}$$

as in the definition of a cobordism §1.6.3 satisfying the same allowability condition

$$\operatorname{codim} \sigma^{-1}(X_c) < \frac{c}{2}$$

⟨**10.3**⟩ **Definition.** The *equivariant intersection homology* $IH_i(G \circlearrowright X)$ is the allowable i-cycles modulo allowable cobordism. The ordinary (non-equivariant) intersection homology $IH_i(X)$ is $IH_i(1 \circlearrowright X)$, where 1 is the one element group.

⟨**10.4**⟩ **Remark.** The allowability condition, and particularly the appearance of $\frac{c}{2}$, is unintuitive at first. As usual, the solution is to look at lots of examples.

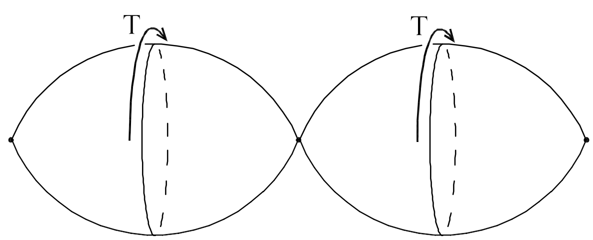
⟨**10.5**⟩ As before, we will take real coefficients by tensoring with $\mathbb{R}$. The intersection cohomology $IH^i(G \circlearrowright X)$ is the vector space dual of the intersection homology. It is no longer a ring, but $IH^*(G \circlearrowright X)$ is a graded module over the graded ring $H^*(G \circlearrowright X)$. We get the ordinary intersection homology by taking the group to be 1: $IH_i(X) = IH_i(1 \circlearrowright X)$.

⟨**10.6**⟩ **Caveat.** This definition should give the right answer ([**10**], [**23**]) but it is unproved at present. What can be proved to give the right answer at the moment differs from this in two respects that are conceptually minor but technically major:

(1) The spaces $S_c \subset X$ are taken as strata in some appropriate stratification theory. (There are various possible choices.) The strata are provably a finer decomposition than the decomposition by orbits of $\mathbf{H} \circlearrowright X$.
(2) The space X and the cycles P are taken to have extra structure like a subanalytic structure or a piecewise linear structure, and the maps preserve this structure. Nevertheless, the resulting groups are provably homeomorphism invariants.

A precise statement, discovered jointly with T. Braden, is in [**13**].

⟨**10.7**⟩ **Exercise.** Consider the example where X is two spheres with the North pole of one glued to the South pole of the other, and T is circle group which rotates both spheres simultaneously.



The various types of homology groups of X that have been defined in this lecture are given in the following table:

Type of homology	i odd	$i=0$	$i=2$	i even, $i \geq 4$
$H_i(X)$	0	$\mathbb{R}$	$\mathbb{R}^2$	0
$H_i(T \circlearrowright X)$	0	$\mathbb{R}$	$\mathbb{R}^3$	$\mathbb{R}^3$
$IH_i(X)$	0	$\mathbb{R}^2$	$\mathbb{R}^2$	0
$IH_i(T \circlearrowright X)$	0	$\mathbb{R}^2$	$\mathbb{R}^4$	$\mathbb{R}^4$

Give explicit cycles generating these groups, and give plausibility arguments that these calculations are correct. (The hardest ones are the 4 generators of $IH_i(T \circlearrowright X)$ for $i \geq 2$ and even. Each generator of $IH_2(T \circlearrowright X)$ may be represented as a 3-sphere with a free T action, mapped into X is such a way that the inverse image of a fixed point in X is a single T orbit. This has codimension 2 in the 3-sphere, so it satisfies the allowability condition 1.10.2.)

This example actually comes up. It is a generalized Schubert variety §4.7.3, and it is a Springer variety §5.4.6. The calculation methods of Lectures 3, 4, and 5 all apply to this example.

LECTURE 2
Moment Graphs

(Geometry of Orbits)

⟨**0.1**⟩ In this Lecture we will consider a space X with an action of a torus T satisfying certain conditions. We will associate to $T \circlearrowright X$ a linear graph called its moment graph. (Or more accurately, we will associate to $T \circlearrowright X$ an equivalence class of linear graphs). It turns out that interesting torus actions give rise to beautiful linear graphs. This is perhaps the first indication that the moment graph construction is a natural one to consider. We will construct the moment graphs of several classes of spaces: projective spaces, quadric hypersurfaces, Grassmannians, Lagrangian Grassmannians, flag manifolds, and toric varieties.

⟨**0.2**⟩ **Notation.** Our torus is $\mathbb{T}/L$ where $\mathbb{T}$ is an n-dimensional real vector space and L is a lattice, as in §0.1. We denote by t an element of $\mathbb{T}$ and by $\bar{t}$ its coset in $T = \mathbb{T}/L$. We reserve the symbol $\mathbb{V}$ for the dual vector space to $\mathbb{T}$ so we have an evaluation map

$$\begin{array}{ccccc} \mathbb{T} & \times & \mathbb{V} & \longrightarrow & \mathbb{R} \\ t & \times & v & \mapsto & < t, v > . \end{array}$$

In most of our examples, $\mathbb{T}$ will naturally be $\mathbb{R}^n$ and L will be $\mathbb{Z}^n$. In this case, $\mathbb{V}$ is also naturally $\mathbb{R}^n$. We write $t = (t_1, \ldots, t_n)$ and $v = (v_1, \ldots, v_n)$ so

$$< t, v >= t_1 v_1 + \cdots + t_n v_n$$

We can also think of T as $(S^1)^n$. We will denote an element of $(S_1)^n$ by $z = (z_1, \ldots, z_n)$, so $z_j = e^{2\pi i t_j}$.

2.1. Assumptions on the Action of T on X

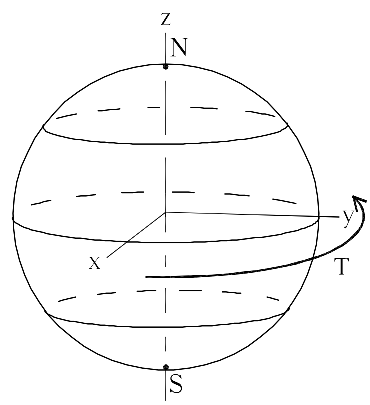
The action of the n-torus T on X decomposes it into orbits Tx each of which has a dimension that is at most n. (In fact, each orbit Tx is a k-torus where $k \leq n$.)

⟨**1.1**⟩ **Definition.** The *k-skeleton* of $T \circlearrowright X$ is the union of all the orbits of X that are of dimension at most k.

For example, the 0-skeleton is the union of the fixed points $\{x \in X \mid tx = x \text{ for all } t \in T\}$. The n-skeleton is X itself. The k-skeleton is preserved as a set by the action of T on X, so the k-skeleton is itself a space with a T action.

⟨**1.2**⟩ **Definition.** A *balloon* $T \circlearrowright B$ is a 2-sphere $B = \{x, y, z \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$ together with a linear function $D(B) : \mathbb{T} \longrightarrow \mathbb{R}$ taking L to $\mathbb{Z}$ such that $T = \mathbb{T}/L$ acts on B as follows: If $t \in \mathbb{T}$, then the projection $\bar{t}$ of t in T rotates the sphere about the z axis by an angle of $2\pi D_B(t)$, i.e.

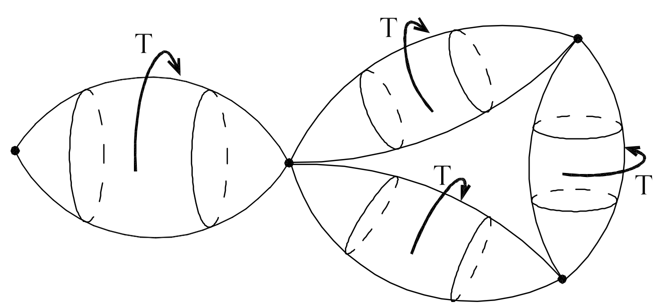
$$\bar{t} = \begin{bmatrix} \cos 2\pi D_B(t) & -\sin 2\pi D_B(t) & 0 \\ \sin 2\pi D_B(t) & \cos 2\pi D_B(t) & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$



⟨**1.3**⟩ **Exercise.** Suppose T acts on S^2 so that the orbits are the North pole $N = (0, 0, 1)$, the South pole $S = (0, 0, -1)$ and the circles of constant latitude $z = c$ where c is a constant between -1 and 1. Show that $T \circlearrowright S^2$ is equivalent to a balloon.

⟨**1.4**⟩ **Exercise*.** Show that any action of a torus T on S^2 is either a balloon or else it's the trivial action, where every point of T leaves every point of S^2 fixed.

⟨**1.5**⟩ **Definition.** A *balloon sculpture* is a space with a torus action such that is a finite union of balloons B_j such that any two balloons are either disjoint or intersect a fixed point of the torus action.

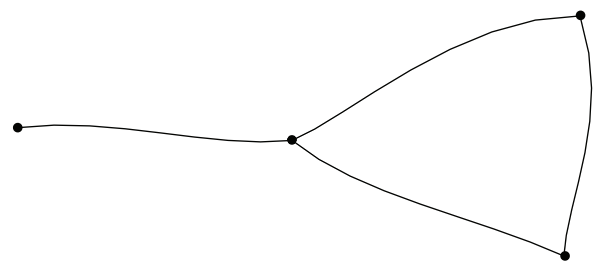


A balloon sculpture Y

⟨**1.6**⟩ **Assumption.** *We will assume that the* 1*-skeleton of* $T \circlearrowright X$ *is a balloon sculpture* (until Lecture 5). ⟨**1.7**⟩ **Exercise.** If X is compact, show that this assumption is equivalent to the assumption that the T acts with finitely many fixed points, and the 1-skeleton with the 0-skeleton deleted is a 2-dimensional manifold.

2.2. The Moment Graph

⟨**2.1**⟩ If $T \circlearrowright Y$ is a balloon sculpture, then the quotient space Y/T is a graph whose vertices correspond to the fixed points of T, and whose edges correspond to the balloons. The graph Y/T is obtained by collapsing each balloon down to a line segment.



The graph Y/T

⟨**2.2**⟩ We want to enhance the graph Y/T to a linear graph in a vector space $\mathbb{V}$, as in §0.2. To do this, we need direction data §0.2.5: To each edge of the graph, we need to give a direction $D \in \mathbb{V}$. The idea is to use D_{B_j} as the direction. It is a vector in the vector space $\mathrm{Hom}(\mathbb{T}, \mathbb{R})$, the space of all of linear maps $\mathbb{T} \longrightarrow \mathbb{R}$, i.e. $\mathbb{V}$ is the dual vector space $\mathbb{T}^*$. We will call D_{B_j} the *direction vector* of the balloon B_j or of the corresponding edge of the graph Y/T

Definition [**13**]. The moment graph of $T \circlearrowright X$ is the linear graph in $\mathbb{V} = \mathrm{Hom}(\mathbb{T}, \mathbb{R})$ obtained from the graph Y/T, where Y is the 1-skeleton of $T \circlearrowright X$, by associating the direction vector D_{B_j} to the edge corresponding to the balloon B_j. We notate the moment graph $\mathcal{G}(T \circlearrowright X)$.

⟨**2.3**⟩ **Existence and uniqueness.** The moment graph can be defined only up to equivalence §0.2.4 because we have specified it by direction data.

The direction data for the moment graph is well defined. By changing the identification of B_j with S^2, the actual direction vector D_{B_j} could be changed, but the direction in $\mathbb{V}$ would still be the same, by §2.2.6.

A moment graph will not always exist (§2.2.7). Remarkably, it does exist for the most interesting examples. We will construct it for many examples in this lecture. The general phenomenon of existence of the moment graph will be discussed in §2.9.

⟨**2.4**⟩ **Notation in $\mathbb{V}$.** When $\mathbb{T}$ is identified with $\mathbb{R}^n$, we will identify $\mathbb{V} = \mathbb{T}^* = (\mathbb{R}^n)^*$ as well. We denote the standard basis for $\mathbb{V}$ by $e_1, \ldots, e_n$, where e_i is the point where $v_i = 1$ and all of the other v_j are 0. Considered a linear map $\mathbb{T} \longrightarrow \mathbb{R}$, we have $<(t_1, \ldots, t_j, \ldots, t_n), e_j> = t_j$.

⟨**2.5**⟩ **Exercise.** Suppose that $T \circlearrowright B$ is a balloon and that $D'_B : \mathbb{T} \longrightarrow \mathbb{R}$ is a nonzero linear map such that if $D'_B(t) = 0$ then $\bar{t}$ fixes every point of B. Then D'_B is some scalar multiple of D_B, so D'_B and D_B determine the same direction in $\mathbb{V}$.

⟨**2.6**⟩ **Exercise.** Suppose that B is displayed as a balloon in two different ways, i.e. there are two different homeomorphisms equivariant homeomorphisms from B to a sphere as in the definition of a balloon. Suppose that D_B and D'_B are the corresponding functions from $\mathbb{T} \longrightarrow \mathbb{R}$. Show that D_B and D'_B determine the same direction. (You can use exercise 2.2.5.) In fact, $D'_B = \pm D_B$.

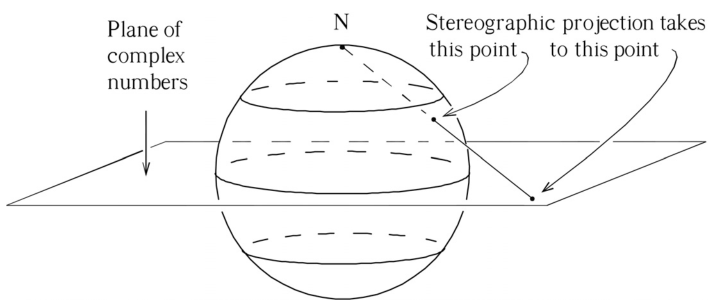
⟨2.7⟩ Exercise. Construct an example $T \circlearrowright X$ where the moment graph does not exist. (Hint: Take X to be the balloon sculpture whose direction data coincides with that of exercise 0.2.8.)

2.3. Complex Projective Line and the Line Segment

Most of the rest of this Lecture will be devoted to explicit computation of moment graphs for specific torus actions $T \circlearrowright X$.

⟨3.1⟩ Definition. The *complex projective k-space* $\mathbb{P}^k$ is the quotient space $\mathbb{C}^{k+1} - \{0\}/\mathbb{C}^\times$ where $\mathbb{C}^\times$ is multiplicative group of the complex numbers acting on $\mathbb{C}^n$ by scalar multiplication. A point in $\mathbb{P}^k$ is denoted by *homogeneous coordinates* $(x_1 : x_2 : \cdots : x_{k+1})$ where the x_j are complex numbers, not all of which are zero, and $(\lambda x_1 : \lambda x_2 : \cdots : \lambda x_{k+1})$ represents the same point in $\mathbb{P}^K$ as $(x_1 : x_2 : \cdots : x_{k+1})$ if λ is a nonzero complex number.

⟨3.2⟩ The projective line. Complex 1-space is called the projective line. Topologically, it is a 2-sphere, called the *Riemann sphere* in complex analysis. We may identify $\mathbb{P}^1 - (0 : 1)$ with the complex plane $\mathbb{C}$ by sending $(x_1 : x_2)$ to x_2/x_1. We may identify the 2-sphere minus the north pole N with the complex plane $\mathbb{C}$ by stereographic projection.



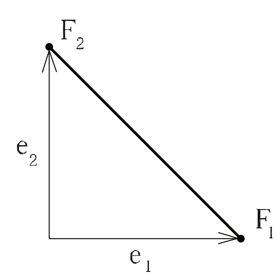
Stereographic projection takes rotation about the z axis to rotation in the complex plane about 0, i.e. to multiplication by a complex number on the unit circle S^1.

Now, suppose that the 2 torus acts on the projective line by the formula

$$z(x_1 : x_2) = (z_1 x_1 : z_2 x_2).$$

Proposition. With this action, $\mathbb{P}^1$ is a balloon B where the direction vector D_B is $e_1 - e_2$ in $\mathbb{V}$. (Here e_i is the standard basis as in §2.2.4.)

This proposition gives us the moment graph of $\mathbb{P}^1$. We must send the two vertices corresponding to the two fixed points $F_1 = (1 : 0)$ and $F_2 = (0 : 1)$ to points p_1 and p_2 in $\mathbb{V} = \mathbb{R}^2$ so that the straight line from p_2 to p_1 is parallel to the direction vector $e_1 - e_2$. An obvious choice is $p_1 = e_1$ and $p_2 = e_2$, so the moment graph is a line segment between e_1 and e_2, as in this picture.



$\mathcal{G}(T \circlearrowright \mathbb{CP}^1)$

Proof. We compute the action of T in $\mathbb{P} - (0:1) = \mathbb{C}$

$$\frac{z_2 x_2}{z_1 x_1} = \frac{e^{2\pi i t_2} x_2}{e^{2\pi i t_1} x_1} = e^{2\pi i (t_2 - t_1)} \left(\frac{x_2}{x_1}\right) = e^{2\pi i (e_2 - e_1)(t_1, t_2)} \left(\frac{x_1}{x_2}\right) = e^{2\pi i (e_2 - e_1) t} \left(\frac{x_1}{x_2}\right)$$

which means that $\bar{t}$ gives a rotation of $(e_1 - e_2)(t)$ Alternatively, the proposition can be seen by §2.2.5: If $(e_2 - e_1)(t_1, t_2) = 0$, then $t_1 = t_2$ so $(e^{2\pi i t_1} x_1 : e^{2\pi i t_2} x_2)$ is the same point as $(x_1; x_2)$ because both homogeneous coordinates are multiplied by the same number.

