The Alexander duality functors and local duality with monomial support

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Abstract

Alexander duality is made into a functor which extends the notion for monomial ideals to any finitely generated $\mathbb{N}^n$-graded module. The functors associated with Alexander duality provide a duality on the level of free and injective resolutions, and numerous Bass and Betti number relations result as corollaries. A minimal injective resolution of a module M is equivalent to the injective resolution of its Alexander dual, and contains all of the maps in the minimal free resolution of M over every $\mathbb{Z}^n$-graded localization. Results are obtained on the interaction of duality for resolutions with cellular resolutions and lcm-lattices. Using injective resolutions, theorems of Eagon-Reiner and Terai are generalized to all $\mathbb{N}^n$-graded modules: the projective dimension of M equals the *support-regularity* of its Alexander dual, and M is Cohen-Macaulay if and only if its Alexander dual has a *support-linear* free resolution. Alexander duality is applied in the context of the $\mathbb{Z}^n$-graded local cohomology functors $H^i_I(-)$ for squarefree monomial ideals I in the polynomial ring S, proving a duality directly generalizing local duality, which is the case when $I = \mathfrak{m}$ is maximal. In the process, a new flat complex for calculating local cohomology at monomial ideals is introduced, showing, as a consequence, that Terai's formula for the Hilbert series of $H^i_I(S)$ is equivalent to Hochster's for $H^{n-i}_{\mathfrak{m}}(S/I)$.

1 Introduction

Alexander duality has a long history in combinatorial commutative algebra, beginning in earnest with the influential paper of Hochster [Hoc]. Recently, Alexander duality has been revived by the breakthrough of Eagon-Reiner [ER] revealing the duality for resolutions of squarefree monomial ideals between the conditions of being Cohen-Macaulay and having linear resolution. After generalizations by Terai [Ter], who proved that regularity is dual to projective dimension, and Herzog-Reiner-Welker [HRW], who proved that sequential Cohen-Macaulayness is dual to componentwise linearity, the question was raised in [HRW] whether Alexander duality makes sense for larger classes of modules than squarefree monomial ideals. This question was answered in the affirmative by [Mil], which extended the notion to arbitrary monomial ideals and their resolutions. More precisely, although the qualitative duality statements for squarefree ideals cannot hold verbatim for general monomial ideals, the quantitative versions involving Betti numbers introduced in [BCP] to generalize [Ter] remain valid. In a different direction, Yanagawa [Yan2] defined *squarefree modules* to study free resolutions and cohomology of squarefree monomial ideals (obtaining some results also obtained by [Mus]), and posed the problem of defining Alexander duality for them.

After the present overview, this paper starts by reviewing Alexander duality for squarefree monomial ideals in Section 1.1. An elementary introduction to Matlis duality is then presented in the context of the $\mathbb{Z}^n$-graded polynomial ring $S = k[x_1, \ldots, x_n]$ over a field k. After some combinatorial motivation in Section 1.3, Alexander duality is defined in Section 2 for all finitely generated $\mathbb{N}^n$-graded S-modules. Duality for the squarefree modules of

Yanagawa is obtained as a special case, as are all of the notions of Alexander duality defined thus far. In fact, Alexander duality is captured in a collection of *functors* (Definition 2.12) intimately related to Matlis duality and local duality (which is reviewed in Section 6). That the functors of Definition 2.12 are the natural extensions to $\mathbb{N}^n$-graded modules of Alexander duality for squarefree monomial ideals is demonstrated by their uniqueness (Theorem 2.6), which also shows that the duality for squarefree modules defined independently and simultaneously by Römer [Röm] via the exterior algebra is the appropriate special case here.

One of the primary goals of this paper is to demonstrate the utility and concreteness of injective resolutions for $\mathbb{Z}^n$-graded modules by exploring their structure and relation with free resolutions via Alexander and local duality. In Section 4 it is shown that Alexander duality provides a duality between free and injective resolutions: the maps in a (minimal) free resolution of a module M are dual to certain maps in a (minimal) injective resolution of its Alexander dual $M^{\mathbf{a}}$, and equal, in the appropriate sense, to certain maps in a (minimal) injective resolution of M (Theorem 4.5). However, the injective resolution contains much more information: a minimal injective resolution of M contains in it every minimal free resolution of every $\mathbb{Z}^n$-graded localization of M (Corollary 4.15). This abundance of information is also seen via the lcm-lattice of [GPW] (Section 4.3) as well as the Scarf and coScarf complexes of [BPS, Stu, MSY] (Example 4.8, Figures 1 and 2). In fact, the minimal injective resolution of a module M contains so much information that it is *equivalent* to the minimal injective resolution of its Alexander dual $M^{\mathbf{a}}$, in that there is a *generalized Alexander duality functor* (Definition 2.15) taking one to the other (Corollary 4.9). Thus, the functorial nature of Alexander duality introduced in Section 2 extends from modules to resolutions, and captures for $\mathbb{N}^n$-graded modules the essence of the breakthrough in [ER] in its most general form. This is quintessentially reflected in Theorem 4.20, where the duality theorems of [ER, Ter] for squarefree monomial ideals are generalized to all $\mathbb{N}^n$-graded modules, using properties of injective resolutions in the proof (but not the statement).

