APPENDIX: HILBERT SCHEMES OF POINTS IN THE PLANE

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Consider the polynomial ring $\mathbb{C}[x,y]$ in two variables over the complex numbers. As a set, the *Hilbert scheme* $H_n = \mathrm{Hilb}^n(\mathbb{C}^2)$ of n points in the plane consists of those ideals $I \subseteq \mathbb{C}[x,y]$ such that the quotient $\mathbb{C}[x,y]/I$ has dimension n as a vector space over $\mathbb{C}$. This appendix provides some background on how this set can be considered naturally as a smooth algebraic variety of dimension $2n$. The goal is to orient the reader rather than to give a complete introduction. Therefore some details are omitted from the exposition to make the intuition more clear (and short). The material here, which is based loosely on the introductory parts of [Hai98], reflects what was presented at the help session for Haiman's lectures; in particular, the Questions were all asked by participants at the help session.

To begin, let's get a feeling for what an ideal I of colength n can look like. If $P_1, \ldots, P_n \in \mathbb{C}^2$ are distinct (reduced) points, for example, then the ideal of functions vanishing on these n points has colength n. This is because the ring of functions on n points has a vector space basis $\{f_1, \ldots, f_n\}$ in which $f_i(P_j) = 0$ unless $i = j$, and $f_i(P_i) = 1$. Ideals of the form $I(P_1, \ldots, P_n)$ are called *generic* colength n ideals.

At the opposite end of the spectrum, I could be an ideal whose (reduced) zero set consists of only one point $P \in \mathbb{C}^2$. In this case, $\mathbb{C}[x,y]/I$ is a local ring with lots of nilpotent elements. In geometric terms, this means that P carries a nonreduced scheme structure. Such a nonreduced scheme structure on P is far from unique; in other words, there are many length n local rings $\mathbb{C}[x,y]/I$ supported at P. In fact, we shall see below that in a precise sense they come in an n-dimensional family.

Among the ideals supported at single points, the *monomial ideals* are the most special. These ideals have the form $I = \langle x^{a_1}y^{b_1}, \ldots, x^{a_m}y^{b_m}\rangle$ for some nonnegative integers $a_1, b_1, \ldots, a_m, b_m$, and are supported at $(0,0) \in \mathbb{C}^2$. Note that if $x^h y^k$ is a monomial outside of I and $x^{h'}y^{k'}$ is a monomial dividing $x^h y^k$ (so $h' \leq h$ and $k' \leq k$), then $x^{h'}y^{k'}$ also lies outside of I. This makes it convenient to draw the monomials outside of I as the boxes "under a staircase".

Example 1. For the ideal $I = \langle x^2, xy, y^3\rangle$ of colength $n = 4$, the diagram of boxes under the staircase has an 'L'-shape:

$$\begin{array}{|c|c}\hline y^2 & \\ \cline{1-1} y & \\ \hline 1 & \multicolumn{1}{c|}{x} \\ \hline\end{array}$$

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Note that the monomial x^2 would be the first box after the bottom row, while xy would nestle in the nook of the 'L', and y^3 would lie atop the first column. Thus the minimal generators of I specify where to draw the staircase. □

If the diagram of monomials outside I has λ_i boxes in row i under the staircase, then $\sum_i \lambda_i = n$ is by definition a *partition* λ of n, and we write $I = I_\lambda$.

Example 2. In Example 1, there are 2 boxes in row 0, and 1 box in each of rows 1 and 2, yielding the partition $2+1+1=4$ of $n=4$. Thus the ideal is $I = I_{2+1+1}$. □

In full generality, the quotient $\mathbb{C}[x,y]/I$ is a product of local rings with maximal ideals corresponding to a finite set $P_1, \ldots, P_r$ of distinct points in $\mathbb{C}^2$, with the lengths $\ell_1, \ldots, \ell_r$ of these local rings satisfying satisfying $\ell_1 + \cdots + \ell_r = n$ (do not confuse this partition of n with the partitions obtained from monomial ideals, where $r = 1$). When $r = n$ it must be that $\ell_i = 1$ for all i, so the ideal I is generic.

Question 1. Is there some transformation of the plane so that every colength n ideal has a basis of monomials?