⟨**3.3**⟩ **Exercise.** More generally, show that if an n-torus T acts on the projective line by $\bar{t}(x_1 : x_2) = (e^{2\pi i \phi_1(t)} x_1 : e^{2\pi i \phi_2(t)} x_2)$ for $\phi_1, \phi_2 : \mathbb{T} \longrightarrow \mathbb{R}$, and $\phi_1 \neq \phi_2$, then $\mathbb{P}^1$ is a balloon with $D_B = \phi_1 - \phi_2$.

⟨**3.4**⟩ Almost all of the balloons in the 1-skeleta of the $T \circlearrowright X$ we will consider in this Lecture are themselves a copy of $\mathbb{P}^1$ embedded in the space X. So the analysis of this section will be used repeatedly in what follows.

2.4. Projective $(n-1)$-Space and the Simplex

We generalize the discussion $\mathbb{P}^1$ above. The "standard" action of the n-torus on $\mathbb{P}^{n-1}$ is

$$z(x_1 : \cdots : x_n) = (z_1 x_1 : \cdots : z_n x_n)$$

where $z = (z_1, \ldots, z_n)$ and $z_j \in S^1 \subset \mathbb{C}$.

⟨**4.1**⟩ **The fixed points** are the n points F_i where all the homogeneous coordinates are zero except the i-th one.

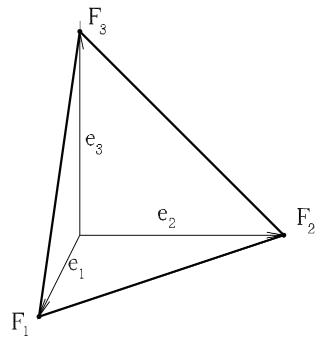
⟨**4.2**⟩ **The balloons.** Let i and j be any pair of distinct indices $1 \le i, j \le n$. Then the balloon B_{ij} is where all the homogeneous coordinates are zero except the i-th one or the j-th one. It connects the fixed points F_i and F_j.

⟨**4.3**⟩ **Remark: balloons and $\mathbb{C}^*$ orbits.** The action of $T = S_1^n$ on $\mathbb{P}^{n-1}$ extends to an action of $T_{\mathbb{C}} = (\mathbb{C}^*)^n$ where $\mathbb{C}^*$ is the nonzero complex numbers considered as a group under multiplication. The action of $T_{\mathbb{C}}$ is given by the same formula $z(x_1 : \cdots : x_n) = (z_1 x_1 : \cdots : z_n x_n)$ where $z_j \in \mathbb{C}^*$. Each balloon consists of three $T_{\mathbb{C}}$ orbits: the two fixed points and one more, of complex dimension 1. So the classification of balloons is the same as the classification of complex one dimensional orbits of the $T_{\mathbb{C}}$ action.

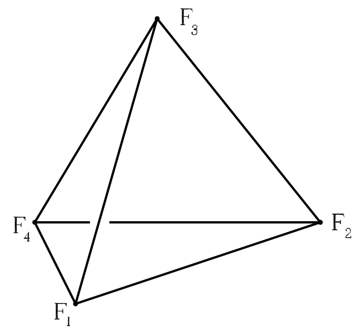
⟨**4.4**⟩ **The direction vector** of $B_{i,j}$ is $e_i - e_j$.

⟨**4.5**⟩ **The moment graph.** If we send F_i to e_i, then the straight line connecting F_i to F_j is parallel to $e_i - e_j$. So the moment graph $\mathcal{G}(T \circlearrowright \mathbb{P}^{n-1})$ of $\mathbb{P}^{n-1}$ is the 1-skeleton of the $(n-1)$-simplex Δ_{n-1}. (The $(n-1)$-simplex Δ_{n-1} is the convex hull of the basis vectors $e_1, \ldots, e_n$, or alternatively

$$\Delta_{n-1} = \{(v_1, \ldots v_n) \mid v_1 + \cdots + v_n = 1 \text{ and } v_j \ge 0\}).$$



The 2 dimensional simplex



The 3 dimensional simplex pictured in a 3 dimensional subspace of the 4 dimensional space V

⟨4.6⟩ **The proofs.** The points F_i are fixed by the equivalence relation on homogeneous coordinates. The sets $B_{i,j}$ are projective lines, so they are spheres. The action of T on $B_{i,j}$ is very similar to the action of §2.3.2, so the direction vectors can be computed in a similar way. Alternatively, §2.3.3 can be used directly. The only real challenge is to show that the 1-skeleton is the union of the balloons B_{ij}. This follows from the following exercise.

⟨4.7⟩ **Exercise.** Show that if $x \in \mathbb{P}^n$ has k nonzero homogeneous coordinates, then the dimension of the orbit Tx is $k-1$.

2.5. Quadric Hypersurfaces and the Cross-Polytope

⟨5.1⟩ **Definition of $T \circlearrowright X$.** The $(2n-2)$-*dimensional quadric hypersurface* Q_{2n-2} is the subset of $\mathbb{P}^{2n-1}$ with homogeneous coordinates $(x_1 : \cdots : x_n : y_1 : \cdots : y_n)$ cut out by the equation $x_1y_1 + x_2y_2 + \cdots + x_ny_n = 0$. (This makes sense because if $(x_1 : \cdots : x_n : y_1 : \cdots : y_n)$ satisfies the equation, then so will $(\lambda x_1 : \cdots : \lambda x_n : \lambda y_1 : \cdots : \lambda y_n)$.)

The n-torus T acts on $X = Q_{2n-2}$ by the formula

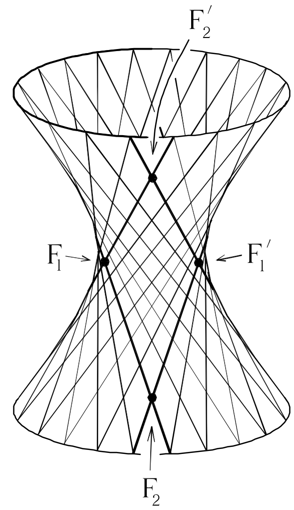
$$z(x_1 : \cdots : x_n : y_1 : \cdots : y_n) = (z_1x_1 : \cdots : z_nx_n : z_1^{-1}y_n : \cdots : z_n^{-1}y_n)$$

You can check that this formula is compatible with the equivalence relation on homogeneous coordinates defining $\mathbb{P}^{2n-1}$ and that it preserves the equation for the hypersurface Q_{2n-2}.

⟨5.2⟩ **The fixed points** are the points where exactly one homogeneous coordinate is nonzero. Let's call F_i the point where x_i is nonzero and F_i' the point where y_i is nonzero.

⟨5.3⟩ **The balloons.** For every pair of homogeneous coordinates except the n pairs $\{x_i, y_i\}$, the points in X where only that pair of homogeneous coordinates is nonzero is a projective line. These are the balloons. So there is a balloon connecting any pair of fixed points except the n pairs with the same index, F_i and F_i'. So the number of balloons is $\binom{2n}{2} - n = \frac{(2n)(2n-1)}{2} - n$, where $\binom{2n}{2}$ is the number of 2 element subsets of a $2n$ element set.

⟨5.4⟩ **Real picture for $n = 2$.** We can't draw any interesting complex quadrics, because their dimensions are too large. However, we can draw the real quadric $Q^{\mathbb{R}}_{2n-2}$ for $n = 2$. It is the surface $x_1y_1 + x_2y_2 = 0$ in $\mathbb{RP}^3$. The real projective space $\mathbb{RP}^3$ contains the real affine space $\mathbb{R}^3$ as a dense subspace. The intersection of the quadric with $\mathbb{R}^3$ is pictured at the right. It is doubly ruled surface. The four fixed points F_1, F_2, F_1', F_2' lie in $Q^{\mathbb{R}}_2 \subset Q_2$. Each of the four balloons in Q_2 is a $\mathbb{CP}^1$, it intersects $\mathbb{RP}^3$ in a $\mathbb{RP}^1$, which intersects $\mathbb{R}^3$ in a straight line. These 4 points and 4 lines are shown on the picture. Just as the balloons are the closures of the complex 1-dimensional $T_{\mathbb{C}}$ orbits §2.4.3, these 4 real lines are the closures of the real 1-dimensional $T_{\mathbb{R}}$ orbits, where $T_{\mathbb{R}} = (\mathbb{R}^*)^2$ acts by the same formulas as in the complex case.



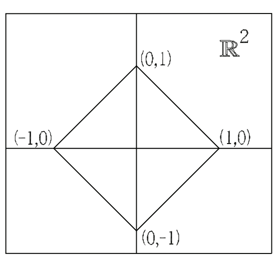
⟨5.5⟩ **The direction vectors.** For the balloon B joining F_i and F_j, the direction vector D_B is $e_i - e_j$. For the balloon B joining F_i' and F_j', D_B is $-e_i + e_j$. For the balloon B joining F_i and F_j' for $i \neq j$, D_B is $e_i + e_j$.

⟨5.6⟩ **Exercise.** Verify this calculation of direction vectors. Hint: use §2.3.3.

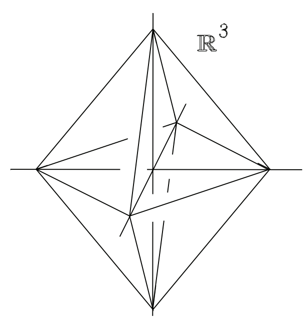
⟨5.7⟩ **The n-dimensional cross-polytope** O_n is the polyhedron in $\mathbb{V} = \mathbb{R}^n$ defined by the relation that the sum of the absolute values of the coordinates is at most 1.

$$O_n = \{(v_1, \dots, v_n) \in \mathbb{R}^n \mid |v_1| + \cdots + |v_n| \leq 1\}$$

The cross-polytopes in dimensions 2 and 3 are the square and the octahedron.



The square, O_2



The octahedron, O_3

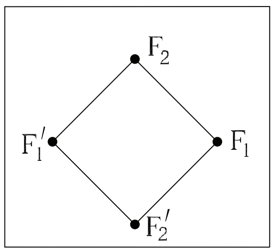
The vertices of the cross-polytope O_n are the $2n$ points $\{e_1, \dots, e_n; -e_1, \dots, -e_n\}$ where the e_i are the standard basis vectors for $\mathbb{R}^n$. The convex polyhedron O_n can be defined alternatively as the convex hull of this set of vertices.

⟨5.8⟩ **Exercise.** Show that there is an edge between any pair of vertices except for the n pairs $\{e_i, -e_i\}$ so that the number of edges is $\binom{2n}{2} - n = \frac{(2n)(2n-1)}{2} - n$, where $\binom{2n}{2}$ is the number of 2 element subsets of a $2n$ element set.

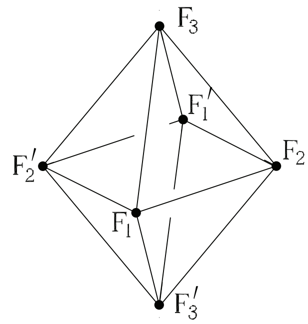
⟨5.9⟩ Exercise. Show that the number of faces of dimension i in the cross-polytope O_n is the coefficient of q^{i+1} in the polynomial $(1+2q)^n$.

⟨5.10⟩ The moment graph of the $(2n-2)$-dimensional quadric hypersurface X is the 1-skeleton of the cross-polytope O_n.

More explicitly, we define a map from the set of fixed points to $\mathbb{V} = \mathbb{R}^n$ by sending F_i to e_i and sending F_i' to $-e_i$. Then for every pair of fixed points connected by a balloon, the direction vector of that balloon is parallel to the line connecting the corresponding points in $\mathbb{V}$.

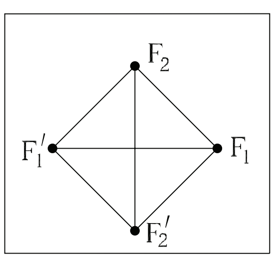


Moment graph $\mathcal{G}(T \circlearrowright Q_2)$

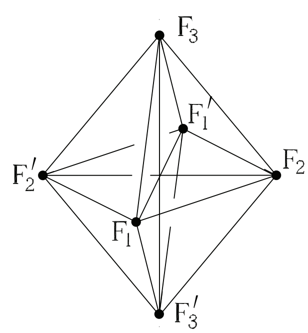


Moment graph $\mathcal{G}(T \circlearrowright Q_4)$

⟨5.11⟩ Odd dimensional quadric hypersurfaces. The $(2n-1)$-*dimensional quadric hypersurface* Q_{2n-1} is the subset of $\mathbb{P}^{2n}$ with homogeneous coordinates $(w : x_1 : \cdots : x_n : y_1 : \cdots : y_n)$ cut out by the equation $w^2 + x_1y_1 + x_2y_2 + \cdots + x_ny_n = 0$. The n-torus T acts on the x and the y coordinates as before, and it acts trivially on w. So it contains the $(2n-2)$-dimensional quadric hypersurface Q_{2n-2} as the T invariant subspace where $w = 0$. The fixed points of Q_{2n-1} are the same as the fixed points of Q_{2n-2}, but there are n additional balloons: namely, the subspace where only x_i, y_i, and w are nonzero is a balloon connecting F_i and F_i'. The moment graph for Q_{2n-1} is the moment graph for Q_{2n-2} with n additional straight lines connecting F_i to F_i'.



Moment graph $\mathcal{G}(T \circlearrowright Q_3)$



Moment graph $\mathcal{G}(T \circlearrowright Q_5)$

(The moment graph $\mathcal{G}(T \circlearrowright Q_5)$ is the PCMI logo.)

2.6. Grassmannians and Hypersimplices

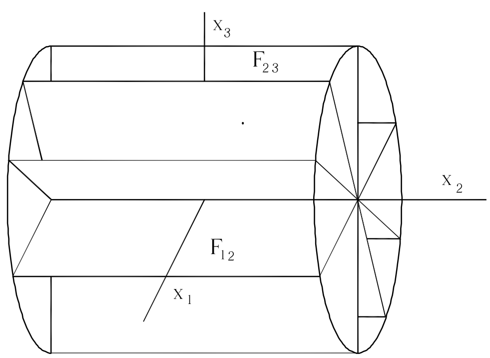
A point in projective space $\mathbb{P}^{n-1}$ represents a line through the origin in the vector space $\mathbb{C}^n$: The points in the line are the different homogeneous coordinates that represent the point in projective space. Similarly, we can make a space whose points represent subspaces of higher dimension in $\mathbb{C}^n$. This leads to various kinds of Grassmannian varieties.

⟨**6.1**⟩ **The space** G_i^n is the Grassmannian variety whose points are the i-dimensional subspaces of the n-dimensional complex vector space $\mathbb{C}^n$. The n-torus acts on it through its action on $\mathbb{C}^n$: $z(x_1, \dots, x_n) = (z_1 x_1, \dots, z_n x_n)$.

⟨**6.2**⟩ **The fixed points.** Suppose that S is a subset of $\{1, 2, \dots, n\}$. Let P_S be the coordinate plane corresponding to S, i.e. P_S is the $|S|$-plane defined by the condition that only the coordinates $\{x_j \mid j \in S\}$ can be nonzero. Here $|S|$ is the number of elements of S. The fixed points in G_i^n are the planes P_S where $|S| = i$. We denote P_S by F_S when thinking of it as a fixed point in G_i^n.

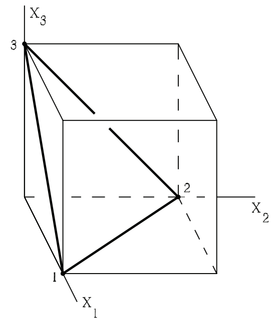
⟨**6.3**⟩ **The balloons.** Suppose S' is obtained from S by deleting the number j and adding number k, for $j \neq k$. Then the set of i-dimensional subspaces that contain $P_{S \cap S'}$ and are contained in $P_{S \cup S'}$ is a balloon connecting F_S and $F_{S'}$. The **direction vector** of this balloon is $e_j - e_k$.

⟨**6.4**⟩ Here is a picture of the planes in the balloon connecting $F_{\{1,2\}}$, and $F_{\{2,3\}}$ in G_2^3. Since we can't visualize $\mathbb{C}^3$, we're using a real picture, i.e. real planes in the real vector space $\mathbb{R}^3$ instead of complex planes in the complex vector space $\mathbb{C}^3$.

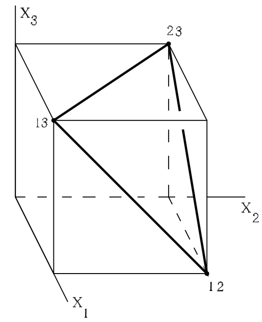


Points in a balloon in G_2^3

⟨**6.5**⟩ **The hypersimplex** Δ_i^n is the intersection of the n-cube $[0,1]^n \subset \mathbb{R}^n = \mathbb{V}$ with the plane $v_1 + v_2 + \cdots + v_n = i$. It is a convex polyhedron with vertices $\nu_S = \Sigma_{j \in S} e_j$ where S is an i element subset of $\{1, \dots, n\}$. The vertices ν_S and $\nu_{S'}$ are connected by an edge if S' is obtained from S by deleting the number i and adding number j, for $i \neq j$. Then the edge is parallel to $e_i - e_j$.