In order to deal efficiently with maps and complexes of flat and injective modules in the category of $\mathbb{Z}^n$-graded S-modules, *monomial matrices* are coined in Section 3, along with a definition of what it means for a map of ($\mathbb{Z}^n$-graded) flat or injective modules to be *minimal*. The manner in which monomial matrices reflect the functors in Definition 2.7 (which are used to define the Alexander duality functors) is systematically explored. The important class of cellular monomial matrices is introduced in Section 3.3, mainly for the purpose of easily visualized examples for the theory, such as Scarf and coScarf complexes (especially Figures 1 and 2 along with Examples 3.11, 3.14, 3.16, 3.22, 4.8, 4.17, 4.24, and 5.5).

The bookkeeping ability of monomial matrices is exploited in later sections to keep track of $\mathbb{Z}^n$-graded shifts and isomorphism classes of summands for the purposes of Betti and Bass number comparisons, localization, and local duality. In particular, the quantitative aspects of Alexander duality for $\mathbb{N}^n$-graded modules are collected in full generality in Section 5. These take the form of equalities and inequalities between Betti and Bass numbers of a module and those of its dual, and are immediate corollaries to Section 4.

Section 6 contains the generalization of local duality to the case of local cohomology with support on a radical monomial ideal I (Corollary 6.8). These local cohomology functors $H_I^{\cdot}(-)$ are important for computation of sheaf cohomology on toric varieties via the Cox homogeneous coordinate ring [EMS], and the duality proved here might simplify such computation in a manner similar to the way usual local duality simplifies computation of sheaf cohomology on projective space. The duality here is also related to the duality between $H^i_I(-)$ and local cohomology (at $\langle x_1,\ldots,x_n\rangle$) of the formal scheme completed along I [Ogus].

The local cohomology $H^{\cdot}_I(-)$ is calculated in Theorem 6.2 via a new complex of flat modules, called the *canonical Čech complex of* I, which are defined via the *Čech hull* (the most important functor of Definition 2.7). The canonical Čech complex is related to I the same way that the usual Čech complex is related to $\mathfrak{m} = \langle x_1,\ldots,x_n\rangle$, and its ($\mathbb{Z}^n$-graded) Matlis dual is related to $H^{\cdot}_I(-)$ in a manner similar to the way a dualizing complex is related to $H^{\cdot}_{\mathfrak{m}}(-)$. The canonical Čech complex is perhaps even more important than the duality theorem itself (Corollary 6.8), since it can also be used to calculate $H^{\cdot}_I(-)$ in an ungraded setting. In the $\mathbb{Z}^n$-graded setting the canonical Čech complex yields immediately the module structure and Hilbert series of $H^i_I(S)$ in terms of $H^{n-i}_{\mathfrak{m}}(S/I)$ (Corollary 6.7). These last calculations simplify those of [Mus] and [Ter2].

The rest of this introductory section is devoted to reviewing Alexander duality for radical monomial ideals and Matlis duality for $\mathbb{Z}^n$-graded modules $M = \bigoplus_{\mathbf{b}\in\mathbb{Z}^n} M_{\mathbf{b}}$ with the purpose of motivating the functors and categorical equivalences of Section 2.