Answer 1. This question can be interpreted in two different ways, because the word "basis" has multiple meanings. Thinking of "basis" as "generating set", the question asks if given I, there is a coordinate system for $\mathbb{C}^2$ in which I is a monomial ideal. The answer is no, in general; for instance, if $\mathbb{C}[x,y]/I$ is not a local ring, then I can't be a monomial ideal in any coordinates. The second meaning of "basis" is "$\mathbb{C}$-vector space basis". Even though I itself may not be expressible in some coordinates as a monomial ideal, the quotient $\mathbb{C}[x,y]/I$ always has a $\mathbb{C}$-vector space basis of (images of) monomials. This observation will be crucial later on.

If all colength n ideals were generic, then the set H_n would be easy to describe, as follows. Every unordered list of n distinct points in $\mathbb{C}^2$ corresponds to a set of $n!$ points in $(\mathbb{C}^2)^n$, or alternatively to a single point in the quotient $S^n\mathbb{C}^2 := (\mathbb{C}^2)^n/S_n$ by the symmetric group. Of course, not every point of $S^n\mathbb{C}^2$ corresponds to an unordered list of *distinct* points; for that, one needs to remove the diagonals

$$\{(P_1, \ldots, P_n) \in (\mathbb{C}^2)^n \mid P_i = P_j\} \tag{1}$$

of $(\mathbb{C}^2)^n$ before quotienting by S_n. Since S_n acts freely on the complement $((\mathbb{C}^2)^n)^\circ$ of the diagonals (1), the complement $(S^n\mathbb{C}^2)^\circ$ of the diagonals in the quotient $S^n\mathbb{C}^2$ is smooth. Therefore, whatever variety structure we end up using, H_n will contain an open smooth subvariety $(S^n\mathbb{C}^2)^\circ$ of dimension $2n$ parametrizing generic ideals.

The variety structure on H_n arises by identifying it as an algebraic subvariety of a more familiar variety: a grassmannian. Consider the vector subspace V_n inside of $\mathbb{C}[x,y]$ spanned by the $\binom{n+2}{2}$ monomials of degree at most n.

Lemma 3. *Given any colength n ideal I, the image of V_n spans the quotient $\mathbb{C}[x,y]/I$ as a vector space.*

Proof. The n monomials outside any initial ideal of I span the quotient $\mathbb{C}[x,y]/I$, and these monomials must lie inside V_n. □

The intersection $I \cap V_n$ is a vector subspace of codimension n. Thus H_n is (as a set, at least) contained in the grassmannian $\mathrm{Gr}^n(V_n)$ of codimension n subspaces of V_n.

Definition 4. Given a partition λ of n, write $U_\lambda \subset H_n$ for the set of ideals I such that the monomials outside I_λ map to a vector space basis for $\mathbb{C}[x,y]/I$.

The set of codimension n subspaces $W \subset V_n$ for which the monomials outside I_λ span V_n/W consitutes a standard open affine subvariety of $\mathrm{Gr}^n(V_n)$, defined by the nonvanishing of the corresponding Plücker coordinate. This means that W has a unique basis consisting of vectors of the form

$$x^r y^s - \sum_{hk \in \lambda} c_{hk}^{rs} x^h y^k. \tag{2}$$

Here, we write $hk \in \lambda$ to mean $x^h y^k \notin I_\lambda$, so the box labeled (h,k) lies under the staircase for I_λ. The affine open inside $\mathrm{Gr}^n(V_n)$ is actually a cell—namely, the variety whose coordinate ring is the polynomial ring in the coefficients c_{hk}^{rs} from (2).

The intersection of each ideal $I \in U_\lambda$ with V_n is a codimension n subspace of V_n spanned by vectors of the form (2), by definition of U_λ. Of course, if $W \subset V_n$ is to be expressible as the intersection of V_n with some ideal I, the coefficients c_{hk}^{rs} can't be chosen completely at will. Indeed, the fact that I is an ideal imposes relations on the coefficients that say "multiplication by x takes $x^r y^s$ to $x^{r+1}y^s$ and preserves I, and similarly for multiplication by y."