The hypersimplex Δ_1^3

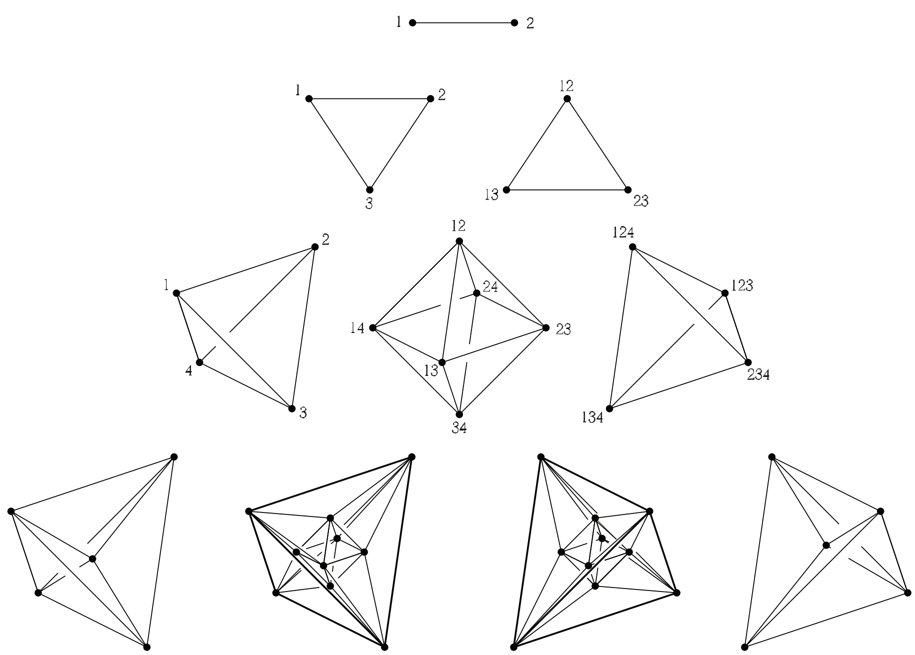


The hypersimplex Δ_2^3

⟨**6.6**⟩ **The moment graph** of the Grassmannian G_i^n is the 1-skeleton of the hypersimplex Δ_i^n. There is a rich theory surrounding hypersimplices and Grassmannians [**12**], [**11**], [**6**].

⟨**6.7**⟩ **Exercise.** Show that the hypersimplices can be arranged in a polyhedral version of Pascal's triangle where the faces of each polyhedron are isomorphic to one of the two polyhedra lying above it.

For dimension up to 4, this is illustrated in the following picture. The labels of vertices show which coordinates are 1 (or equivalently, which coordinate axes are in the corresponding plane representing a T fixed point of the Grassmannian). The figures in last line, representing 4-dimensional hypersimplices, are projections to $\mathbb{R}^3$ called Schlegel diagrams. Note that the polyhedra on the two upper edges of the picture are ordinary simplices.



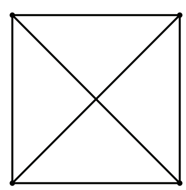
Pascal's triangle of hypersimplices

⟨6.8⟩ The Lagrangian Grassmannian and the cube. Consider $\mathbb{C}^{2n}$ with coordinates $x_1, \ldots, x_n, y_1, \ldots, y_n$ and the alternating form $\Sigma_i x_i y_i' - x_i' y_i$. The *Lagrangian Grassmannian* L_n is the subvariety of the Grassmannian G_n^{2n} consisting of n planes on which this alternating form vanishes identically. The torus T acts on L_n by through its action on $\mathbb{C}^{2n}$ by the formula

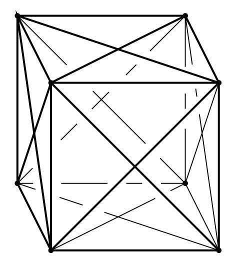
$$z(x_1, \ldots, x_n, y_1, \ldots, y_n) = (z_1 x_1, \ldots, z_n x_n, z_1^{-1} y_n, \ldots, z_n^{-1} y_n).$$

The fixed points F_S are the coordinate planes that lie in L_n. For any subset $S \subset \{1, \ldots, n\}$, F_S is the plane whose nonzero coordinates are the x_i for $i \in S$ and the y_i for $i \notin S$. There are 2^n of them.

Exercise. Show that the vertices of the moment graph of L_n are the vertices of the n-cube $[0,1]^n \subset \mathbb{V}$ and the edges of the moment graph are the edges of the cube together with the diagonals of the 2-dimensional faces.



The moment graph $\mathcal{G}(T \circlearrowleft L_2)$



The moment graph $\mathcal{G}(T \circlearrowleft L_3)$

2.7. The Flag Manifold and the Permutahedron

⟨7.1⟩ The flag manifold. Consider $\mathbb{C}^n$ as $\mathbb{R}^{2n}$ in the usual way, with the standard real dot product $\cdot_{\mathbb{R}}$ on it. A point in the *flag manifold* F_n is an ordered set of n mutually orthogonal complex lines through the origin in $\mathbb{C}^n$. Here mutually orthogonal means that if x is in one of the complex lines and y is in another one, then $x \cdot_{\mathbb{R}} y = 0$. (This is the same as their being orthogonal with respect to the standard Hermitian inner product.)

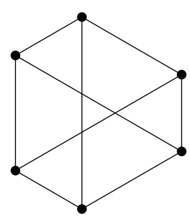
The n-torus T acts on F_n through its standard action on $\mathbb{C}^n$. This action preserves the orthogonality condition.

⟨7.2⟩ Fixed points. A point is fixed if the n mutually orthogonal lines coincide with the complex coordinate axes in $\mathbb{C}^n$. There are $n!$ of them, one for each ordering of the coordinate axes.

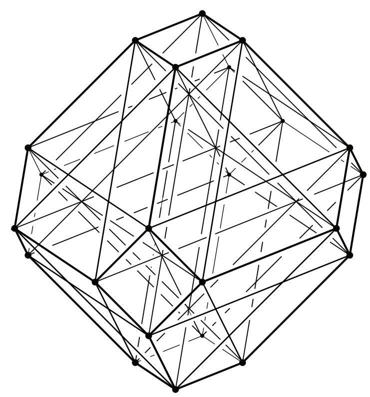
⟨7.3⟩ The balloons. Pick two coordinate axes of $\mathbb{C}^n$, say the x_i axis and the x_j axis. A balloon is the set where all but two of the mutually orthogonal complex lines are required to lie on a coordinate axis that is not the x_i axis or the x_j axis. The remaining two complex lines are free to wander (staying orthogonal to each other) in the 2-dimensional plane spanned by the x_i axis and the x_j axis.

⟨7.4⟩ The permutahedron. Fix n distinct real numbers $a_1, \ldots, a_n$. The permutahedron is the convex hull in $\mathbb{R}^n$ of the $n!$ points $(a_{\sigma(1)}, \ldots a_{\sigma(n)})$ where σ runs through the $n!$ permutations of the numbers $\{1, \ldots, n\}$. It is an $(n-1)$-dimensional polytope because it lies in a hyperplane in $\mathbb{R}^n$ where the sum of the coordinates is constant since the sum of the coordinates of the vertices is constant.

$\langle$**7.5**$\rangle$ **The moment graph of the flag manifold.** The vertices of the moment graph for F_n are the vertices of the permutahedron. Two vertices are connected by an edge if one is a reflection of the other in one of the $\binom{n}{2}$ hyperplanes defined by an equation $v_i = v_j$.



Moment graph
$\mathcal{G}(T \circlearrowright F_3)$



Moment graph $\mathcal{G}(T \circlearrowright F_4)$

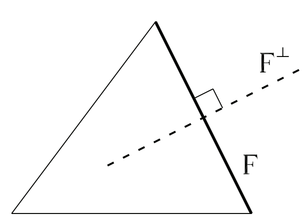
2.8. Toric Varieties and Convex Polyhedra

So far, we have begun with a space with a torus action $T \circlearrowright X$ and we have computed the moment graph $\mathcal{G}(T \circlearrowright X)$. In this section we go the other way. We give ourselves a rational convex polyhedron in a vector space $\mathbb{V}$, and we associate to it a space with a torus action $T \circlearrowright T(P)$ called the toric variety associated to P. The moment graph of $T \circlearrowright T(\mathbb{V})$ is the 1-skeleton of P.

We recall our notational conventions: $\mathbb{V}$ is the real vector space $\mathbb{R}^n$, $\mathbb{T}$ is its dual vector space, also $\mathbb{R}^n$, and L is the lattice $\mathbb{Z}^n \subset \mathbb{T}$.

$\langle$**8.1**$\rangle$ **Rational polyhedra.** A convex n-dimensional polyhedron P in the real vector space $\mathbb{V} = \mathbb{R}^n$ is called *rational* if all of its vertices lie in $\mathbb{Q}^n$, i.e. all the coordinates of its vertices are rational numbers.

$\langle$**8.2**$\rangle$ $F^\perp$**.** Given a face F of the polyhedron $P \subset \mathbb{V}$, we will denote by $F^\perp$ the vector subspace of $\mathbb{T} = \mathbb{V}^*$ consisting of vectors which are perpendicular F, i.e. the set of all $t \in \mathbb{T}$ such that $<t, v - v'> = 0$ for every pair of points $v, v' \in F$. If F is a vertex of P, then $F^\perp = \mathbb{V}$. If F is P itself, then $F^\perp$ is just the zero vector $0 \in \mathbb{T}$.



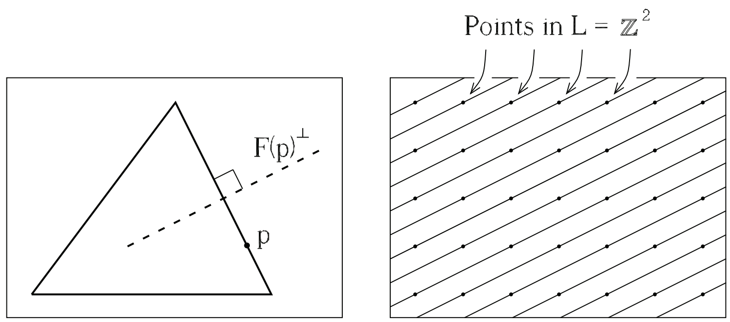
⟨8.3⟩ $F(p)$. Given a point $p \in P$ of a polyhedron, we write $F(p)$ for the smallest face of P containing p. If p is a vertex, then $F(p)$ is p itself. $F(p) = P$, if and only if p is an interior point of P.

⟨8.4⟩ The toric variety $\mathbf{T}(P)$ is the quotient space

$$\mathbf{T}(P) = \frac{P \times \mathbb{T}}{\sim}$$

where $\sim$ is the following equivalence relation:

$$(p, t) \sim (p', t') \quad \text{if and only if} \quad p = p' \ \text{ and } \ t \cong t' \bmod (F(p)^{\perp} + L)$$



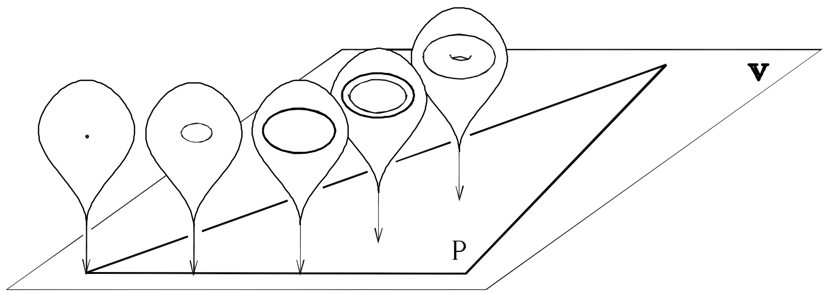
$p \in P \subset \mathbb{V}$ The subgroup $F(p)^{\perp} + L$ in $\mathbb{T}$

⟨8.5⟩ The T action. The torus $T = \mathbb{T}/L$ acts on the toric variety $\mathbf{T}(P)$ as follows: $\mathbb{T}$ acts on $P \times \mathbb{T}$ by vector addition $t(p, t') = (p, t' + t)$, and this action passes to an action of $\mathbb{T}$ on the quotient space $\mathbf{T}(P)$. On the quotient space, L acts trivially, since if $t \in L$, then $t(p, t') \sim (p, t')$ So the quotient group $\mathbb{T}/L$ acts on the quotient space $\mathbf{T}(P)$.

⟨8.6⟩ The moment map. There is a map $\mu : \mathbf{T}(P) \longrightarrow P$ called the *moment map* which is induced from the projection $(P \times \mathbb{T}) \longrightarrow P$. The reason the projection passes to the quotient $\mathbf{T}(P)$ is that the equivalence relation $\sim$ is compatible with this map — it identifies points only if they lie in the same fiber. In fact, there is an identification $\mathbf{T}(P)/T \approx P$, the moment map $\mathbf{T}(P) \longrightarrow \mathbf{T}(P)/T$ is the quotient map for the group action $T \circlearrowright \mathbf{T}(P)$.

Proposition. The fiber $\mu^{-1}p \subset \mathbf{T}(P)$ over a point $p \in P$ is a torus of the same dimension as the face $F(p)$.

So we can think of the toric variety $\mathbf{T}(P)$ as a family of tori over the polyhedron P whose fiber dimensions decrease as you get to smaller faces. To visualize it, here are some pictures of fibers at various points of P.

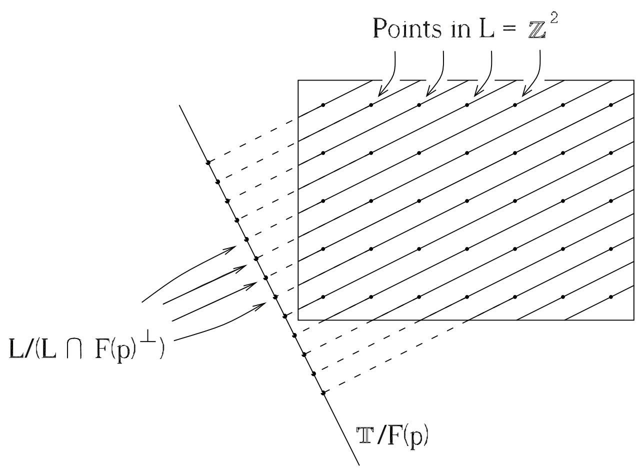


Moment map fibers $\mu^{-1}(p)$ for various $p \in P$

The torus $\mu^{-1}(p)$ over the a point p in the interior of the polyhedron P becomes thinner, looking more like a bicycle tire than a car tire, as p approaches an edge. It collapses into a circle when p reaches the edge. The circle $\mu^{-1}(p)$ over a point p in an edge becomes smaller as p approaches a vertex, and it collapses into a point when p reaches the vertex.

⟨**8.7**⟩ **Proof of the proposition.** Why is the fiber $\mu^{-1}(p)$ a torus? It is a single orbit of the action of the torus T, so it must be a torus if it is a Hausdorff space. But why is it Hausdorff? We have

$$\mu^{-1}p = \mathbb{T}/(L + F(p)^{\perp}) = \frac{\mathbb{T}/F(p)}{L/(L \cap F(p)^{\perp})}$$



We must show that $L/(L \cap F(p)^{\perp})$ is a lattice in $\mathbb{T}/F(p)$. Since this quotient space will itself be a torus: it will be the vector space $\mathbb{T}/F(p)^{\perp}$ modulo the lattice $L/(L \cap F(p)^{\perp})$

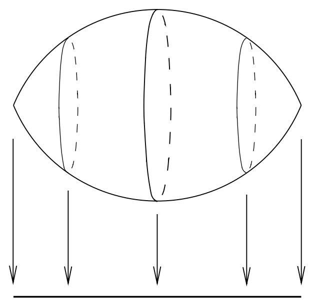
⟨**8.8**⟩ **Exercise.** Show that the following conditions are equivalent, and they all hold if the polytope P is rational:

(1) The vector space $F^{\perp}$ is a rational subspace of $\mathbb{T}$ for all faces F.
(2) The vector space $F^{\perp}$ is spanned by $F^{\perp} \cap L$ for all faces F.
(3) The quotient space $\mathbb{T}/(L + F^{\perp})$ is Hausdorff for all faces F.
(4) The subgroup $L/(L \cap F(p)^{\perp})$ is a lattice in the vector space $\mathbb{T}/F(p)$.

(5) The toric variety $(P \times \mathbb{T})/\sim$ is Hausdorff.

⟨8.9⟩ Proposition. The moment graph of the toric variety $T(P)$ is the 1-skeleton P^1 of P.

Since the dimension of the orbit $\mu^{-1}(p)$ is the dimension of $F(p)$, the 1-skeleton of $T \circlearrowright \mathbf{T}(P)$ is the inverse image of the 1-skeleton of P. The inverse image of an edge of P is a balloon.



It remains to see that the direction vector of this balloon is parallel to the edge. This follows from §2.2.5.

⟨8.10⟩ Exercise. Show that the projective $(n-1)$-space $\mathbb{P}^{n-1}$ is a toric $\mathbf{T}(P)$ where P is an $(n-1)$-simplex.

⟨8.11⟩ Simple polytopes. A polytope is *simple* if the edges coming in to every vertex, considered as vectors, are linearly independent. For example, a tetrahedron and a cube are simple, whereas an octahedron is not. All 2-dimensional polyhedra are simple.

Toric varieties of simple polytopes play a special role that will become apparent later (§3.8.1).

2.9.* Moment Maps

This is a * starred section, meaning that its prerequisites go beyond those of the other sections, and its results are not needed for the rest of what we will do. The purpose is to provide an orientation for going further in the subject, and to show how the material ties in to other mathematical ideas.

⟨9.1⟩ The Lie algebra. Our torus T is a compact Lie group. The vector space $\mathbb{T}$ is its Lie algebra. The map $\mathbb{T} \longrightarrow T$ is the exponential map of Lie theory and the lattice L is its kernel. In general, the exponential map is not a group homomorphism, but it is for the Lie group T, since T is Abelian. If $T \circlearrowright X$ and X is smooth, every $t \in \mathbb{T}$ gives rise to a vector field on X which we notate $x \mapsto t(x)$.