1.1 Alexander duality

To each squarefree monomial ideal $I \subset S := k[x_1,\ldots,x_n]$ is associated a simplicial complex Δ_I on the set of vertices $\{1,\ldots,n\}$. The faces of Δ_I correspond to the squarefree monomials not in I:

$$\Delta_I \quad = \quad \{F \subseteq \{1,\ldots,n\} \mid \mathbf{x}^F \notin I\},$$

where $\mathbf{x}^F := \Pi_{i\in F} x_i$. Identifying a subset $F \subseteq \{1,\ldots,n\}$ and its characteristic vector in $\{0,1\}^n \subset \mathbb{Z}^n$, the *Alexander dual ideal* $I^{\mathbf{1}}$ is characterized as the squarefree ideal satisfying

$$(S/I)_F \neq 0 \quad \Longleftrightarrow \quad (I^{\mathbf{1}})_{\mathbf{1}-F} \neq 0\,, \tag{1}$$

where $\mathbf{1} = (1,\ldots,1)$. In other words, $I^{\mathbf{1}} = \langle \mathbf{x}^{\overline{F}} | F \in \Delta_I\rangle$, where $\overline{F} = \mathbf{1} - F = \{1,\ldots,n\} \setminus F$. It is not difficult to show that $(I^{\mathbf{1}})^{\mathbf{1}} = I$; see, for instance, [BCP] or [Mil]. There are many equivalent definitions of Alexander dual ideal, which may also be found in these references.

1.2 Matlis duality

For a $\mathbb{Z}^n$-graded ring R (usually the polynomial ring S or the field k, here), the functor $\underline{\mathrm{Hom}}_R(-,-)$ is defined on $\mathbb{Z}^n$-graded modules by letting its graded piece in degree $\mathbf{b} \in \mathbb{Z}^n$ consist of the homogeneous morphisms of degree $\mathbf{b}$:

$$\underline{\mathrm{Hom}}_R(M,N)_{\mathbf{b}} \quad = \quad \mathrm{Hom}_R(M, N[\mathbf{b}]) \quad = \quad \mathrm{Hom}_R(M[-\mathbf{b}], N).$$

(The nonunderlined Hom denotes homogeneous homomorphisms of degree zero, and the shift $N[\mathbf{b}]$ is defined from N by $N[\mathbf{b}]_{\mathbf{c}} = N_{\mathbf{b}+\mathbf{c}}$.) The *Matlis dual* $M^\vee$ is the $\mathbb{Z}^n$-graded module defined by the first equality below:

$$M^\vee \quad = \quad \underline{\mathrm{Hom}}_S(M, E(k)) \quad \cong \quad \underline{\mathrm{Hom}}_k(M,k),$$

where $E(k) = k[x_1^{-1}, \ldots, x_n^{-1}] \cong \underline{\mathrm{Hom}}_k(S,k)$ is the *injective of hull of* k as an S-module. The second isomorphism follows from the following, with $J = S$:

$$\underline{\mathrm{Hom}}_k(- \otimes_S J, k) \quad \cong \quad \underline{\mathrm{Hom}}_S(-, \underline{\mathrm{Hom}}_k(J,k))\,. \tag{2}$$

The degree $\mathbf{b}$ part of the Matlis dual is

$$(M^\vee)_{\mathbf{b}} \quad = \quad \mathrm{Hom}_k(M_{-\mathbf{b}}, k),$$

so Matlis duality "reverses the grading". This notion of Matlis dual agrees with that in [GW] and [BH, Section 3.6], and is applicable to modules M which are not necessarily finitely generated or artinian.

Let $\mathcal{M}$ denote the category of $\mathbb{Z}^n$-graded S-modules, the morphisms in $\mathcal{M}$ being homogeneous of degree $\mathbf{0}$. Matlis duality is an exact, contravariant, S-linear functor from $\mathcal{M}$ to $\mathcal{M}$. If, furthermore, $M_{\mathbf{b}}$ is a finite-dimensional k-vector space for all $\mathbf{b} \in \mathbb{Z}^n$, then $(M^\vee)^\vee \cong M$. Matlis duality interchanges noetherian modules and artinian ones because it turns ascending chains into descending chains. On the category of modules that are both artinian and noetherian—i.e. those with finite length—Matlis duality is a therefore a *dualizing functor* by definition. Matlis duality also interchanges flat objects in $\mathcal{M}$ with injective objects (relative to $\underline{\mathrm{Hom}}$, not Hom); indeed, the functor on the right in (2) is $\underline{\mathrm{Hom}}_S(-, J^\vee)$, and is exact $\Leftrightarrow$ the functor on the left is exact $\Leftrightarrow$ $- \otimes_S J$ is because k is a field.

1.3 Duality in order lattices

The "degree reversal" of Matlis duality can be viewed as duality in the order lattice $\mathbb{Z}^n$. To see this, suppose that $M \subseteq T := S[x_1^{-1}, \ldots, x_n^{-1}]$ is an S-submodule of the Laurent polynomial ring (i.e. a *monomial module*, in the language of [BS]). Then $M^\vee$ is the quotient of T characterized by

$$M_{\mathbf{b}} \neq 0 \quad \Longleftrightarrow \quad (M^\vee)_{-\mathbf{b}} \neq 0. \tag{3}$$

Thus, while M corresponds to a dual order ideal in $\mathbb{Z}^n$ (consisting of the exponent vectors on monomials in M), the Matlis dual $M^\vee$ corresponds to an order ideal in $\mathbb{Z}^n$ (consisting of the negatives of the exponent vectors on monomials in M).