Explicitly, if $x^{r+1}y^s \in V_n$, then multiplying (2) by x yields another polynomial $x^{r+1}y^s - \sum_{hk\in\lambda} c_{hk}^{rs} x^{h+1}y^k$ inside $I \cap V_n$. Some of the terms $x^{h+1}y^k$ no longer lie outside I_λ, so we have to expand them again using (2) to get

$$x^{r+1}y^s - \Bigg(\sum_{h+1,k\in\lambda} c_{hk}^{rs} x^{h+1} y^k + \sum_{h+1,k\notin\lambda} c_{hk}^{rs} \sum_{h'k'\in\lambda} c_{h'k'}^{h+1,k} x^{h'} y^{k'} \Bigg) \in I. \tag{3}$$

Equating the coefficients on $x^h y^k$ in (3) to those in

$$x^{r+1}y^s - \sum_{hk\in\lambda} c_{hk}^{r+1,s} x^h y^k$$

from (2) yields relations in the polynomial ring $\mathbb{C}[\{c_{hk}^{rs}\}]$. These relations, taken along with their counterparts that result by switching the roles of x and y, cut out U_λ. Though we have yet to see that these relations generate a radical ideal, we can at least conclude that U_λ is an algebraic subset of an open cell in the grassmannian.

Theorem 5. *The affine varieties U_λ cover the subset $H_n \subset \mathrm{Gr}^n(V_n)$, thereby endowing H_n with the structure of quasiprojective algebraic variety.*

Proof. The sets U_λ cover H_n by Lemma 3, and each set U_λ is locally closed in $\mathrm{Gr}^n(V_n)$ by the discussion above. □

In summary: H_n is a quasiprojective variety because it is locally obtained by the intersection of a Zariski open condition (certain monomials span mod I) and a Zariski closed condition ($W \subset V_n$ is closed under multiplication by x and y). Having endowed H_n naturally with the structure of an algebraic variety, let us explore its properties.

Lemma 6. *Every point $I \in H_n$ is connected to a monomial ideal by a rational curve.*

Proof. Choosing a term order and taking a Gröbner basis of I yields a family of ideals parametrized by the coordinate variable t on the affine line. When $t = 1$ we get I back, and when $t = 0$ we get the initial ideal of I, which is a monomial ideal. □

This proof is stated somewhat vaguely, but can be made quite precise using the notion of flat family and the fact that Gröbner degenerations are flat families over the affine line [Eis95, Proposition 15.17]. Here is an example, for more concrete intuition.

Example 7. Suppose $I = \langle x^2 + 7x, xy + \sqrt{2}y, y^3 - y^2\rangle$, and consider the ideal

$$I = \langle x^2 + 7tx, xy + \sqrt{2}ty, y^3 - ty^2\rangle \subset \mathbb{C}[x, y][t].$$

This new ideal should be thought of as a family of ideals in $\mathbb{C}[x, y]$, parametrized by the coordinate t. The ideal at $\alpha \in \mathbb{C}$ is obtained by setting $t = \alpha$ in the generators for I_t. Every one of these ideals has colength 4, because they all have the ideal $\langle x^2, xy, y^3\rangle$ from Example 1 as an initial ideal. It follows that this family of ideals (or better yet, the family $\mathbb{C}[x, y][t]/I_t$ of quotients) is flat over $\mathbb{C}[t]$. □

Lemma 6 allows us to conclude the following.

Proposition 8. *The Hilbert scheme H_n is connected.*

Question 2. Lemma 6 only says that every ideal connects to some monomial ideal. How do you know that you can get from one monomial ideal to another?

Answer 2. They're all connected to generic ideals:

Lemma 9. *For every partition λ of n, the point $I_\lambda \in H_n$ lies in the closure of the generic locus $(S^n\mathbb{C}^2)^\circ$.*

Proof. Consider the set of exponent vectors (h, k) on monomials $x^h y^k$ outside I as a subset of $\mathbb{Z}^2 \subset \mathbb{C}^2$. These exponent vectors constitute a collection of n points in $\mathbb{C}^2$. The colength n ideal of these points is called the *distraction* I'_λ of I_λ. If $I_\lambda = \langle x^{a_1}y^{b_1}, \ldots, x^{a_m}y^{b_m}\rangle$, then $I'_\lambda = \langle f_1, \ldots, f_m\rangle$, where