⟨9.2⟩ Complex algebraic varieties. All of the spaces X we have constructed in this section are complex projective algebraic varieties. Our torus $T = (S^1)^n$ is the maximal compact subgroup of a complex torus $T_{\mathbb{C}} = (\mathbb{C}^*)^n$ which is an algebraic group. The action of T extends to an algebraic action of $T_{\mathbb{C}}$. The fixed points of T are still fixed under $T_{\mathbb{C}}$. The real dimension of a T orbit Tx is the complex dimension of the $T_{\mathbb{C}}$ orbit $T_{\mathbb{C}}x$. If B is one of the balloons and N and S are the two fixed points on it, then $B - N - S$ is a single orbit of $T_{\mathbb{C}}$ of complex dimension

1. These are all the 1 complex dimensional orbits of $T_{\mathbb{C}}$. If we are given a complex algebraic action of $T_{\mathbb{C}}$ on X, then our hypothesis that the 1-skeleton of $T \circlearrowright X$ is a balloon sculpture is equivalent to the hypothesis that $T_{\mathbb{C}}$ has finitely many orbits of complex dimension 0 and 1.

⟨9.3⟩ The moment map. If X is nonsingular and projective, then it has a real symplectic form ω called the Kähler form. By Weyl's trick of averaging over T, we can choose ω to be T invariant. We define a $\mathbb{V}$-valued differential 1-form θ on X as follows: For $t \in \mathbb{T}$, let ξ_t be the corresponding vector field on X. If $\tau \in T_x X$ is a tangent vector to X at x, then $t \mapsto \omega(\tau, \xi_t(x))$ is a linear map $\mathbb{T} \longrightarrow \mathbb{R}$, so it is an element of $\mathbb{V} = \mathbb{T}^*$. That element is $\theta(\tau)$. The moment map $\mu : X \longrightarrow \mathbb{V}$ is defined by the formula

$$\mu(x) = \int_{x_0}^{x} \theta$$

where x_0 is a base point chosen in X. (If X is not connected, we define μ on each connected component separately by this procedure.)

If X is singular, we proceed a little differently. We embed X in a complex projective space in a way that is $T_{\mathbb{C}}$ equivariant. Then we take the moment map on the ambient complex projective space as constructed above, and restrict it to X.

If $T \circlearrowright X$ is a toric variety, then the moment map as defined here will coincide with the moment map from its definition as a toric variety.

⟨9.4⟩ Proposition. If $T_{\mathbb{C}}$ acts algebraically on X with finitely many orbits of dimension 0 and 1, then the moment graph $\mathcal{G}(T \circlearrowright X)$ is $\mu(X^1)$, the moment map image of its 1-skeleton. The set of vertices of the moment graph is $\mu(X^0)$. The image $\mu(X)$ of all of X will be the convex hull of the moment graph. There were

several choices in constructing the moment map (choice of a Kähler form, choice of a base point). Different choices will result in different but equivalent linear graphs.

⟨9.5⟩ Exercise*. Suppose X is nonsingular and compact, and that $T_{\mathbb{C}}$ acts algebraically on X with finitely many fixed points F. Suppose further that at each fixed F, the representation $T_{\mathbb{C}} \circlearrowright T_F X$ on the tangent space has no representation of multiplicity greater than 1. Show that $T_{\mathbb{C}}$ acts with finitely many one dimensional orbits, so that the 1-skeleton of $T \circlearrowright X$ is a balloon sculpture.

LECTURE 3
The Cohomology of a Linear Graph

(Polynomial and Linear Geometry)

We will attach a cohomology ring to any linear graph $\mathcal{G}$. Most of this section is a study of this ring and how to compute it. Then section 3.8 contains the main theorem: if the linear graph $\mathcal{G}$ is a moment graph $\mathcal{G}(T \circlearrowright X)$, then the cohomology ring of $\mathcal{G}$ is the equivariant cohomology ring of $T \circlearrowright X$.

3.1. The Definition of the Cohomology of a Linear Graph

⟨1.1⟩ Notations. Suppose $\mathcal{G}$ is a linear graph. We will call the vertices $\nu, \nu', \ldots$ If ν and ν' are connected by an edge, we will call it $\ell_{\nu\nu'}$. The graph is embedded in a real n-dimensional vector space $\mathbb{V}$, whose dual vector space is $\mathbb{T}$. For every edge $\ell_{\nu\nu'}$, let $\ell_{\nu\nu'}^{\perp}$ be the $(n-1)$-dimensional subspace of $\mathbb{T}$ consisting of vectors that are orthogonal to the straight line $\ell_{\nu\nu'}$. Let $\mathcal{O}(\mathbb{T})$ be the ring of real polynomial functions $f : \mathbb{T} \longrightarrow \mathbb{R}$ graded so that the grading degree is twice the degree of the function.

Definition [13]. Consider the ring

$$\bigoplus_{\text{vertices } \nu \text{ of } \mathcal{G}} \mathcal{O}(\mathbb{T}).$$

An element of this ring is a polynomial function $f_\nu : \mathbb{T} \longrightarrow \mathbb{R}$ for each vertex ν of $\mathcal{G}$. We can notate such an element $\{f_\nu, f_{\nu'}, \ldots\}$ The *cohomology* of $\mathcal{G}$, $\mathcal{H}^*(\mathcal{G})$ is the subring of this cut out by the requirement that for every edge $\ell_{\nu\nu'}$ of $\mathcal{G}$, we have the *restriction condition:*

$$f_\nu | \ell_{\nu\nu'}^{\perp} = f_{\nu'} | \ell_{\nu\nu'}^{\perp}.$$

In other words, the restriction condition requires that if the vertices ν and ν' are connected by an edge $\ell_{\nu\nu'}$, then the polynomial f_ν and the polynomial $f_{\nu'}$ must have the same restriction to the space $\ell_{\nu\nu'}^{\perp}$.

For a useful reformulation of this definition, see §4.3.5.

⟨1.2⟩ Exercise. Show that $\mathcal{H}^*(\mathcal{G})$ is a subring of $\bigoplus_\nu \mathcal{O}(\mathbb{T})$.

⟨1.3⟩ Graded structure. The ring $\mathcal{H}^*(\mathcal{G})$ is a graded ring

$$\mathcal{H}^*(\mathcal{G}) = \bigoplus_{i \geq 0} \mathcal{H}^i(\mathcal{G})$$

where $\mathcal{H}^i(\mathcal{G}) = 0$ if i is odd, and $\mathcal{H}^{2k}(\mathcal{G})$ is the set of elements represented by sets of polynomials $\{f_\nu, \ldots\}$, each of which is homogeneous of degree k (i.e. every term of f_ν is of degree k). If $\alpha \in \mathcal{H}^i(\mathcal{G})$ and $\beta \in \mathcal{H}^j(\mathcal{G})$, then the product $\alpha\beta \in \mathcal{H}^{i+j}(\mathcal{G})$.

⟨**1.4**⟩ **Module structure.** The ring $\mathcal{H}^*(\mathcal{G})$ is a graded module over the graded ring $\mathcal{O}(\mathbb{T})$ of polynomial functions on $\mathbb{T}$. The module action of $g \in \mathcal{O}(\mathbb{T})$ sends $\{f_\nu, f_{\nu'}, \ldots\} \in \mathcal{H}^*(\mathcal{G})$ to $\{gf_\nu, gf_{\nu'}, \ldots\}$.

⟨**1.5**⟩ **Restriction.** Suppose that we have an inclusion of linear graphs $\mathcal{G}' \subset \mathcal{G}$, i.e. $\mathcal{G}'$ has some of the vertices of $\mathcal{G}$ and some of the edges. Then the projection

$$\bigoplus_{\text{vertices } \nu \text{ of } \mathcal{G}} \mathcal{O}(\mathbb{T}) \longrightarrow \bigoplus_{\text{vertices } \nu \text{ of } \mathcal{G}'} \mathcal{O}(\mathbb{T})$$

induces a map

$$\mathcal{H}^*(\mathcal{G}) \longrightarrow \mathcal{H}^*(\mathcal{G}').$$

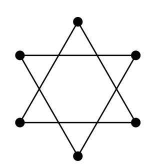
⟨**1.6**⟩ **Exercise.** Show that the graded structure, the module structure, and the restriction, as defined above, make sense – for example that they respect the condition for each edge of $\mathcal{G}$ in the definition of $\mathcal{H}^*(\mathcal{G})$.

⟨**1.7**⟩ Sections 3.2 to 3.7 will be devoted to the study of the cohomology ring of a graph. The definition is simple enough, but it is not immediately clear from the definition how you would compute it or how to think about it. The papers of Guillemin, Holm, and Zara are recommended for further reading [**18**], [**19**], [**20**], [**21**], [**22**].

3.2. Interpreting $\mathcal{H}^i(\mathcal{G})$ for Small i

In this section, we will give interpretations for $i = 0, 2,$ or 4.

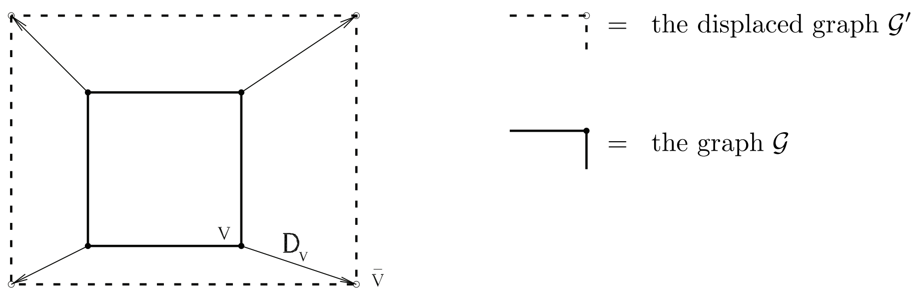
⟨**2.1**⟩ **The degree 0 part of the cohomology.** The dimension of the vector space $\mathcal{H}^0(\mathcal{G})$ is the number of connected components of the topological graph associated to $\mathcal{G}$. (The topological graph of the figure at the right consists of two disjoint triangles.) **Exercise.** Prove this.



$H^0(\mathcal{G}) = \mathbb{R} \oplus \mathbb{R}$

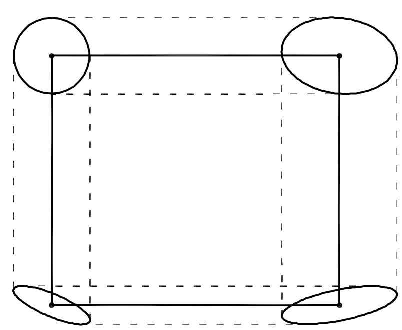
⟨**2.2**⟩ **The degree 2 part of the cohomology.** The dimension of the vector space $\mathcal{H}^2(\mathcal{G})$ is the dimension of the space of graphs in $\mathbb{V}$ that are equivalent to $\mathcal{G}$ (see §0.2.4).

⟨**2.3**⟩ **Proof.** An element of $\bigoplus_{\text{vertices } \nu \text{ of } \mathcal{G}} \mathcal{O}(\mathbb{T})^2$ is a linear function on $\mathbb{T}$ for every vertex ν of the graph $\mathcal{G}$. But a linear function on $\mathbb{T}$ is a vector D_ν in $\mathbb{V}$. Draw the vector D_ν as an arrow, and put its tail at the vertex ν and call its head $\bar{\nu}$. We will consider D_ν a displacement of ν to a new vertex $\bar{\nu}$ of a new graph $\bar{\mathcal{G}}$. For every edge $\ell_{\nu\nu'}$ the restriction condition that $f_\nu|\ell^\perp_{\nu\nu'} = f_{\nu'}|\ell^\perp_{\nu\nu'}$ translates in to the condition that the line $\ell_{\bar{\nu}\bar{\nu}'}$ connecting the head of D_ν to the head of $D_{\nu'}$, is parallel to $\ell_{\nu\nu'}$. But that is exactly the condition that the displaced graph $\bar{\mathcal{G}}$ should be in the same equivalence class of linear graphs as $\mathcal{G}$.



⟨**2.4**⟩ **Exercise.** Determine the dimension of $\mathcal{H}^2(\mathcal{G})$ for all of the linear graphs pictured in Lecture 2.

⟨**2.5**⟩ **The degree 4 part of the cohomology.** The dimension of the vector space $\mathcal{H}^4(\mathcal{G})$ is the dimension of the space $C(\mathcal{G})$ of configurations of the following sort: For each vertex ν of the graph $\mathcal{G}$, we give an ellipsoid E_ν in $\mathbb{V}$ centered at ν. For each edge $\ell_{\nu\nu'}$, we ask that when you take the projection along the direction of $\ell_{\nu\nu'}$ to an $(n-1)$-dimensional quotient space of $\mathbb{V}$, the two ellipsoids E_ν and $E_{\nu'}$ should have the same image. (Recall that an ellipsoid is the zero set of a degree two polynomial that is compact.) I am indebted to Victor Guillemin for this interpretation of $\mathcal{H}^4(\mathcal{G})$.



A configuration of ellipses in $C(\mathcal{G})$

⟨**2.6**⟩ **Exercise.** Prove this statement. More precisely, prove that the tangent space to $C(\mathcal{G})$ at any point is canonically $\mathcal{H}^4(\mathcal{G})$.

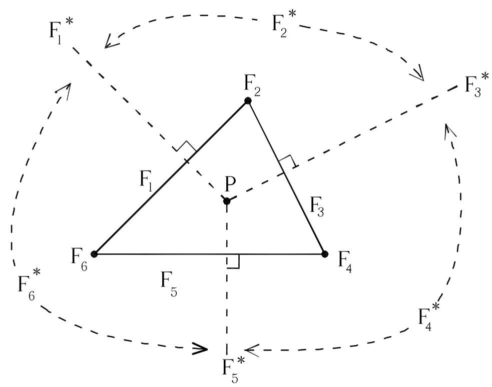
3.3. Piecewise Polynomial Functions

Suppose that $\mathcal{G}$ is the 1-skeleton of a convex polyhedron P. We will give an interpretation of the ring $\mathcal{H}^*(\mathcal{G}(P))$.

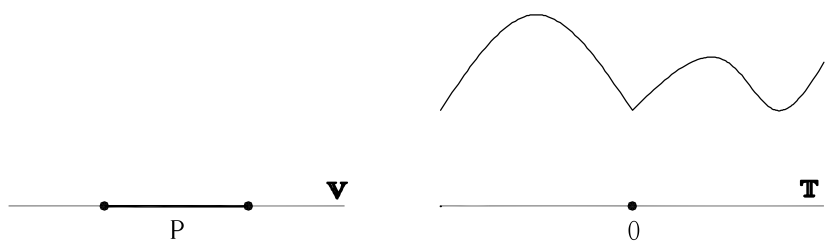
⟨**3.1**⟩ **The dual cone decomposition.** If P is a convex polyhedron in $\mathbb{V}$, then the dual space $\mathbb{T}$ is partitioned into subsets F^* corresponding to faces F of P as follows: If $t \in \mathbb{T}$, suppose that $c \in \mathbb{R}$ is the maximum value that the image $t(P)$ can take. Then $t^{-1}(c)$ will be some face F of P. We say that $t \in F^*$.

For example, $0 \in \mathbb{T}$ is always in P^* (P is a face of itself). If P has the same dimension as $\mathbb{V}$, then $0 = P^*$. The set F^* is an open subset of $\mathbb{T}$ if and only if F is a vertex of P.

If we identify $\mathbb{V} = \mathbb{T} = \mathbb{R}^2$ and $t(v) =< t, v >$ where $< \cdot, \cdot >$ is the usual inner product, then we can picture the dual cone decomposition like this:



⟨**3.2**⟩ **Piecewise polynomial functions.** A function $f : \mathbb{T} \longrightarrow \mathbb{R}$ is called *piecewise polynomial* with respect to the dual cone decomposition $\mathbb{T} = \bigcup V^*$ if it is continuous and its restriction to each set V^* is given by a polynomial function.



A polyhedron in $V = \mathbb{R}^1$

The graph of a function that is piecewise polynomial with respect to the dual cone decomposition

⟨**3.3**⟩ **Exercise.** Show that a continuous function is piecewise polynomial if its restriction to F^* is given by a polynomial function for every vertex F.

⟨**3.4**⟩ **Interpretation of the cohomology of the 1-skeleton of P.** If $\mathcal{G}$ is the 1-skeleton of the polyhedron P, then its cohomology ring $\mathcal{H}^*(\mathcal{G})$ is the ring of functions on $\mathbb{T}$ that are piecewise polynomial with respect to the dual cone decomposition.

⟨**3.5**⟩ **Exercise.** Prove this. Use the lemma that a polynomial is entirely determined on its values on any open set.

⟨**3.6**⟩ **Remark.** When the polyhedron is simple, this ring is called the *Reisner Stanley ring* of the dual simplicial polyhedron.

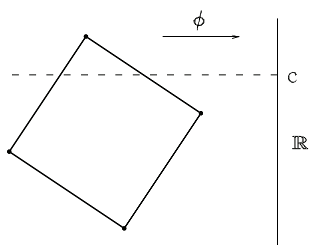
3.4. Morse Theory

Morse theory is the main tool we have for understanding the cohomology of a graph. The idea of Morse theory is to break the computation of the cohomology down into a series of simpler computations.

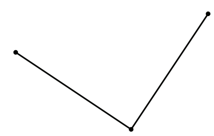
⟨**4.1**⟩ **Morse functions.** Suppose we have a linear graph $\mathcal{G}$ in a vector space $\mathbb{V}$. Consider a linear function $\phi : \mathbb{V} \longrightarrow \mathbb{R}$. The values $\phi(\nu)$ where ν is a vertex of $\mathcal{G}$ are called the *critical values* of ϕ. The function ϕ is called a *Morse function* if all of the critical values are distinct, i.e. for any pair of vertices ν and ν' of $\mathcal{G}$, $\phi(\nu) \neq \phi(\nu')$. It follows that ϕ is not constant on any edge of $\mathcal{G}$.

Morse functions exist for any linear graph. In fact, if you choose a linear function $\phi : \mathbb{V} \longrightarrow \mathbb{R}$ at random, you have to be infinitely unlucky to get one that is not Morse.