Example 1.1 For instance, if for some $F \subseteq \{1, \ldots, n\}$ the module M is a localization

$$M = S[\mathbf{x}^{-\overline{F}}] := S[x_i^{-1} \mid i \notin F] \quad \text{then} \quad M^\vee = k[x_1^{-1}, \ldots, x_n^{-1}][x_i \mid i \notin F] =: \underline{E}(S/\mathfrak{m}^F).$$

Here, $\mathfrak{m}^F = \langle x_i \mid i \in F \rangle$, and the module $\underline{E}(S/\mathfrak{m}^F)$ is called the *injective hull in* $\mathcal{M}$ *of* $S/\mathfrak{m}^F$. Note that $S[\mathbf{x}^{-\overline{F}}]$ is a flat S-module, so that $S[\mathbf{x}^{-\overline{F}}]^\vee$ is indeed an injective object of $\mathcal{M}$. See [GW] for more on injectives and injective hulls in $\mathcal{M}$. □

In case $M \in \mathcal{M}$ is arbitrary, the Matlis dual $M^\vee$ can still be described in terms of lattice duality in $\mathbb{Z}^n$ by taking an injective resolution of M, which Matlis duality transforms into a flat resolution of $M^\vee$; see [Mil, Section 3] and Section 4, below.

Alexander duality for squarefree monomial ideals can also be thought of as duality in an order lattice, but this time in the much smaller lattice $\{0,1\}^n \subset \mathbb{Z}^n$. Of course, the duality in $\{0,1\}^n$ is $F \mapsto \mathbf{1} - F = \overline{F}$, so this comes from Equation (1). Comparing Equations (1) and (3), we find that Alexander duality looks just like Matlis duality except that for Alexander duality:

1. We only care about degrees F in the interval $[\mathbf{0}, \mathbf{1}] = \{0,1\}^n \subset \mathbb{Z}^n$;
2. We assume that $I^{\mathbf{1}}$ is a squarefree ideal; and
3. We have to shift $I^{\mathbf{1}}$ by $\mathbf{1}$; that is, instead of $F \mapsto -F$, we have $F \mapsto \mathbf{1} - F$.

To sum this up algebraically, define $B_{\mathbf{1}}M = \bigoplus_{\mathbf{0} \preceq \mathbf{b} \preceq \mathbf{1}} M_{\mathbf{b}}$ to be the part of $M \in \mathcal{M}$ which is *bounded in the interval* $[\mathbf{0}, \mathbf{1}]$. The precise relation between Matlis duality and Alexander duality for squarefree monomial ideals is then:

> For each squarefree ideal I, the Alexander dual ideal is the unique squarefree monomial ideal $I^{\mathbf{1}}$ such that $B_{\mathbf{1}}I^{\mathbf{1}}$ is the Matlis dual of $(B_{\mathbf{1}}(S/I))[\mathbf{1}]$.

This characterization of Alexander duality is the model for the definitions in Section 2. For instance, "I is a squarefree ideal" is replaced by "M is a positively $\mathbf{1}$-determined module", which means essentially that M is $\mathbb{N}^n$-graded and can be recovered from $B_{\mathbf{1}}M$. The functor which performs the recovery of M from $B_{\mathbf{1}}M$ is called $P_{\mathbf{1}}$, the *positive extension* with respect to $\mathbf{1}$. The characterization above then becomes

$$I^{\mathbf{1}} \quad = \quad P_{\mathbf{1}}\Big(B_{\mathbf{1}}(S/I)[\mathbf{1}]^{\vee}\Big),$$

which says that $I^{\mathbf{1}}$ is gotten from S/I by (i) restricting S/I to the interval $[\mathbf{0}, \mathbf{1}]$; (ii) shifting the result down by $\mathbf{1}$ (which puts it in the interval $[-\mathbf{1}, \mathbf{0}]$); (iii) flipping the result by Matlis duality (so that it sits again in $[\mathbf{0}, \mathbf{1}]$); and then (iv) extending positively. The vector $\mathbf{1}$ will be replaced below by an arbitrary vector $\mathbf{a} \in \mathbb{N}^n$.