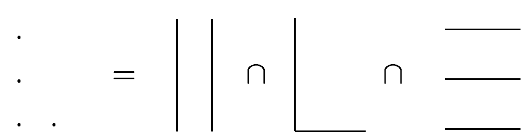
$$f_i = x(x-1)(x-2)\cdots(x-a_i+2)(x-a_i+1)y(y-1)\cdots(y-b_i+1).$$

Indeed, this ideal has colength n because every term of f_i divides its leading term $x^{a_i}y^{b_i}$, forcing I_λ to be the unique initial ideal of $\langle f_1, \ldots, f_m\rangle$; and each polynomial f_i clearly vanishes on the exponent set of I_λ, so each f_i lies in I'_λ. □

Example 10. The distraction of $I_{2+1+1} = \langle x^2, xy, y^3\rangle$ is the ideal

$$I'_{2+1+1} = \langle x(x-1),\ xy,\ y(y-1)(y-2)\rangle.$$

The zero set of every generator of the distraction is a union of lines, namely integer translates of one of the two coordinate axes in $\mathbb{C}^2$. The zero set of our ideal I'_{2+1+1} is



The groups of lines on the right hand side are the zero sets of $x(x-1)$, xy, and $y(y-1)(y-2)$, respectively. □

Remark 11. Proposition 8 holds for Hilbert schemes of n points in $\mathbb{C}^d$ even when d is arbitrary, with the same proof. The connectedness theorem of Hartshorne [Har66] says that it holds more generally for Hilbert schemes of $\mathbb{Z}$-graded ideals. However, the result does not extend to general Hilbert schemes of $\mathbb{Z}^n$-graded ideals [HS02, San02]. □

Proposition 12. *For each λ, the local ring $(H_n)_{I_\lambda}$ of H_n at I_λ has embedding dimension at most $2n$; that is, the maximal ideal $\mathfrak{m}_{I_\lambda}$ satisfies $\dim_{\mathbb{C}}(\mathfrak{m}_{I_\lambda}/\mathfrak{m}_{I_\lambda}^2) \le 2n$.*

Proof. Identify each variable c_{hk}^{rs} with an arrow pointing from the box $hk \in \lambda$ to the box $rs \notin \lambda$ (see Example 13). Allow arrows starting in boxes with $h < 0$ or $k < 0$, but set them equal to zero. The arrows lie inside—and in fact generate—the maximal ideal $\mathfrak{m}_{I_\lambda}$ at the point $I_\lambda \in H_n$. As each term in the double sum in (3) has two c's in it, the double sum lies inside $\mathfrak{m}_{I_\lambda}^2$. Moving both the tail and head of any given arrow one box to the right therefore does not change the arrow's residue class modulo $\mathfrak{m}_{I_\lambda}^2$, as long as the tail of the original arrow does not lie in the last box in a row of λ. Switching the roles of x and y, we conclude that an arrow's residue class mod $\mathfrak{m}_{I_\lambda}^2$ is unchanged by moving vertically or horizontally, as long as the tail stays under the staircase and the head stays above it. This analysis includes the case where the tail of the arrow crosses either axis, in which case the arrow is zero.

Every arrow can be moved horizontally and vertically until either

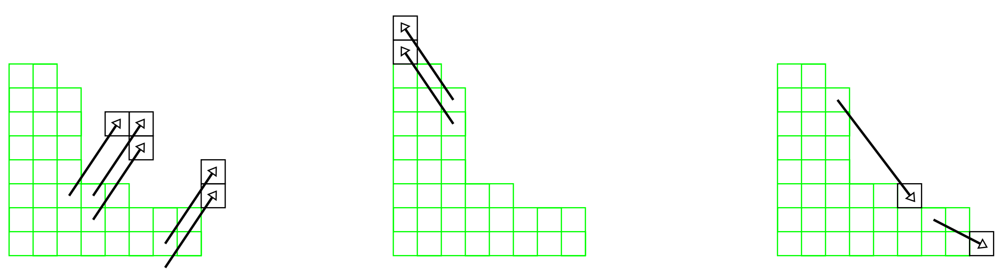
(i) the tail crosses an axis; or

(ii) there is a box $hk \in \lambda$ such that the tail lies just inside row k of λ while the head lies just above column h outside λ; or

(iii) there is a box $hk \in \lambda$ such that the tail lies just under the top of column h in λ while the head lies in the first box to the right outside row k of λ.