$\langle$**4.2**$\rangle$ **The truncated graph.** Now suppose c is a real number, which we call the "cut-off value". We define $\mathcal{G}^{\leq c}$ to be the subgraph of $\mathcal{G}$ consisting of those vertices ν such that the critical value $\phi(\nu) \leq c$, together with all the edges $\ell_{\nu\nu'}$ connecting vertices ν and ν' both of which have critical values $\leq c$.



A Morse function ϕ


The truncated graph $\mathcal{G}^{\leq c}$

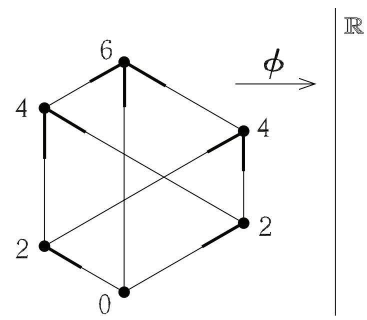
If we have a Morse function ϕ, we can label the vertices of $\mathcal{G}$ by $\nu_1, \nu_2, \ldots \nu_k$ so that their critical values are increasing $\phi(\nu_1) < \phi(\nu_2) < \cdots < \phi(\nu_k)$. Call c_j the critical value $\phi(\nu_j)$. For $c < c_1$, we have $\mathcal{G}^{\leq c}$ is empty. As the number c increases, $\mathcal{G}^{\leq c}$ grows by jumps every time c reaches a critical value c_j until finally for $c \geq c_k$, $\mathcal{G}^{\leq c} = \mathcal{G}$. The idea of Morse theory is to trace the growth of $\mathcal{H}^*(\mathcal{G}^{\leq c})$ as c increases.

$\langle$**4.3**$\rangle$ **The Morse module.** Suppose that $c_1 < c_2 < \cdots < c_k$ are the critical values of the Morse function ϕ, and c_0 is a real number less than the smallest critical value c_1. Then for all integers $j \in \{1, \ldots, k\}$, we have

$$\begin{array}{ccccccc} \mathcal{G}^{\leq c_{j-1}} & \overset{i_j}{\subset} & \mathcal{G}^{\leq c_j} & & & & \\ & & & & & & \\ \mathcal{H}^*(\mathcal{G}^{\leq c_{j-1}}) & \overset{i_j^*}{\longleftarrow} & \mathcal{H}^*(\mathcal{G}^{\leq c_j}) & \longleftarrow & M_j & \longleftarrow & 0 \end{array}$$

Here i_j^* is the map on cohomology induced by the inclusion of graphs i_j and M_j the kernel of the map i_j^*. The kernel M_j is a graded module over $\mathcal{O}(\mathbb{T})$ because it is the kernel of a map of graded modules. It is a graded ideal in $\mathcal{H}^*(\mathcal{G}^{\leq c_j})$, but it is more useful to think of it as a $\mathcal{O}(\mathbb{T})$-module. The module M_j is called *the Morse module* of the vertex ν_j whose critical value is c_j.

⟨4.4⟩ The Morse index. For any vertex $\nu \in \mathcal{G}$, let $\mathcal{L}(\nu)$ denote the set of edges coming in to ν. The Morse function ϕ splits the edges in $\mathcal{L}(\nu)$ into two types: $\mathcal{L}^-(\nu)$ is the edges going down from ν as measured by ϕ, i.e. the edges connecting ν to vertices ν' with $\phi(\nu') < \phi(\nu)$. The others are in $\mathcal{L}(\nu)^+$, the edges going up from ν. We define the *Morse index* Index(ν) to be twice the number of edges in $\mathcal{L}^-(\nu)$.



Morse indices of vertices

⟨4.5⟩ Calculation of the Morse module M_j. The graph $\mathcal{G}^{\leq c_j}$ has exactly one more vertex than the graph $\mathcal{G}^{\leq c_{j-1}}$, namely ν_j. Therefore for an element $\{\ldots, f_\nu, \ldots\}$ of $M_j \subset \mathcal{H}^*(\mathcal{G}^{\leq c_j})$, all of the f_ν will be zero except for f_{ν_j} corresponding to ν_j. This polynomial $f_{\nu_j} : \mathbb{T} \longrightarrow \mathbb{R}$ will vanish on all of the hyperplanes $\ell^\perp$ for $\ell \in \mathcal{L}^-$. For each $\ell \in \mathcal{L}^-$, let g_ℓ be a nonzero linear function on $\mathbb{T}$ that is zero on $\ell^\perp$.

Proposition. The Morse module M_j is the principal ideal in $\mathcal{O}(\mathbb{T})$ generated by the homogeneous element

$$g_{\nu_j} = \prod_{\ell \in \mathcal{L}^-(\nu_j)} g_\ell.$$

As a module over $\mathcal{O}(\mathbb{T})$, M_j is a free module generated by g_{ν_j}, which lies in the graded piece $\mathcal{O}(\mathbb{T})^{\mathrm{Index}(\nu_j)}$.

⟨4.6⟩ Exercise. Finish the proof of this proposition.

⟨4.7⟩ Exercise*. Suppose that $\mathcal{G}$ is the 1-skeleton of a simple polyhedron. Show that the ordering of the vertices given by a Morse function corresponds to a linear shelling of the dual simplicial polytope.

3.5. Perfect Morse Functions

Having a Morse function isn't much help unless the cokernel of the map

$$\mathcal{H}^*(\mathcal{G}^{\leq c_{j-1}}) \xleftarrow{i_j^*} \mathcal{H}^*(\mathcal{G}^{\leq c_j})$$

is zero, because in general it can be very difficult to compute this cokernel. If it is zero, the Morse function is called perfect:

⟨5.1⟩ Definition. The Morse function φ is called *perfect* if i_j^* is surjective for all j.

⟨5.2⟩ Hilbert series. One of our goals is to compute the dimensions of the cohomology groups $\mathcal{H}^*(\mathcal{G})$, or equivalently, to compute the Hilbert series of the cohomology of a graph Hilb$(\mathcal{H}^*(\mathcal{G}))$(see §0.3.4). This determines the isomorphism class of $\mathcal{H}^*(\mathcal{G})$ except for the ring structure and the structure as a module over $\mathcal{O}(\mathbb{T})$.

If ϕ is perfect, then we see by induction that the dimension of the i-th graded piece of $\mathcal{H}^*(\mathcal{G})$ is the sum of the dimensions of the i-th graded pieces of the Morse modules. Expressed in Hilbert series,

$$\mathrm{Hilb}(\mathcal{H}^*(\mathcal{G})) = \sum_{1\le j\le k} \mathrm{Hilb}(M_j).$$

But since the Morse module M_j is a free $\mathcal{O}(\mathbb{T})$ module on a generator of degree $\mathrm{Index}(\nu_j)$, and the Hilbert series of $\mathcal{O}(\mathbb{T})$ is computed in §0.3.5, we have the following:

⟨**5.3**⟩ **Proposition.** If ϕ is a perfect Morse function, the Hilbert series of the cohomology of the graph is given by

$$\mathrm{Hilb}(\mathcal{H}^*(\mathcal{G})) = \sum_{i=1}^{k} x^{\mathrm{Index}(\nu_i)} \left(\frac{1}{1-x^2}\right)^n = \sum_{i=1}^{k} q^{\mathrm{Index}(\nu_i)/2} \left(\frac{1}{1-q}\right)^n.$$

⟨**5.4**⟩ **The Betti numbers of a graph.** Suppose that $\mathcal{G}$ has a perfect Morse function ϕ. Then we define the *Betti numbers* B_i of $\mathcal{G}$ to be the number of vertices of $\mathcal{G}$ whose Morse index is i. Note that B_i is automatically zero if i is odd.

We define the Poincaré polynomial P to be

$$P(x) = \sum_i B_i x^i.$$

so we have

$$\mathrm{Hilb}(\mathcal{H}^*(\mathcal{G})) = P(x)\left(\frac{1}{1-x^2}\right)^n$$

and

$$P(x) = \mathrm{Hilb}(\mathcal{H}^*(\mathcal{G}))\left(1-x^2\right)^n$$

where the last expression, which is *a priori* an infinite power series, is actually a polynomial.

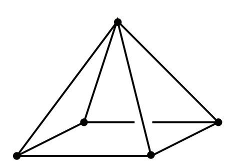
⟨**5.5**⟩ **Exercise.** Show that if the graph $\mathcal{G}$ has more than one different perfect Morse function, the Betti numbers (and the Poincaré polynomials) are independent of the choice of the Morse function.

⟨**5.6**⟩ **Exercise.** Show that the sum of the Betti numbers $\sum_i B_i$ is the number of vertices of the graph $\mathcal{G}$.

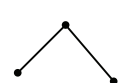
⟨**5.7**⟩ **Exercise.** Show that the sum $\sum_i (i/2)B_i$ is the number of edges of the graph $\mathcal{G}$.

⟨**5.8**⟩ **Exercise*.** Show that the homology groups of the topological graph $\mathcal{G}$ are determined by the Betti numbers of $\mathcal{H}^*(\mathcal{G})$.

⟨**5.9**⟩ **Exercise.** Let $\mathcal{G}$ be the 1-skeleton of the Egyptian pyramid in 3-space. Show that not all of the Morse functions on $\mathcal{G}$ are perfect by showing that they would lead to different Betti numbers. Can you identify which ones are not perfect?



⟨**5.10**⟩ **Exercise.** Show that the height function on the linear graph in the plane displayed on the right is not perfect. We call this graph the "inverted V".



⟨**5.11**⟩ **Remark.** It can be difficult to tell whether a given Morse function ϕ is perfect. There is one deep general theorem about this, due to Guillemin and Zara [**18**]. However, as we will see in §3.8.8, most of the cases we are considering have perfect Morse functions for topological reasons.

⟨**5.12**⟩ **Exercise.** It is known that all Morse functions of the graphs pictured in Lecture 2 are perfect. Calculate their Betti numbers.

3.6. Determining $\mathcal{H}^*(\mathcal{G})$ as a $\mathcal{O}(\mathbb{T})$ Module

For many purposes, we want more than the dimensions of the cohomology groups $\mathcal{H}^i(\mathcal{G})$. A Morse function ϕ enables us to determine it as a $\mathcal{O}(\mathbb{T})$-module:

⟨**6.1**⟩ **Proposition.** If $\mathcal{G}$ has a perfect Morse function, its cohomology $\mathcal{H}^*(\mathcal{G})$ is a free graded $\mathcal{O}(\mathbb{T})$-module. The number of free generators of degree $2i$ is B_i.

This proposition can be proved inductively, using §3.4.5.

⟨**6.2**⟩ **Proposition.** If ϕ is perfect, the cohomology $\mathcal{H}^*(\mathcal{G})$ is a free graded module over $\mathcal{O}\mathbb{T}$, i.e.

$$\mathcal{H}^*(\mathcal{G}) = \bigoplus_j \overline{g_j}\mathcal{O}(\mathbb{T})$$

where $\overline{g_j}$ is a lift of g_i to $\mathcal{H}^*(\mathcal{G})$.

⟨**6.3**⟩ **Exercise.** Prove this.

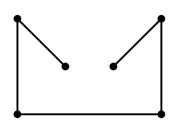
⟨**6.4**⟩ **Definition.** We say that a linear graph has the *free module property* if its cohomology is a free graded module over $\mathcal{O}(\mathbb{T})$.

If a graph has the free module property, we may define its Poincaré polynomial by

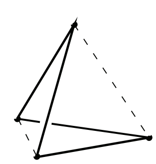
$$P(\mathcal{G}) = \mathrm{Hilb}(\mathcal{H}^*(\mathcal{G}))(1-q)^n$$

which will necessarily be a polynomial. Graphs with a perfect Morse function have the free module property, but the converse isn't true:

⟨**6.5**⟩ **Exercise.** Show that the graph to the right has no perfect Morse function, but has the free module property.



⟨**6.6**⟩ **Exercise.** Show that a nonplanar quadrilateral in 3-space does not have the free module property (example due to T. Braden).



3.7. Poincaré Duality

Suppose that $\mathcal{G}$ is k-valent: it has k edges coming out of every vertex ($\#|\mathcal{L}(\nu)| = k$ for all vertices $\nu \in \mathcal{G}$). The simplest form of Poincaré duality is the numerical statement that the Betti numbers B_j and B_{2k-j} are equal.

This numerical Poincaré duality holds whenever there is a perfect Morse function whose negative is also perfect.

⟨7.1⟩ Exercise. Suppose that $\mathcal{G}$ is k-valent and that it has a perfect Morse function ϕ such that the Morse function $-\phi$ is also perfect. Show that

$$B_j(\mathcal{G}) = B_{2k-j}(\mathcal{G}).$$

As usual in mathematics, it is better to have a canonical isomorphism or a duality of vector spaces than an equality of their dimensions. We want something of the kind for Poincaré duality. First, we need some preliminaries on graded rings.

⟨7.2⟩ The canonical filtration of a graded R-module. Consider a graded module M over a graded ring R. Let $M^{\leq k}$ be the sum of the graded pieces $M^0 \oplus M^1 \oplus \cdots \oplus M^k$. This is not an R-module, but it generates one; call it $F_k M = R \cdot M^{\leq k}$. Then M has an increasing filtration of R-submodules $F_0 M \subseteq F_1 M \subseteq \cdots$.

⟨7.3⟩ Exercise. If $\mathcal{G}$ has the free module property, then B_i is the dimension of the i graded piece of $F_i\mathcal{H}^*(\mathcal{G})/F_{i-1}\mathcal{H}^*(\mathcal{G})$.

⟨7.4⟩ Internal Hom. Suppose that M and N are two graded R modules. Then the space $\mathrm{Hom}_R(M, N)$ has the structure of a graded R module. The i-th graded piece is the elements of $\mathrm{Hom}_R(M, N)$ that map each M^j into N^{j+i}.

⟨7.5⟩ Proposition. *Functorial Poincaré duality.* Now, suppose that $\mathcal{G}$ is connected, k-valent, and that it is universally perfect (i.e. all Morse functions are perfect). Then $\mathcal{H}^*(\mathcal{G})/F_{2k-1}\mathcal{H}^*(\mathcal{G})$ is a free $\mathcal{O}(\mathbb{T})$ module on one generator in degree $2k$. Call it $\mathcal{D}$. The pairing

$$\begin{array}{ccccccccc} \mathcal{H}^*(\mathcal{G}) & \otimes_{\mathcal{O}(\mathbb{T})} & \mathcal{H}^*(\mathcal{G}) & \longrightarrow & \mathcal{H}^*(\mathcal{G}) & \longrightarrow & \frac{\mathcal{H}^*(\mathcal{G})}{F_{2k-1}\mathcal{H}^*(\mathcal{G})} & = \mathcal{D} \\ x & \times & y & \mapsto & xy & & & \end{array}$$

is perfect in the sense that the induced map

$$\mathcal{H}^*(\mathcal{G}) \longrightarrow \mathrm{Hom}(\mathcal{H}^*(\mathcal{G}), \mathcal{D})$$

is an isomorphism of $\mathcal{O}(\mathbb{T})$ modules.

⟨7.6⟩ Exercise. Show that functorial Poincaré duality implies numerical Poincaré duality.

3.8. The Main Theorems

This section relates the cohomology of a linear graph to torus actions and the moment graph construction.

⟨8.1⟩ Assumptions. We consider a torus acting on a space $T \circlearrowright X$ such that the moment graph $\mathcal{G}(T \circlearrowright X)$ exists. (This means, in particular, that the 1-skeleton of the action is a balloon sculpture.) *We further assume that X has only even dimensional real cohomology*, i.e. $H_i(X; \mathbb{R}) = 0$ for i odd.

These assumptions hold for complex projective spaces, quadric hypersurfaces, Grassmannians, Lagrangian Grassmannians, flag manifolds and their and toric varieties based on simple polyhedra. In other words, the assumptions hold for all spaces considered in Lecture 2, except for toric varieties of some non-simple polyhedra.

⟨**8.2**⟩ **Theorem** [**13**]. The T equivariant cohomology ring of X is the cohomology ring of the moment graph of X, i.e.

$$H^*(T \circlearrowright X) = \mathcal{H}^*(\mathcal{G}(T \circlearrowright X)).$$

⟨**8.3**⟩ **Theorem.** The ordinary (non-equivariant) cohomology ring of X, calculated from the moment graph by

$$H^*(X) = \frac{\mathcal{H}^*(\mathcal{G}(X))}{\text{the ideal generated by } \mathcal{O}(\mathbb{T})^{>0}}$$

where $\mathcal{O}(\mathbb{T})^{>0}$ is the positive degree part of $\mathcal{O}(\mathbb{T})$.

⟨**8.4**⟩ **Theorem.** The moment graph has the free module property, i.e. $\mathcal{H}^*(\mathcal{G}(X))$ is a free graded module over the polynomial algebra $\mathcal{O}(\mathbb{T})$ and the Poincaré polynomial of the graph $P(\mathcal{G}) = \mathrm{Hilb}(\mathcal{H}^*(\mathcal{G}))(1-q)^n$ is the Poincaré polynomial $\sum_i x^i \dim H_i(X)$ of X.

⟨**8.5**⟩ We can pause to marvel at the statements. The data in moment graph of $T \circlearrowright X$ depends only on a very small part of X – its 1-skeleton. Yet by these theorems, all of the homology and equivariant homology of X is encoded in this data.

The proofs of these three propositions are beyond our ambitions here. The reader is referred to [**13**] and the references given there. However, we have given enough information in our explicit construction of generators and relations for the equivariant cohomology of the 2-sphere, we have to construct the map

$$H^*(T \circlearrowright X) \overset{\sim}{\longleftarrow} \mathcal{H}^*(\mathcal{G}(T \circlearrowright X))$$

in Theorem 3.8.2.

⟨**8.6**⟩ **Exercise.** Construct a map $H^*(T \circlearrowright X) \mapsto \mathcal{H}^*(\mathcal{G}(T \circlearrowright X))$.