2 The Alexander duality functors

For the duration of this paper $\mathbf{a} \in \mathbb{N}^n$ will denote a fixed element, satisfying $\mathbf{a} \succeq \mathbf{1}$.

2.1 Categories of modules

Let $S = k[x_1, \ldots, x_n]$ for a field k, and let $\mathcal{M}$ be the category whose objects are $\mathbb{Z}^n$-graded S-modules and whose morphisms are of degree $\mathbf{0}$. Denote the i^{th} basis vector of $\mathbb{Z}^n$ by $\mathbf{e}_i$, so that multiplication by x_i gives a homomorphism of k-vector spaces $\cdot x_i : M_{\mathbf{b}} \to M_{\mathbf{b}+\mathbf{e}_i}$ for any $M = \bigoplus_{\mathbf{b} \in \mathbb{Z}^n} M_{\mathbf{b}} \in \mathcal{M}$. If $M \in \mathcal{M}$ has homogeneous components that are finite-dimensional k-vector spaces, M will be called $\mathbb{Z}^n$*-finite*; this condition prevents Matlis duals from becoming too large. Recall that M is $\mathbb{N}^n$-graded if $M_{\mathbf{b}} = 0$ for $\mathbf{b} \notin \mathbb{N}^n$. By analogy, a module $M \in \mathcal{M}$ will be called $(\mathbf{a} - \mathbb{N}^n)$*-graded* if $M_{\mathbf{b}} = 0$ unless $\mathbf{b} \preceq \mathbf{a}$.

The Alexander duality functors will be defined on certain full subcategories of $\mathcal{M}$.

Definition 2.1 *Let $M \in \mathcal{M}$ be $\mathbb{Z}^n$-finite.*

1. *M is* $\mathbf{a}$-determined *if $\cdot x_i : M_{\mathbf{b}} \to M_{\mathbf{b}+\mathbf{e}_i}$ is an isomorphism unless $0 \le b_i \le a_i - 1$.*
2. *M is* positively $\mathbf{a}$-determined *if it is $\mathbb{N}^n$-graded and $\cdot x_i : M_{\mathbf{b}} \to M_{\mathbf{b}+\mathbf{e}_i}$ is an isomorphism whenever $a_i \le b_i$.*
3. *M is* negatively $\mathbf{a}$-determined *if it is $(\mathbf{a} - \mathbb{N}^n)$-graded and $\cdot x_i : M_{\mathbf{b}} \to M_{\mathbf{b}+\mathbf{e}_i}$ is an isomorphism whenever $b_i \le -1$.*

Denote the full subcategories of $\mathcal{M}$ on the $\mathbf{a}$*-determined, positively* $\mathbf{a}$*-determined, and negatively* $\mathbf{a}$*-determined modules by* $\mathcal{M}^{\mathbf{a}}$, $\mathcal{M}^{\mathbf{a}}_{+}$, *and* $\mathcal{M}^{\mathbf{a}}_{-}$, *respectively. Finally, let* $\overline{\mathcal{M}}^{\mathbf{a}}$ *be the full subcategory of* $\mathbb{Z}^n$*-finite modules whose degree* $\mathbf{b}$ *components are zero unless* $\mathbf{0} \preceq \mathbf{b} \preceq \mathbf{a}$.

Example 2.2 The most important modules in this paper will be shifts of S and shifts of injective hulls $\underline{E}(S/\mathfrak{m}^F)$, which are defined in Example 1.1. For these, we have:

1. A shift $S[-\mathbf{b}]$ is positively $\mathbf{a}$-determined if and only if $\mathbf{0} \preceq \mathbf{b} \preceq \mathbf{a}$, and $\mathbf{a}$-determined if $\mathbf{1} \preceq \mathbf{b}$. However, $S[-\mathbf{b}]$ is never negatively $\mathbf{a}$-determined.
2. A shift $\underline{E}(S/\mathfrak{m}^F)[-\mathbf{b}]$ is $\mathbf{a}$-determined if and only if $\mathbf{0} \preceq \mathbf{b} \cdot F \preceq \mathbf{a} - \mathbf{1}$, where $\mathbf{b} \cdot F = \sum_{i \in F} b_i \mathbf{e}_i$. The reason why we can forget about the coordinates b_j for $j \notin F$ is that $\cdot x_j$ is an isomorphism on every graded piece of $\underline{E}(S/\mathfrak{m}^F)$ whenever $j \notin F$. On the other hand, $\cdot x_i$ for $i \in F$ is an isomorphism on $\underline{E}(S/\mathfrak{m}^F)[-\mathbf{b}]_{\mathbf{c}}$ unless $c_i = b_i$, in which case $\cdot x_i$ is the zero map on a 1-dimensional k-vector space.
3. A shift $\underline{E}(S/\mathfrak{m}^F)[-\mathbf{b}]$ is negatively $\mathbf{a}$-determined if and only if $F = \{1, \ldots, n\}$ and $\mathbf{0} \preceq \mathbf{b} \preceq \mathbf{a} - \mathbf{1}$. A shift $\underline{E}(S/\mathfrak{m}^F)[-\mathbf{b}]$ is never positively $\mathbf{a}$-determined. □