Arrows of the first sort do not contribute at all to $\mathfrak{m}_{I_\lambda}/\mathfrak{m}_{I_\lambda}^2$. On the other hand, there are exactly n northwest-pointing arrows of the second sort, and exactly n southeast-pointing arrows of the third sort. Therefore $\mathfrak{m}_{I_\lambda}/\mathfrak{m}_{I_\lambda}^2$ has dimension at most $2n$. □

Example 13. All three figures below depict the same partition λ: $8+8+5+3+3+3+3+2 = 35$. In the left figure, the middle of the five arrows represents $c_{31}^{54} \in \mathfrak{m}_{I_\lambda}$. As in the proof of Proposition 12, all of the arrows in the left figure are equal modulo $\mathfrak{m}_{I_\lambda}^2$. Since the bottom one is manifestly zero as in item (i) from the proof of Proposition 12, all of the arrows in the left figure represent zero in $\mathfrak{m}_{I_\lambda}/\mathfrak{m}_{I_\lambda}^2$.



The two arrows in the middle figure are equal, and the bottom one c_{25}^{08} provides an example of a regular parameter in $\mathfrak{m}_{I_\lambda}$ as in (ii). Finally, the two arrows in the rightmost figure represent unequal regular parameters as in (iii). □

Now we finally have enough prerequisites to prove the main result.

Theorem 14. *The Hilbert scheme H_n is smooth and irreducible of dimension $2n$.*

Proof. Since the intersection of two irreducible components would be contained in the singular locus of H_n, it is enough by Proposition 8 to prove smoothness. Lemma 9 implies that the dimension of the local ring of H_n at any monomial ideal I_λ is at least $2n$, because the generic locus has dimension $2n$. On the other hand, Proposition 12 shows that the maximal ideal of that local ring can be generated by $2n$ polynomials. Therefore H_n is regular in a neighborhood of any point I_λ.

The two-dimensional torus acting on $\mathbb{C}^2$ by scaling the coordinates has an induced action on H_n. Under this action, Lemma 6 and its proof say that every orbit on H_n contains a monomial ideal (= torus-fixed point) in its closure. By general principles, the singular locus of H_n must be torus-fixed (though not necessarily pointwise, of course) and closed. Since every torus orbit on H_n contains a smooth point of H_n in its closure, the singular locus must be empty. □

The proof of Theorem 14 used the fact that Gröbner degenerations are accomplished by taking limits of one-parameter torus actions on H_n. In plain language, this means simply that if appropriate powers of t are used in the equations defining the family I_t, the variable t can be thought of as a coordinate on $\mathbb{C}^*$ for nonzero values of t.

Remark 15. Theorem 14 fails for Hilbert schemes $\mathrm{Hilb}^n(\mathbb{C}^d)$ of points in spaces of dimension $d \geq 3$, as proved by Iarrobino [Iar72]. If it were irreducible, then $\mathrm{Hilb}^n(\mathbb{C}^d)$ would have dimension dn, the dimension of the open subset of configurations of n distinct points. But Iarrobino constructed a dimension e family of ideals of colength n in the polynomial ring, where e is proportional to $n^{(2-2/d)}$. It follows that $\mathrm{Hilb}^n(\mathbb{C}^d)$ is in fact reducible for $d \geq 3$ and n sufficiently large. On the other hand, $\mathrm{Hilb}^n(\mathbb{C}^d)$ is connected by reasoning as in the case $n = 2$ (Lemma 6 and Lemma 9). □

Question 3. Is the open set $U_\lambda \subset H_n$ the locus of colength n ideals having I_λ as an initial ideal?

Answer 3. When λ is the partition $1 + \cdots + 1 = n$, then yes. Otherwise, no, since the set of such ideals has dimension strictly less than $2n$. However, the locus in H_n of ideals having initial ideal I_λ is cell—that is, isomorphic to $\mathbb{C}^d$ for some d. Lemma 6 can be interpreted as saying that H_n is the disjoint union of these cells. This is the *Białynicki-Birula decomposition* of H_n [BB76, ES87]. It exists essentially because H_n has an action the torus $(\mathbb{C}^*)^2$ with isolated fixed points. Knowledge of the Białynicki-Birula decomposition allows one to compute the cohomology ring of H_n, which was the purpose of [ES87].

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