⟨**8.7**⟩ **Exercise.** Show that the free generators $\alpha_1, \alpha_2, \ldots$ for $\mathcal{H}^*(\mathcal{G}(X))$ as a module over $\mathcal{O}(\mathbb{T})$ pass in the quotient to generators of $H^*(X)$ as a vector space, i.e. as a module over $\mathbb{R}$.

⟨**8.8**⟩ **Morse theory and Poincaré duality for our examples.** In Lecture 2, we gave many examples of spaces with a torus action: projective spaces, quadric hypersurfaces, Grassmann manifolds, Lagrangian Grassmannians, flag manifolds, and toric varieties for simple polyhedra. These examples all satisfy the hypotheses of the theorems above. Furthermore, they are all universally perfect (every Morse function is perfect), so they satisfy Poincaré duality. (This may be seen using topological methods.) Many other examples in this favorable class will be mentioned in §4.1.2.

⟨**8.9**⟩*** Morse theory and moment maps.** Suppose that X is a nonsingular algebraic variety, and the action $T \circlearrowright X$ and the moment map $\mu : X \longrightarrow \mathbb{V}$ are as in §2.9. If $\varphi : V \longrightarrow \mathbb{R}$ is a Morse function for the moment graph of $T \circlearrowright X$ in the sense of this Lecture, then $\varphi \circ \mu : X \longrightarrow \mathbb{R}$ is a Morse function in the usual sense of differential topology. In this case, the Morse function will be perfect. In this case, Morse theory we have described is a reflection of the usual topological Morse theory.

⟨8.10⟩* The Schubert basis. Suppose X is a generalized flag manifold, i.e. a projective space, a quadric hypersurface, a Grassmann manifold, a Lagrangian Grassmannian, a flag manifold, or more generally a space of §4.7.2. Then the Morse function $\varphi \circ \mu$ is perfect on ordinary cohomology $H^*(X)$. The basis of cohomology it provides is called the *Schubert basis*, and the study of the properties of this basis in the ring $H^*(X)$ is called *Schubert calculus*, an interesting combinatorial study involving such things as the Littlewood-Richardson rule, Schubert polynomials, etc. By Exercise 3.8.7, the $H^*(X)$ and its Schubert basis is encoded in the moment graph, so in principle questions in Schubert calculus reduce to questions about the moment graph.

⟨8.11⟩* A general Lie group. Here's a brief account. Suppose $G \circlearrowright X$ is an action of a general connected Lie group. Then $H^*(G \circlearrowright X) = H^*(K \circlearrowright X)$ where K is a maximal compact subgroup of G. Then, by a theorem of Borel, $H^*(K \circlearrowright X) = H^*(T \circlearrowright X)^W$ where T is a maximal torus of K and W is the Weyl group of K and the superscript means taking the invariants. Now, suppose the T action satisfies our hypotheses, so it has moment graph in $\mathcal{G}(T \circlearrowright X) \subset \mathbb{V}$. The Weyl group W acts on $\mathbb{V}$ preserving the moment graph, so we can calculate $H^*(G \circlearrowright X) = \mathcal{H}^*(\mathcal{G}(T \circlearrowright X))^W$.

LECTURE 4
Computing Intersection Homology

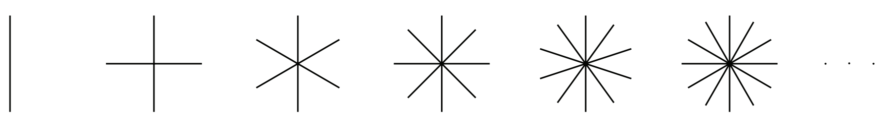
(Polynomial and Linear Geometry II)

In the last lecture, we saw the value of perfect Morse functions. In this lecture, we consider some linear graphs $\mathcal{G}$ with Morse functions that are not perfect. By changing the cohomology theory, the Morse functions become perfect again. When $\mathcal{G}$ arose as the moment graph of $T \circlearrowright X$, the new cohomology theory turns out to be the equivariant intersection cohomology of $T \circlearrowright X$.

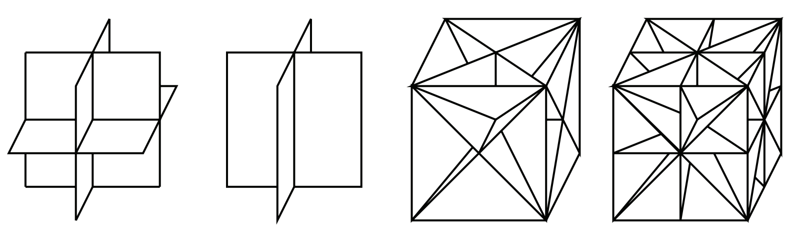
All of the ideas of this Lecture are joint work with Tom Braden.

4.1. Graphs Arising from Reflection Groups

⟨1.1⟩ Finite reflection groups. Consider a finite configuration of hyperplanes $\mathcal{H}$ in $\mathbb{V}$ that pass through the origin. Suppose that reflection in each hyperplane H in $\mathcal{H}$ takes the configuration $\mathcal{H}$ to itself. Then we call $\mathcal{H}$ a set of reflecting hyperplanes. These are all classified. For example here are the sets of reflecting hyperplanes when $\mathbb{V}$ has dimension 2:

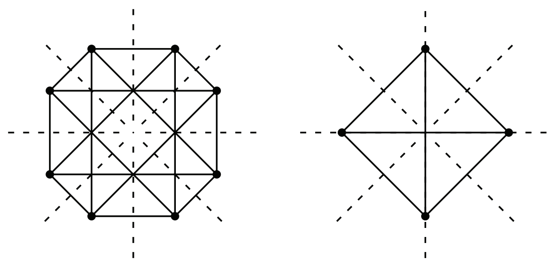


and here are some when $\mathbb{V}$ has dimension 3:



Or, when $\mathbb{V} = \mathbb{R}^n$, $\mathcal{H}$ could be the $\binom{n}{2}$ planes $x_i = x_j$ in $\mathbb{R}^n$ where two coordinates are equal. A *finite reflection group* W is the group of maps of $\mathbb{V}$ to itself generated by reflections in hyperplanes in $\mathcal{H}$.

⟨**1.2**⟩ **The linear graph associated to $\mathcal{H}$.** Choose any point $v \in \mathbb{V}$. We get a linear graph $\mathcal{G}(\mathcal{H}, v)$ as follows: The set of vertices of $\mathcal{G}(\mathcal{H}, v)$ is the orbit Wv of the point v. Two vertices ν and ν' are connected by an edge whenever ν' is the reflection of ν through one of the hyperplanes of $\mathcal{H}$. (So the edge will be perpendicular to the hyperplane.) Here are two of the possible graphs associated to a single $\mathcal{H}$ where $\mathbb{V}$ has dimension 2:



⟨**1.3**⟩ **Exercise.** All of the linear graphs pictured in sections 2.3 to 2.7 arise in this way. Construct the family of hyperplanes $\mathcal{H}$ for each of them.

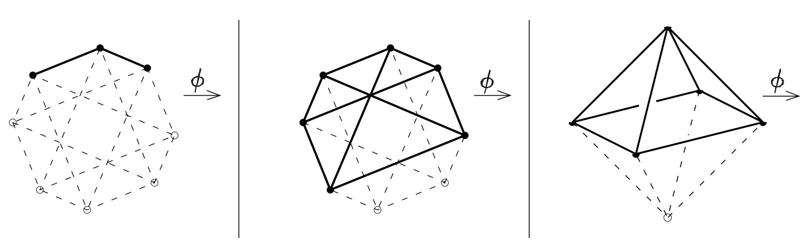
⟨**1.4**⟩ **Crystallographic reflection groups.** If there is some lattice $L \subset \mathbb{V}$ such that reflection in each of the hyperplanes in $\mathcal{H}$ takes this lattice into itself, then $\mathcal{H}$ is called *crystallographic*. This is true for most, but not all of the possible choices for $\mathcal{H}$. If $\mathcal{H}$ is crystallographic, then $\mathcal{G}(\mathcal{H}, v)$ arises as a moment graph, as described in §4.7.2.

⟨**1.5**⟩ **The linear graphs $\mathcal{G}(\mathcal{H}, v)$ are all universally perfect.** In other words, all Morse functions on these graphs are perfect. (This may be seen using a topological argument if $\mathcal{H}$ is crystallographic. In general, it follows from [**18**].) In fact, every graph we have considered so far is universally perfect, with the exception of a few counterexamples and 1-skeleta of non-simple polytopes. We will now construct a large class of examples with non-perfect Morse functions.

4.2. Upward Saturated Subgraphs

⟨**2.1**⟩ Consider a linear graph arising from a finite reflection group $\mathcal{G}(\mathcal{H}, v) \subset \mathbb{V}$ and a Morse function $\phi : \mathbb{V} \longrightarrow \mathbb{R}$ (a linear function that takes distinct values on different vertices of $\mathcal{G}$). Recall (§3.4.4) that if ν is a vertex of $\mathcal{G}$, we define $\mathcal{L}^-(\nu)$ to be the edges going down from ν and $\mathcal{L}^+(\nu)$ to be the edges going up from ν, where "up" and "down" are measured by ϕ.

⟨**2.2**⟩ **Definition.** We call a subgraph $\mathcal{G}'$ of $\mathcal{G}$ *upward saturated* with respect to ϕ if whenever ν is in $\mathcal{G}'$ then every edge in $\mathcal{L}^+(\nu)$ is in $\mathcal{G}'$.



Upward saturated subgraphs

⟨**2.3**⟩ The Morse function ϕ is not usually perfect on upward saturated subgraphs. In fact, for two of the examples above, the inverted V §3.5.10 and the Egyptian pyramid §3.5.9 the function ϕ has already been shown not to be perfect. However,

⟨**2.4**⟩ **Exercise.** Show that $-\phi$ is perfect for an upward saturated subgraph. (Use the fact that $\mathcal{G}(\mathcal{H}, v)$ is universally perfect.)

⟨**2.5**⟩ **Exercise*.** The Morse function ϕ turns the set of vertices of $\mathcal{G}(\mathcal{H}, v)$ into a poset where $\nu \leq \nu'$ if there is a sequence of edges from ν to ν' such that ϕ increases along each edge. The partial order of this poset is called the *Bruhat order*. Show that an upward saturated subgraph can be characterized as a complete subgraph on a set of vertices that is an ideal in this poset.

4.3. Sheaves on Graphs

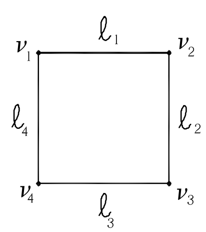
We introduce the notion of a sheaf on a graph. This will give us another interpretation of the cohomology of a linear graph. It will also give us a better understanding of when a Morse function is perfect.

Definition [**7**]**.** Suppose $\mathcal{G}$ is a topological graph. A *sheaf of graded rings* $\underline{S}$ on $\mathcal{G}$ is the following data.

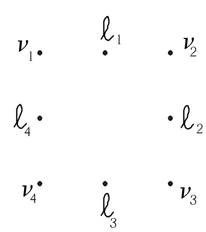
(1) A graded ring S_ν for every vertex ν of $\mathcal{G}$;
(2) A graded ring S_ℓ for ever edge ℓ of $\mathcal{G}$; and
(3) A graded ring homomorphism $s_{\nu\ell} : S_\nu \longrightarrow S_\ell$ whenever ν lies on ℓ.

There is a similar definition replacing "rings" by any other category, such as graded modules over a graded ring.

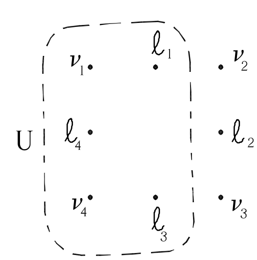
⟨**3.1**⟩ **Definition.** Consider the set $E(\mathcal{G})$ which is the union of the set of vertices of the graph with the set of edges of the graph. An *open subset* of $E(\mathcal{G})$ is a subset U with the property that if a vertex ν is in U, then all the edges in $\mathcal{L}(\nu)$ are in U (where $\mathcal{L}(\nu)$ is the set of edges containing ν).



Linear graph $\mathcal{G}$



The finite set $E(\mathcal{G})$



An open subset U

⟨**3.2**⟩ **Definition.** Let U be an open subset of $E(\mathcal{G})$. A *section* of a sheaf $\underline{S}$ over U is the choice of an element e_ν for every vertex ν in $E(\mathcal{G})$ and element e_ℓ for every edge ℓ in $E(\mathcal{G})$ such that if ν lies in ℓ, then $s_{\nu\ell}(e_\nu) = e_\ell$. We will notate the set of such sections $\Gamma(\underline{S}, U)$.

If $U' \subset U$, then we have a restriction homomorphism $\Gamma(\underline{S}, U) \longrightarrow \Gamma(\underline{S}, U')$ defined by restricting the data.

⟨**3.3**⟩ **Exercise*.** Show that the definition of open set makes $E(\mathcal{G})$ into a (finite) topological space. Show that the function $U \mapsto \Gamma(\underline{S}, U)$ satisfies the sheaf axioms

$E(\mathcal{G})$. Establish an equivalence between sheaves as usually defined on the finite topological space $E(\mathcal{G})$ and the notion of a sheaf on a graph.

$\langle$**3.4**$\rangle$ **The sheaf $\underline{A}$.** Now suppose that $\mathcal{G}$ is a linear graph. Then it has a canonical sheaf of graded rings $\underline{A}$ on it defined as follows.

(1) The graded ring A_ν is $\mathcal{O}(\mathbb{T})$, the ring of polynomial functions on $\mathbb{T} = \mathbb{V}^*$ for every vertex ν of $\mathcal{G}$;
(2) The graded ring A_ℓ for the edge ℓ of $\mathcal{G}$ is $\mathcal{O}(\ell^\perp)$, the ring of polynomial functions on $\ell^\perp \subset \mathbb{T}$.
(3) The homomorphism $a_{\nu\ell}$ is the restriction of polynomial functions.

$\langle$**3.5**$\rangle$ **Proposition.** The cohomology $\mathcal{H}^*(\mathcal{G})$ of the graph $\mathcal{G}$ is the global sections $\Gamma(\underline{A}, E(\mathcal{G}))$ of the sheaf $\underline{A}$.

This is just a slightly disguised presentation of the definition of $\mathcal{H}^*(\mathcal{G})$.

4.4. A Criterion for Perfection

We mix the language of sheaves on graphs with Morse theory.

$\langle$**4.1**$\rangle$ Recall from §3.5.1 that the criterion for a Morse function ϕ to be perfect is a surjectivity condition for each vertex of the graph. We will focus on this criterion for a single vertex ν. Suppose that ϕ is a Morse function for the linear graph $\mathcal{G}$, ν is the vertex with the largest critical value $\phi(\nu) = c$, and $c' < c$ is the next to the largest critical value. The Morse function ϕ is perfect at ν if the map

$$\mathcal{H}^*(\mathcal{G}) = \mathcal{H}^*(\mathcal{G}^{\leq c}) \longrightarrow \mathcal{H}^*(\mathcal{G}^{\leq c'})$$

is a surjection.

$\langle$**4.2**$\rangle$ Consider the following open cover of the finite set $E(\mathcal{G})$.

- $E^{<c} = E(\mathcal{G}) - \{\nu\}$
- $\mathcal{S}(\nu) = \{\nu\} \cup \mathcal{L}^-(\nu)$ ($\mathcal{S}$ for star)

so that

- $E^{<c} \cap \mathcal{S}(\nu) = \mathcal{L}^-(\nu)$
- $E^{<c} \cup \mathcal{S}(\nu) = E(\mathcal{G})$

We have a diagram

$$\underline{A}_\nu = \Gamma(\underline{A}, \mathcal{S}(\nu)) \longrightarrow \Gamma(\underline{A}, \mathcal{L}^-(\nu)) \longleftarrow \Gamma(\underline{A}, E^{<c}) = \mathcal{H}^*(\mathcal{G}^{\leq c'})$$

where both maps are restriction maps.

$\langle$**4.3**$\rangle$ **Proposition.** The image of $\underline{A}_\nu \longrightarrow \Gamma(\underline{A}, \mathcal{L}^-(\nu))$ is contained in the image of $\Gamma(\underline{A}, \mathcal{L}^-(\nu)) \longleftarrow \Gamma(\underline{A}, E^{<c})$. The graph is perfect at ν if and only if these two images coincide, i.e. if and only if the map

$$\underline{A}_\nu \longrightarrow \text{Image}\left[\Gamma(\underline{A}, \mathcal{L}^-(\nu)) \longleftarrow \Gamma(\underline{A}, E^{<c})\right]$$

is a surjection.

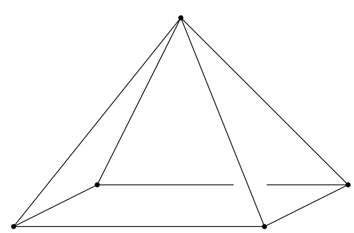
$\langle$**4.4**$\rangle$ **Exercise.** Prove this. You may want to prove the following lemmas first:
• The map $\Gamma(\underline{A}, E(\mathcal{G})) \longrightarrow \Gamma(\underline{A}, \mathcal{S}(\nu))$ is surjective.
• The maps $\Gamma(\underline{A}, \mathcal{L}^-(\nu)) \longleftarrow \Gamma(\underline{A}, E^{<c})$ and $\Gamma(\underline{A}, E(\mathcal{G})) \longrightarrow \Gamma(\underline{A}, \mathcal{S}(\nu))$ have isomorphic kernels.