Remark 2.3 A positively $\mathbf{a}$-determined module is finitely generated. In particular, the positively $\mathbf{1}$-determined modules are precisely the *squarefree modules* as defined in [Yan2]. It is shown there (though not stated in these terms) that $\mathcal{M}^{\mathbf{1}}_{+}$ is an abelian category. This will follow for all $\mathbf{a}$ from Theorem 2.11, below.

Example 2.4 Let $J \subseteq S$ be a monomial ideal. Then J and S/J are positively $\mathbf{a}$-determined if and only if the generators of J divide $\mathbf{x}^{\mathbf{a}}$. When $\mathbf{b} \preceq \mathbf{a}$ and $J = \langle \mathbf{x}^{\mathbf{b}} \rangle = S[-\mathbf{b}]$ is principal, its *Alexander dual* $J^{[\mathbf{a}]}$ with respect to $\mathbf{a}$ is defined to be the irreducible ideal $\langle x_i^{a_i+1-b_i} \mid b_i \geq 1 \rangle$. For more general J, the ideal $J^{[\mathbf{a}]}$ is defined in [Mil, Definition 1.5] by

$$J^{[\mathbf{a}]} \quad = \quad \bigcap \{ S[-\mathbf{b}]^{[\mathbf{a}]} \mid \mathbf{x}^{\mathbf{b}} \text{ is a minimal generator of } J \}.$$

Since Alexander duality will be reintroduced in Definition 2.12 in a more general setting, the reader is referred to [Mil] for details (and pictures). Given that J is in $\mathcal{M}^{\mathbf{a}}_{+}$, the modules $J^{[\mathbf{a}]}$ and $S/J^{[\mathbf{a}]}$ are also in $\mathcal{M}^{\mathbf{a}}_{+}$. □

If $N \in \mathcal{M}$, the tensor product $- \otimes_S N$ is naturally a functor $\mathcal{M} \to \mathcal{M}$. Let $\underline{\mathrm{Tor}}_i^S(-, N)$ be its left derived functors and $\beta_{i,\mathbf{b}}(M) = \dim_k \underline{\mathrm{Tor}}_i^S(M, k)_{\mathbf{b}}$, the i^{th} *Betti number of M in degree* $\mathbf{b}$. For finitely generated M, $\beta_{i,\mathbf{b}}$ is the number of summands $S[-\mathbf{b}]$ in homological degree i in any minimal free resolution of M. The next proposition clarifies Definition 2.1.2 and extends Example 2.2.1. It will be used in the proofs of Theorem 2.6 and Corollary 4.6.

Proposition 2.5 *A finitely generated module $M \in \mathcal{M}$ is positively* $\mathbf{a}$*-determined if and only if the Betti numbers of M satisfy:* $\beta_{0,\mathbf{b}}(M) = \beta_{1,\mathbf{b}}(M) = 0$ *unless* $\mathbf{0} \preceq \mathbf{b} \preceq \mathbf{a}$.

Proof: If $\beta_{0,\mathbf{b}} = \beta_{1,\mathbf{b}} = 0$ unless $\mathbf{0} \preceq \mathbf{b} \preceq \mathbf{a}$, then any minimal free presentation $\mathbb{F}$ of M is $\mathbb{N}^n$-graded, so M is, too. Furthermore, $\cdot x_i : \mathbb{F}_{\mathbf{b}} \to \mathbb{F}_{\mathbf{b}+\mathbf{e}_i}$ is an isomorphism if $b_i \geq a_i$, so the same is true with M in place of $\mathbb{F}$.

Now assume that M is positively $\mathbf{a}$-determined and let $\mathbf{b} \not\preceq \mathbf{a}$, say $b_i > a_i$. If M has a minimal generator in degree $\mathbf{b}$ then $\cdot x_i : M_{\mathbf{b}-\mathbf{e}_i} \to M_{\mathbf{b}}$ is not surjective. If $\beta_{1,\mathbf{b}}(M) \neq 0$ then since every minimal generator of M is in a degree with i^{th} coordinate $< b_i$, it follows that $\cdot x_i : M_{\mathbf{b}-\mathbf{e}_i} \to M_{\mathbf{b}}$ is not injective. This is a contradiction. □

2.2 Uniqueness of Alexander duality

The next theorem is the main result of Section 2. It says that there is essentially a unique way to extend Alexander duality for monomial ideals (Example 2.4) to a functor on $\mathcal{M}_+^{\mathbf{a}}$.

Theorem 2.6 *There is a unique (up to isomorphism) exact k-linear contravariant functor $A_{\mathbf{a}} : \mathcal{M}_+^{\mathbf{a}} \to \mathcal{M}_+^{\mathbf{a}}$ which, for $\mathbf{0} \preceq \mathbf{b} \preceq \mathbf{c} \preceq \mathbf{a}$, takes canonical inclusions $\iota : S[-\mathbf{c}] \to S[-\mathbf{b}]$ to canonical surjections*

$$A_{\mathbf{a}}(\iota) : \quad S/\langle x_i^{a_i+1-b_i} \mid b_i \geq 1\rangle \quad \to \quad S/\langle x_i^{a_i+1-c_i} \mid c_i \geq 1\rangle\,.$$

Any such functor satisfies $A_{\mathbf{a}}A_{\mathbf{a}} \cong \mathrm{id}_{\mathcal{M}_+^{\mathbf{a}}}$ as well as $A_{\mathbf{a}}(S/J) \cong J^{[\mathbf{a}]}$ for monomial ideals J.

Proof: Existence will be provided in Definition 2.12 and Proposition 2.13. Given existence, all of the remaining statements will then follow from uniqueness, which is treated now.

By Proposition 2.5, any module in $\mathcal{M}_+^{\mathbf{a}}$ has a free presentation in $\mathcal{M}_+^{\mathbf{a}}$. Given a map of modules $\varphi : M \to N$ in $\mathcal{M}_+^{\mathbf{a}}$, choose free presentations (with bases) and a lifting of φ as in the left diagram.

$$\begin{array}{ccccccc} F_1 & \xrightarrow{\partial_M} & F_0 & \longrightarrow & M & \longrightarrow & 0 \\ \varphi_1 \downarrow & & \varphi_0 \downarrow & & \varphi \downarrow & & \\ F_1' & \xrightarrow{\partial_N} & F_0' & \longrightarrow & N & \longrightarrow & 0 \end{array} \qquad \begin{array}{ccccccc} A_{\mathbf{a}}(F_1) & \xleftarrow{\partial^M} & A_{\mathbf{a}}(F_0) & \longleftarrow & A_{\mathbf{a}}(M) & \longleftarrow & 0 \\ \varphi^1 \uparrow & & \varphi^0 \uparrow & & A_{\mathbf{a}}(\varphi) \uparrow & & \\ A_{\mathbf{a}}(F_1') & \xleftarrow{\partial^N} & A_{\mathbf{a}}(F_0') & \longleftarrow & A_{\mathbf{a}}(N) & \longleftarrow & 0 \end{array}$$

Choosing an $A_{\mathbf{a}}$ and applying it to the left diagram gives the right diagram, which has exact rows. The maps φ^1, φ^0, ∂^M, and ∂^N are determined by k-linearity and the action of $A_{\mathbf{a}}$ on $S[-\mathbf{b}]$, and are thus independent of which functor $A_{\mathbf{a}}$ is used to obtain the right diagram. By exactness, $A_{\mathbf{a}}(M) \cong \ker(\partial^M)$, whence $A_{\mathbf{a}}$ is uniquely determined on objects. Furthermore, there is at most one map $A_{\mathbf{a}}(\varphi)$ making the diagram commute, namely $\varphi^0 : \ker(\partial^N) \to \ker(\partial^M)$. Thus the effect of $A_{\mathbf{a}}$ on maps is uniquely determined, as well. □

2.3 The Čech hull and other functors

The goal of this subsection is the equivalences in Theorem 2.11 between the categories of Definition 2.1. The construction of an Alexander duality functor in Section 2.4 will be accomplished by Matlis duality in concert with these equivalences of categories. In order to write down the equivalences, though, we need some intermediate functors. All of the machinery developed in this subsection will be used extensively also in later sections.

Recall that the partially ordered set $(\mathbb{Z}^n, \preceq)$ is a lattice with *meet* $\wedge$ and *join* $\vee$ being the componentwise minimum and maximum.