⟨**4.5**⟩ If you want to understand what makes a Morse function perfect, it is worthwhile to pause to appreciate this proposition. An element e in $\Gamma(\underline{A}, \mathcal{L}^-(\nu))$ is just a collection of polynomials on the hyperplanes $\ell^\perp \subset \mathbb{T}$ for $\ell \in \mathcal{L}^-(\nu)$. The condition that e be in the image of $\Gamma(\underline{A}, \mathcal{L}^-(\nu)) \longleftarrow \Gamma(\underline{A}, E^{<c})$ is a potentially complicated compatibility condition on these polynomials, coming from the structure of the graph. According to the proposition, it is perfect if and only if a set of polynomials $\ell^\perp$ satisfying this compatibility condition is necessarily the restriction of a single polynomial on $\mathbb{T}$.

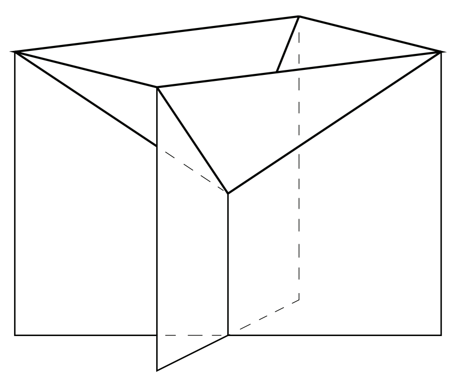
⟨**4.6**⟩ For example, take the Egyptian pyramid. As in §3.3.4, an element of the image of

$$\Gamma(\underline{A}, \mathcal{L}^-(\nu)) \longleftarrow \Gamma(\underline{A}, E^{<c})$$

is a continuous piecewise polynomial function on configuration consisting of the four upper planes on the right below. We are asking whether such a thing is the restriction of a polynomial in 3-space. We can see that it is not, just by a dimension count. For example, there is a 4-dimensional space of linear polynomials on the configuration (it can be anything on each of the 4 lines). But there is only a 3-dimensional space of linear polynomials in 3-space. So the height function is not perfect. (Compare §3.5.9.)



Egyptian pyramid



The dual cone decomposition

4.5. Definition of the Sheaf $\underline{M}$

We define a sheaf of $\mathcal{O}(\mathbb{T})$ modules that repairs the deficiency in perfectness as measured by Proposition 4.4.3.

⟨**5.1**⟩ **Hypotheses on the graph $\mathcal{G}$.** We will assume that the graph $\mathcal{G}$ is an upward saturated subgraph of a graph $\mathcal{G}'$ that arises from a crystallographic reflection group as in §4.1.2. For example, the ambient graph $\mathcal{G}'$ can be any of the graphs of Lecture 2, except for some moment graphs of toric varieties.

⟨**5.2**⟩ **Construction.** [**7**]. We define the sheaf $\underline{M}$ inductively:
• If $\mathcal{L}^-(\nu)$ is empty, then M_ν is a free module over $\mathcal{O}(\mathbb{T})$ generated by a generator g_ν in degree $-|\mathcal{L}^+(\nu)|$, minus the number of edges going up from ν.
• If M_ν has already been constructed, and $\ell \in \mathcal{L}^+(\nu)$ is an edge going up from M_ν then

$$M_\ell = M_\nu \otimes_{\mathcal{O}(\mathbb{T})} \mathcal{O}(\ell^\perp).$$

• If ν is a vertex and $\underline{M}$ has already been constructed on all of the vertices and edges of $\mathcal{G}^{<\nu}$, then M_ν is the free cover of

$$\text{Image}\left[\Gamma(\underline{M}, \mathcal{L}^-(\nu)) \longleftarrow \Gamma(\underline{M}, E^{<\nu}(\mathcal{G}))\right]$$

and the maps $m_{\nu\ell}$ are determined by

$$M_\nu \xrightarrow{F} \text{Image}\left[\Gamma(\underline{M}, \mathcal{L}^-(\nu)) \longleftarrow \Gamma(\underline{M}, E^{<\nu}(\mathcal{G}'))\right] \subset \bigoplus_{\ell \in \mathcal{L}^-(\nu)} M_\ell$$

where F is the free cover map.

⟨**5.3**⟩ **Remark.** As constructed, M_ν is the smallest free module mapping surjectively to $\text{Image}\,[\Gamma(\underline{M}, \mathcal{L}^-(\nu)) \longleftarrow \Gamma(\underline{M}, E^{<\nu}(\mathcal{G}'))]$. By the discussion above, this surjectivity is what we need for ϕ to be a perfect Morse function.

⟨**5.4**⟩ **Proposition.** $\text{Image}\,[\Gamma(\underline{M}, \mathcal{L}^-(\nu)) \longleftarrow \Gamma(\underline{M}, E^{<\nu}(\mathcal{G}'))]$ has the canonical free cover property, as defined in §4.5.6 below, so the construction of M_ν makes sense.

⟨**5.5**⟩ **Exercise.** Show that the induction is possible, i.e. that these three rules determine the sheaf $\underline{M}$ everywhere on $\mathcal{G}$.

⟨**5.6**⟩ **Free covers.** Suppose that R is a graded ring and $\mathcal{M}$ is a finitely generated R-module. A *free cover* of $\mathcal{M}$ is a surjection $\mathcal{F} \xrightarrow{F} \mathcal{M}$ where $\mathcal{F}$ is free a set of generators, and such a surjection can't be found with fewer generators.

Free covers are unique in the following sense: If $\mathcal{F} \xrightarrow{F} \mathcal{M}$ and $\mathcal{F}' \xrightarrow{F'} \mathcal{M}$ are two free covers, then there is always a commutative diagram

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{i} & \mathcal{F}' \\ & {}_{\text{F}}\searrow \quad \swarrow_{\text{F'}} & \\ & \mathcal{M} & \end{array}$$

where i is an isomorphism. In general, however, free covers are not functorial since i is not uniquely determined by the commutativity of this diagram.

Definition. The module $\mathcal{M}$ has *the canonical free cover property* if there is exactly one map i making the diagram above commutative.

⟨**5.7**⟩ **Exercise.** Consider the 2-dimensional module $\mathcal{M}$ over the polynomial ring in one variable $\mathbb{R}[t]$ generated by elements g_1 and g_2, both of degree zero, with the relations $tg_i = 0$. Show that its free cover is the module freely generated by g_1 and g_2, and has the canonical free cover property. On the other hand, consider the module generated by g_1 in degree zero and g_2 in degree 2 with the same relation $tg_i = 0$. Show that its free cover is the free module generated by g_0 and g_2, but that it does not have the canonical free covers property because the map determined by $g_1 \mapsto g_1$ and $g_2 \mapsto g_2 + at^2 g_1$ commutes with the map to $\mathcal{M}$ for any real number a.

⟨**5.8**⟩ **Exercise.** Suppose that $\mathcal{M}$ is generated by homogeneous generators $g_1, g_2, \ldots$ with homogeneous relations $r_1, r_2, \ldots$ and that the degree of each of the r_i is greater than the degree of any of the g_i. Show that $\mathcal{M}$ has the canonical free cover property.

This is the reason for Proposition 4.5.4. In the second example of the last exercise, the relation $tg_1 = 0$ has degree 1, while the generator g_2 has degree 2.

⟨**5.9**⟩ **Exercise.** Carry out the inductive construction for the inverted V graph and the Egyptian pyramid. In both cases, $\underline{M}$ will coincide with $\underline{A}$ until the top vertex. At the top vertex ν, for the inverted V graph, M_ν will be as in the first example of Exercise 4.5.8 above. For the Egyptian pyramid, it will have a generator in degree 2 reflecting the phenomenon for linear functions explained in §4.4.6.

4.6. The Main Results

⟨**6.1**⟩ **Theorem.** [**7**]. Suppose that $T \circlearrowright Y$ is a generalized flag manifold as in §4.7.1 and that $T \circlearrowright X \subset T \circlearrowright Y$ is a generalized Schubert variety as in §4.7.3, and $\mathcal{G}(T \circlearrowright X)$ is its moment graph. Then the equivariant intersection cohomology of $T \circlearrowright X$ is the $\mathcal{O}(\mathbb{T})$ module of global sections of the sheaf $\underline{M}$ on its moment graph

$$IH^*(T \circlearrowright X) = \Gamma(\underline{M}, \mathcal{G}(T \circlearrowright X))$$

and the intersection homology of X is given by

$$IH^*(X) = IH^*(T \circlearrowright X) \otimes_{\mathcal{O}(\mathbb{T})} \mathbb{C} = \Gamma(\underline{M}, \mathcal{G}(T \circlearrowright X)) \otimes_{\mathcal{O}(\mathbb{T})} \mathbb{C}$$

⟨**6.2**⟩ **Theorem.** All Morse functions on $\mathcal{G}(\mathcal{H}, v)$ are perfect for the sheaf $\underline{M}$.

⟨**6.3**⟩ **Theorem.** $\Gamma(\underline{M}, \mathcal{G}(\mathcal{H}, v))$ is a free module over $\mathcal{O}(\mathbb{T})$.

Let IB_i be the number of free generators in degree i.

⟨**6.4**⟩ **Theorem.** $\Gamma(\underline{M}, \mathcal{G}(\mathcal{H}, v))$ satisfies Poincaré duality: $IB_i = IB_{-i}$. Moreover, the canonical pairing

$$\Gamma(\underline{M}, \mathcal{G}(\mathcal{H}, v)) \otimes_{\mathcal{O}(\mathbb{T})} \Gamma(\underline{M}, \mathcal{G}(\mathcal{H}, v)) \quad \longrightarrow \quad \mathcal{O}(\mathbb{T})$$

is perfect in the sense that the induced map

$$\Gamma(\underline{M}, \mathcal{G}(\mathcal{H}, v)) \longrightarrow \mathrm{Hom}(\Gamma(\underline{M}, \mathcal{G}(\mathcal{H}, v)), \mathcal{O}(\mathbb{T}))$$

is an isomorphism of $\mathcal{O}(\mathbb{T})$ modules.

4.7.* Flag Varieties and Generalized Schubert Varieties

⟨**7.1**⟩ **Generalized flag manifolds.** Suppose that G is a connected compact Lie group, T is its maximal torus, and H is a connected compact subgroup of G with the same maximal torus T. Then G/H is a *generalized flag manifold*.

⟨**7.2**⟩ This relates to the linear graph $\mathcal{G}(\mathcal{H}, v)$ constructed in §4.1.2 as follows: If $\mathcal{H}$ is crystallographic §4.1.4, then we can construct a compact group G such whose coroots correspond to $\mathcal{H}$ and whose Weyl group is the reflection group. We can construct a subgroup H corresponding to the choice of $v \in \mathbb{V}$. (It depends only on which hyperplanes in $\mathcal{H}$ the element v lies on.) Then the moment graph of G/H will be $\mathcal{G}(\mathcal{H}, v)$. In the case where v lies on no hyperplane in $\mathcal{H}$, P is the Borel subgroup, the abstract graph corresponding to $\mathcal{G}(\mathcal{H}, v)$ is called the Bruhat graph.

⟨**7.3**⟩ **Generalized Schubert varieties.** The generalized flag manifold G/H also has a description as $G_{\mathbb{C}}/P$ where $G_{\mathbb{C}}$ is the complexification of G and P is a parabolic subgroup.

Suppose we have a Morse function $\phi : \mathbb{V} \longrightarrow \mathbb{R}$ and an upward saturated subgraph $\mathcal{G}'$ of $\mathcal{G}(\mathcal{H}, v)$. We can approximate the linear function by a rational one, without changing the fact that $\mathcal{G}'$ is upward saturated. Then ϕ is a coweight, so it corresponds to a map $\chi : \mathbb{C}^* \longrightarrow T_{\mathbb{C}}$. Now, the generalized Schubert variety whose moment graph is $\mathcal{G}'$ will be those $p \in G_{\mathbb{C}}/P$ such that $\lim_{\lambda \to 0} \chi(\lambda)p$ is a fixed point of $T \circlearrowright (G_{\mathbb{C}}/P)$ corresponding to one of the vertices in $\mathcal{G}'$.

There is an enormous literature on the manifolds $G_{\mathbb{C}}/P$ and on Schubert varieties and their singularities. For introductions with a combinatorial slant, see [**5**] and [**8**].

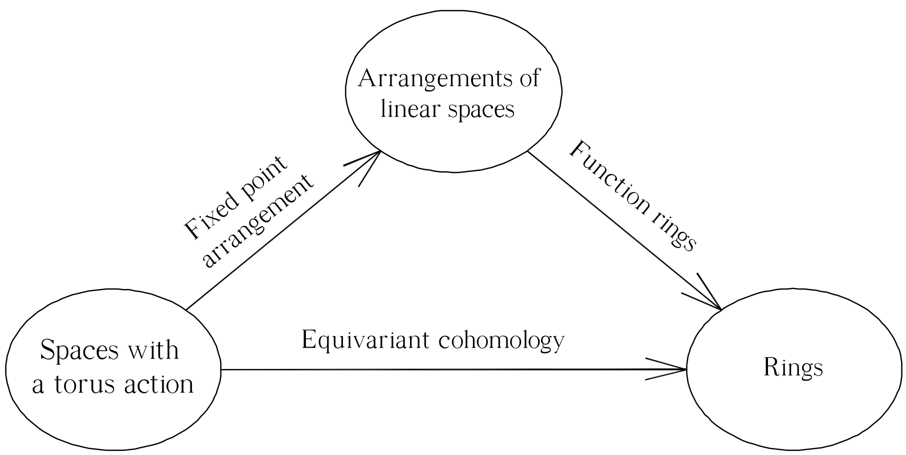
⟨**7.4**⟩ **A mystery.** What about non-crystallographic configurations of reflecting hyperplanes $\mathcal{H}$? For example, k lines at equal angles in $\mathbb{V} = \mathbb{R}^2$ where $k \neq 1, 2, 3, 4,$ or 6 is non-crystallographic. Other examples are constructed out of the icosahedron. All purely graph theoretic theorems we have stated work just as well for these examples. But there is no known topological space corresponding to them. It is reasonable to conjecture that all of the results of this Lecture (in particular the crucial Proposition 4.5.4) hold for the graphs $\mathcal{G}(\mathcal{H}, v)$ associated to non-crystallographic $\mathcal{H}$. Better yet, is there some sort of topological object whose "moment graph" is $\mathcal{G}(\mathcal{H}, v)$?

Another problem is to find general sufficient conditions on a linear graph $\mathcal{G}$ that hold for graphs of the form $\mathcal{G}(\mathcal{H}, v)$, so that the results of this Lecture work for $\mathcal{G}$.

⟨**7.5**⟩ If P is a polyhedron that is rational but not simple, then there is a construction of a sheaf on its 1-skeleton that has a similar relation to equivariant intersection cohomology to the one discussed in this lecture [**4**],[**24**],[**1**],[**2**],[**3**]. A similar mystery applies to 1-skeleta of non-rational polyhedra. This has been the subject of a lot of recent work, e.g. [**9**].

LECTURE 5
Cohomology as Functions on a Variety
(Geometry of Subspace Arrangements)

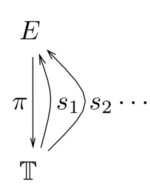
In this lecture, we will describe another paradigm for computing equivariant cohomology via geometry. This paradigm can be described by the following directed graph:



One advantage is that this paradigm treats certain spaces whose 1-skeleton is not a balloon sculpture, as was required up until now. But the main point is that the arrangements of linear spaces that arise in this way seem interesting in themselves.

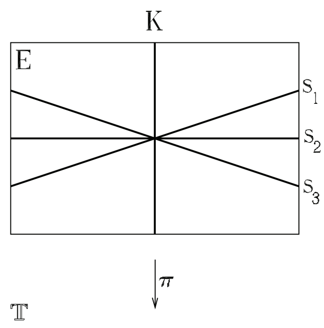
5.1. The Fixed Point Arrangement

⟨1.1⟩ Arrangements of sections. An arrangement of sections is a diagram of vector spaces

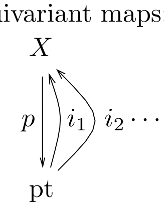


where $\pi : E \longrightarrow \mathbb{T}$ is a surjection and $s_j : \mathbb{T} \longrightarrow E$ are sections, i.e. maps satisfying the section relations: $\pi(s_j(t)) = t$ for all $t \in \mathbb{T}$, i.e. $\pi \circ s_j$ is the identity on $\mathbb{T}$.

⟨**1.2**⟩ Suppose we have fixed $E \xrightarrow{\pi} \mathbb{T}$. To give a section s_i is the same thing as to give its image $s_i(\mathbb{T})$ as a subspace of E. So an arrangement of sections is equivalent to an arrangement of subspaces of E, each of which is transverse to K, the kernel of π.



⟨**1.3**⟩ **A diagram of spaces.** Now suppose we have a space with a torus action $T \circlearrowleft X$ with finitely many fixed points $F_1, F_2, \ldots$ Then we have the following diagram of spaces with a T action and T equivariant maps:



here pt is a point (with a trivial T action), $p : X \longrightarrow$ pt is the only thing it could be, and $i_j : \text{pt} \longrightarrow X$ sends pt to the fixed point F_j.

⟨**1.4**⟩ **The fixed point arrangement.** We apply second equivariant homology functor $H_2(\cdot)$ to this diagram.

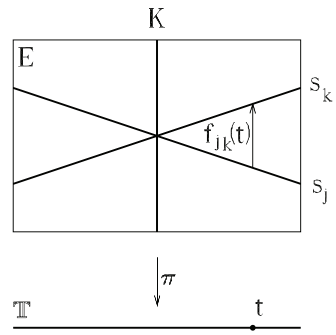
Definition. The *fixed point arrangement* of X is the arrangement of sections $E = H_2(T \circlearrowleft X)$, $\mathbb{T} = H_2(T \circlearrowleft \text{pt})$, $\pi = p_* : H_2(T \circlearrowleft X) \longrightarrow H_2(T \circlearrowleft \text{pt})$ and $s_j = (i_j)_* : H_2(T \circlearrowleft \text{pt}) \longrightarrow H_2(T \circlearrowleft X)$.

5.2. How to Compute the Fixed Point Arrangement

The definition of the fixed point arrangement of $T \circlearrowleft X$ is difficult to work with, because we don't usually have a good handle on the second equivariant homology group. In this section, we give a way to compute the fixed point arrangement without using equivariant homology.

⟨**2.1**⟩ **The spaces K and $\mathbb{T}$.** The space $H_2(T \circlearrowleft \text{pt})$ is what we've been calling $\mathbb{T}$ all along: the vector space such that $\mathbb{T}/L = T$ or, otherwise put, the Lie algebra of T. If X has only even dimensional homology, then the space K is $H_2(X)$, the ordinary second homology group of X. (This follows from equivariant formality, see [**13**].)

⟨2.2⟩ How to specify an arrangement of sections. If we have an arrangement of sections, for every ordered pair of sections s_j and s_k, we have a linear map $f_{jk} : \mathbb{T} \longrightarrow K$ which associates to $t \in \mathbb{T}$ the vector $s_k(t) - s_k(t)$ as in this picture:



Conversely, if we know the linear functions f_{jk}, that determines the configuration up to automorphisms of E commuting with π, which is enough for our purposes. In fact, since $f_{jk} + f_{kl} = f_{kl}$, if we're really efficient about it, we only need to know $m - 1$ of them, where m is the number of fixed points.

⟨2.3⟩ Determining f_{kj}. Suppose that F_j and F_k lie in a balloon B: a 2-sphere that is taken into itself by T. Since there are only finitely many fixed points, T the orbits of T on B will all be circles, except for F_j and F_k. Chose such a circle, S. There will be a linear map $g : T \longrightarrow \mathbb{R}$ and an identification of S with $\mathbb{R}/\mathbb{Z}$ such that the action of $\bar{t}$ corresponds to addition of $g(t)$. The trouble is that there are two such functions g which are negatives of each other, corresponding to opposite identifications of S with $\mathbb{R}/\mathbb{Z}$. If we had an orientation of S, we could specify that the orientation should correspond to the natural orientation of $\mathbb{R}$.

Choose an orientation $\mathbb{O}$ for B. That does two things for us. First, it makes B into a cycle, so it gives us a class $[B]$ in homology. Second, it restricts to an orientation of the disk bounded by S containing F_j. This induces an orientation on its boundary, S, solving the problem above. With these conventions, we get

$$f_{jk} = g[B]$$

⟨2.4⟩ Exercise. Show that this definition is independent of the orientation $\mathbb{O}$ of B chosen. Show directly from this definition that $f_{jk} = -f_{kj}$.

5.3. The Main Result

⟨3.1⟩ The function ring of the configuration. Let A be the union of the linear subspaces in the fixed point configuration. In other words, $a \in A$ if and only if $a = s_j(t)$ for some section s_j in our arrangement and some $t \in \mathbb{T}$. The set A is a real algebraic variety - it has a function ring $\mathcal{O}(A)$, which is the polynomial functions on E, two being considered equivalent if they take the same values on every point of A. In other words,

$$\mathcal{O}(A) = \frac{\mathcal{O}(E)}{I(A)}$$

where $I(A)$ is the ideal of polynomials vanishing on the set A.

⟨**3.2**⟩ The map $\pi : E \longrightarrow \mathbb{T}$ provides an inclusion $\mathcal{O}(\mathbb{T}) \subset \mathcal{O}(A)$. An element f of $\mathcal{O}(\mathbb{T})$ is a polynomial function $\mathbb{T} \xrightarrow{f} \mathbb{R}$ so the composition $A \xrightarrow{\pi} \mathbb{T} \xrightarrow{f} \mathbb{R}$ is in $\mathcal{O}(A)$.

⟨**3.3**⟩ **Theorem.** [**17**] Suppose that $H^*(X)$ is generated as a ring by $H^2(X)$. Then the equivariant cohomology of X is the function ring of the fixed point configuration of X

$$H^*(T \circlearrowright X) = \mathcal{O}(A).$$

The ring $\mathcal{O}(A)$ is a free graded module over the polynomial algebra $\mathcal{O}(\mathbb{T})$. As in §3.8.3,

$$H^*(X) = \frac{\mathcal{O}(A)}{\text{the ideal generated by } \mathcal{O}(\mathbb{T})^{>0}}$$

where $\mathcal{O}(\mathbb{T})^{>0}$ is the positive degree part of $\mathcal{O}(\mathbb{T})$.

⟨**3.4**⟩ **Remark.** The fact that $\mathcal{O}(A)$ is a free module over $\mathcal{O}(\mathbb{T})$ is an interesting and mysterious property of the configuration. Here it follows from theorems in topology. We don't know any geometric characterization of which configurations of sections have this property.

5.4. Springer Varieties

Suppose that $k = k_1 + \cdots + k_j$ is a partition of the integer k. Let γ be the $k \times k$ matrix whose Jordan blocks have size $k_1, \ldots, k_j$. For example if the partition is $6 = 3 + 2 + 1$, then γ is given by

$$\gamma = \begin{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} & & \\ & \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} & \\ & & [0] \end{bmatrix}$$

⟨**4.1**⟩ **Definition.** The *Springer variety* X_γ of γ is the set of complete flags $0 \subset F^1 \subset F^2 \subset \cdots \subset F^{k-1} \subset \mathbb{C}^k$, where F^i is an i-dimensional subspace of $\mathbb{C}^k$, with the property that γ takes each space F^k into itself.

⟨**4.2**⟩ **The torus action on** X_γ. There is a torus T that acts on X_γ. This is the set of diagonal matrices a with the following properties: the determinant of a is 1; the entries of a are complex numbers with absolute value 1; and furthermore the first k_1 diagonal entries are equal to each other; the next k_2 diagonal entries are equal to each other, and so on. For example, for our partition $6 = 3 + 2 + 1$, T consists of the matrices

$$a = \begin{bmatrix} \begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_1 & 0 \\ 0 & 0 & a_1 \end{bmatrix} & & \\ & \begin{bmatrix} a_2 & 0 \\ 0 & a_2 \end{bmatrix} & \\ & & [a_3] \end{bmatrix} \quad \text{such that} \quad |a_i| = 1 \quad \text{and} \quad \det a = 1.$$

⟨**4.3**⟩ **Exercise.** Prove that the torus T preserves the Springer variety X_γ. Use the fact that the matrix t commutes with the matrix γ.

⟨**4.4**⟩ **The arrangement of subspaces.** The action $T \circlearrowright X_\gamma$ of the torus T on the Springer variety satisfies the hypotheses of Theorem 5.3.3 (see [**17**]). (In general, the 1-skeleton of $T \circlearrowright X_\gamma$ is too complicated to satisfy the hypotheses §3.8.1 of Theorem 3.8.2, so the methods of Lecture 3 don't apply to Springer varieties.)

The subspace arrangement A for the Springer variety is quite beautiful configuration that depends, of course, only on the given partition.

The Lie algebra $\mathbb{T}$ of T is the set of $k \times k$ diagonal real matrices with trace 0 such that the first k_1 diagonal entries are equal to each other, the next k_2 diagonal entries are equal to each other, and so on. For $6 = 3 + 2 + 1$, the elements t of $\mathbb{T}$ are

$$t = \begin{bmatrix} \begin{bmatrix} t_1 & 0 & 0 \\ 0 & t_1 & 0 \\ 0 & 0 & t_1 \end{bmatrix} & & \\ & \begin{bmatrix} t_2 & 0 \\ 0 & t_2 \end{bmatrix} & \\ & & [t_3] \end{bmatrix} \quad \text{such that} \quad t_i \quad \text{is real and} \quad \text{trace } t = 0$$

The space $K = H_2(X_\gamma)$ is $(k-1)$-dimensional. It can be identified with the set of all $k \times k$ real matrices with trace 0. (This holds for all partitions except for the trivial partition $k = k$. For the trivial partition, the Springer variety X_γ is just a point, so there isn't much to study.)

There are $k!/(k_1! \cdots k_j!)$ fixed points of the action of $\mathbb{T}$ on X_γ, so the arrangement A consists of $k!/(k_1! \cdots k_j!)$ linear subspaces of $E = \mathbb{T} \oplus K$. Each of these is the graph of a linear map m from $\mathbb{T}$ to K. The linear maps m are constructed as follows: Write a diagonal matrix so that t_1 occurs k_1 times, t_2 occurs k_2 times, etc. There are $k!/(k_1! \cdots k_j!)$ ways to do this. For our partition $6 = 3 + 2 + 1$, here's one of them:

$$\begin{bmatrix} t_1 & & & & & \\ & t_3 & & & & \\ & & t_1 & & & \\ & & & t_2 & & \\ & & & & t_1 & \\ & & & & & t_2 \end{bmatrix}$$

This provides our map m from $\mathbb{T}$ to K.

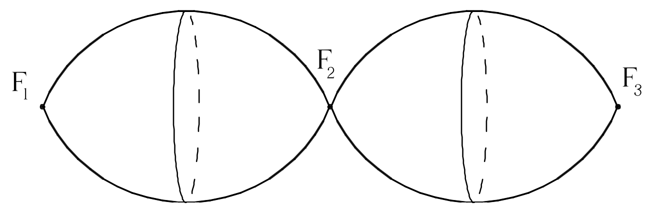
⟨**4.5**⟩ **The Springer action.** The symmetric group S_k on k letters acts on the configuration A of linear subspaces (by permuting the diagonal entries in the last matrix). It follows that S_k acts on the equivariant cohomology of X_γ ([**17**]). The S_k action on A preserves the fibers of the map π to $\mathbb{T}$. Therefore, it passes to an action of S_k on the ordinary cohomology of X_γ. This is the usual Springer action.

A similar equivariant cohomology construction for Springer actions on the cohomology of Springer varieties for loop groups is found in [**14**]. Historically, the effort to solve the problem addressed in [**14**] was what originally led to the whole body of material in this lecture series.

⟨4.6⟩ Example. Suppose we start with the partition $3 = 2+1$ so γ is the nilpotent matrix

$$\gamma = \begin{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} & \\ & [0] \end{bmatrix}.$$

The Springer variety X_γ consists of two 2-spheres joined together. The T action has 3 fixed points F_1, F_2, and F_3. The picture looks like this:



The Springer variety X_γ

(In this case, the 1 skeleton is a balloon sculpture, so Theorem 3.8.2 applies as well.)

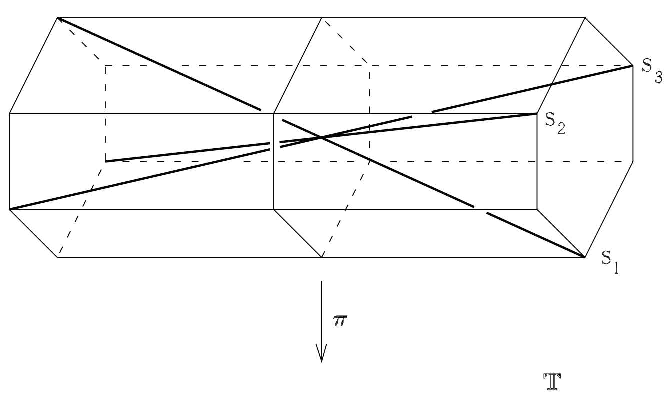
The Lie algebra $\mathbb{T}$ is 1-dimensional. It is the set of real matrices

$$\begin{bmatrix} \begin{bmatrix} t_1 & 0 \\ 0 & t_1 \end{bmatrix} & \\ & [t_2] \end{bmatrix} \quad \text{such that} \quad 2t_1 + t_2 = 0$$

The space K is 2-dimensional. It is the space of diagonal matrices of trace 0. The three maps from $\mathbb{T}$ to K corresponding to the three fixed points are given by

$$\begin{bmatrix} t_1 & & \\ & t_1 & \\ & & t_2 \end{bmatrix}, \begin{bmatrix} t_1 & & \\ & t_2 & \\ & & t_1 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} t_2 & & \\ & t_1 & \\ & & t_1 \end{bmatrix}$$

The resulting configuration of sections looks like this:



Note that this configuration has an obvious action of S_3, the symmetric group which permutes the 3 sections. Thus we get an action of S_3 on the equivariant cohomology of X_γ and also the ordinary cohomology. This is not induced by an action of S_3 on X_γ itself: X_γ is less symmetric than its fixed point configuration.

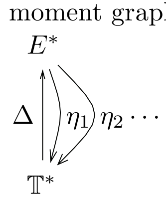
⟨4.7⟩ Exercise. Verify the statements in the last section about the Springer variety for the partition $3 = 2 + 1$.

5.5. Relation with Lecture 3

Suppose $T \circlearrowright X$ satisfies both the hypotheses for Theorem 5.3.3 above and for Theorem 3.8.2. This happens, for example, with flag manifolds and toric varieties for simple polytopes. What is the relation between the two pictures?

Given the moment graph $\mathcal{G}(X \circlearrowright T)$ we will construct the configuration of sections.

Suppose the graph has k vertices $\nu_1, \dots, \nu_k$. The space of all graphs equivalent to $\mathcal{G}(X \circlearrowright T)$ embeds in $\mathbb{V}^k$ by the position of the k vertices. The space $E^* = H^2(T \circlearrowright X)$ is the closure. Also the projections on the factors give k maps η_j from E^* to $\mathbb{V}$. The diagonal gives a map $\Delta : \mathbb{V} \longrightarrow E^*$. As we have assumed from the beginning, $\mathbb{V} = \mathbb{T}^*$. In summary, the moment graph $\mathcal{G}(X \circlearrowright T)$ gives us a diagram



If we dualize this diagram, we get the arrangement of sections of §5.1.1.

⟨5.1⟩ Exercise. In this situation, construct a map

$$\mathcal{O}(A) \longrightarrow \mathcal{H}^*(\mathcal{G}(T \circlearrowright X)).$$

Assuming that $\mathcal{H}^*(\mathcal{G}(T \circlearrowright X))$ is generated by $\mathcal{H}^2(\mathcal{G}(T \circlearrowright X))$, show that this map is an isomorphism.

BIBLIOGRAPHY

1. G. Barthel, J.-P. Brasselet, K.-H. Fieseler, L. Kaup, *Combinatorial Intersection Cohomology for Fans*, Tohoku Math. J. 54 (2002), 1-41.
2. G. Barthel, J.-P. Brasselet, K.-H. Fieseler, L. Kaup, *Equivariant Intersection Cohomology of Toric Varieties*, Contemp. Math. 241, Amer. Math. Soc. (1999), 45-68.
3. G. Barthel, J.-P. Brasselet, K.-H. Fieseler, L. Kaup, *Invariante Divisoren und Schnitthomologie von Torischen Varietäten*, Parameter Spaces (Warsaw, 1994), 9-223, Banach Center Pub. 36.
4. J. Bernstein and V. Lunts, Equivariant Sheaves and Functors, Springer Lect. Notes in Math. 1578 (1994).
5. S. Billey and Lakshmibai, *Singular Loci of Schubert Varieties*, Progress in Math. vol.182 Birkhauser Boston (2000).
6. T. Braden, *Perverse Sheaves on Grassmannians*, Canad. J. Math. 54 (2002) 493-532.
7. T. Braden and R. MacPherson, *From Moment Graphs to Intersection Cohomology*, Math. Ann. 321 (2001), 533-551.
8. F. Brenti, *Kazhdan-Lusztig Polynomials: History, Problems, and Combinatorial Invariance*, Seminaire Lotharingien de Combinatoire 49 (2003), 613-627.
9. P. Bressler and V. Lunts, *Intersection Cohomology on Nonrational Polytopes*, Compositio Math. 135 (2003), 245-278.
10. J.-L. Brylinski, *Equivariant Intersection Cohomology*, Contemp. Math. 139, Amer. Math. Soc., (1992) 5-32.
11. I. Gelfand, M. Goresky, R. MacPherson, V. Serganova, *Combinatorial Geometries, Grassmannians, and the Moment Map*, Advances in Math. 63 (1987) 301-316.
12. I. Gelfand and R. MacPherson *Geometry in Grassmannians and a Generalization of the Dilogarithm*, Advances in Math. 44(1982) 279-312.
13. M. Goresky, R. Kottwitz, and R. MacPherson, *Equivariant Cohomology, Koszul Duality, and the Localization Theorem*, Invent. Math. 131 (1998) 25-83.
14. M. Goresky, R. Kottwitz, and R. MacPherson, *Homology of Affine Springer Fibers in the Unramified Case*, Duke Math. J. 121(2004), 509-561.
15. M. Goresky and R. MacPherson, *Intersection Homology Theory*, Topology 49 (1980), 135-162.

16. M. Goresky and R. MacPherson, *On the topology of Algebraic Torus Actions*, Springer Lect. Notes in Math 1271 (1987) 73-90.
17. M. Goresky and R. MacPherson, *On the spectrum of the equivariant cohomology ring*, to appear.
18. V. Guillemin and K. Zara, *The Existence of Generating Families for the Cohomology Ring of a Graph,* Advances in Math. 174 (2003) 115-173.
19. V. Guillemin and K. Zara, *G-Actions on Graphs*, Int. Math. Res. Notes (2001),519-542.
20. V. Guillemin and K. Zara, 1-*Skeletons, Betti Numbers, and Equivariant Cohomology*, Duke Math. J. 107 (2001), 283-349.
21. V. Guillemin and K. Zara, *Equivariant de Rham Theory and Graphs*, Asian J. of Math. 3 (1999) 49-76.
22. T. Holm, Equivariant Cohomology, Homogeneous Spaces, and Graphs, Thesis, MIT, 2002.
23. R. Joshua, *Equivariant Intersection Cohomology — A Survey*, Contemp. Math. 88, Amer. Math. Soc., (1989), 25-31.
24. V. Lunts, *Equivariant Sheaves on Toric Varieties*, Compositio Math. 96 (1995) 63-83.