Definition 2.7 *Let $M \in \mathcal{M}$. Define the functors $B_{\mathbf{a}}$, $P_{\mathbf{a}}$, and $\check{C}$ as follows.*

1. *Let $B_{\mathbf{a}}M := \bigoplus_{\mathbf{0}\preceq\mathbf{b}\preceq\mathbf{a}} M_{\mathbf{b}}$ be the subquotient* bounded in the interval $[\mathbf{0}, \mathbf{a}]$.
2. *Let $P_{\mathbf{a}}M = \bigoplus_{\mathbf{b}\in\mathbb{Z}^n} M_{\mathbf{a}\wedge\mathbf{b}}$ (that is, $(P_{\mathbf{a}}M)_{\mathbf{b}} = M_{\mathbf{a}\wedge\mathbf{b}}$) with the S-action*

$$\cdot x_i : (P_{\mathbf{a}}M)_{\mathbf{b}} \to (P_{\mathbf{a}}M)_{\mathbf{b}+\mathbf{e}_i} \quad = \quad \begin{cases} \text{identity} & \text{if } b_i \geq a_i \\ \cdot x_i : M_{\mathbf{a}\wedge\mathbf{b}} \to M_{\mathbf{a}\wedge\mathbf{b}+\mathbf{e}_i} & \text{if } b_i < a_i \end{cases}$$

be the positive extension of M. *$P_{\mathbf{a}}$ is usually applied when M is $(\mathbf{a} - \mathbb{N}^n)$-graded.*

3. *Let* $\check{C}M = \bigoplus_{\mathbf{b}\in\mathbb{Z}^n} M_{\mathbf{b}\vee\mathbf{0}}$ *(that is,* $(\check{C}M)_{\mathbf{b}} = M_{\mathbf{b}\vee\mathbf{0}}$*) with the* S*-action*

$$\cdot x_i : (\check{C}M)_{\mathbf{b}} \to (\check{C}M)_{\mathbf{b}+\mathbf{e}_i} \quad = \quad \begin{cases} \text{identity} & \text{if } b_i < 0 \\ \cdot x_i : M_{\mathbf{b}\vee\mathbf{0}} \to M_{\mathbf{e}_i+\mathbf{b}\vee\mathbf{0}} & \text{if } b_i \geq 0 \end{cases}$$

be the negative extension *or* Čech hull of M. $\check{C}$ *is usually applied when* M *is* $\mathbb{N}^n$*-graded.*

This definition of Čech hull generalizes that for monomial ideals in [Mil, Definition 2.5], and will play a similar role. It is an essential extension which looks something like a cross between localization and injective hull. In fact, part 1 of the next example shows that it actually is localization when applied to a free module, while part 3 shows that it takes injective hulls when applied to a quotient of S by a prime ideal. In the example and henceforth, we employ the following notation. For $\mathbf{b} \in \mathbb{Z}^n$ let $\mathrm{supp}(\mathbf{b}) = \{i \in \{1,\ldots,n\} \mid 0 \neq b_i \in \mathbb{Z}\}$, and identify any $F \subseteq \{1,\ldots,n\}$ with its characteristic vector $\sum_{i\in F} \mathbf{e}_i \in \mathbb{Z}^n$. The $\mathbb{Z}^n$-graded prime ideals of S are denoted $\mathfrak{m}^F = \langle x_i \mid i \in F\rangle$, and $\overline{F} = \mathbf{1} - F$, so (for instance) the "homogeneous residue class ring" of $\mathfrak{m}^F$ is $(S/\mathfrak{m}^F)[\mathbf{x}^{-\overline{F}}]$.

Example 2.8

1. If $\mathbf{b} \in \mathbb{N}^n$ and $F = \mathrm{supp}(\mathbf{b})$, then $\check{C}(S[-\mathbf{b}]) \cong S[\mathbf{x}^{-\overline{F}}][-\mathbf{b}]$.
2. If $\mathbf{b} \in \mathbb{N}^n$ then $B_{\mathbf{a}}(S[-\mathbf{b}]) = 0$ unless $\mathbf{b} \preceq \mathbf{a}$, in which case we have that

$$B_{\mathbf{a}}(S[-\mathbf{b}]) \quad \cong \quad \Big(S/\langle x_1^{a_1+1-b_1}, \ldots, x_n^{a_n+1-b_n}\rangle\Big)[-\mathbf{b}]$$

   is the artinian subquotient of S which is nonzero precisely in degrees from the interval $[\mathbf{b}, \mathbf{a}]$. Applying $P_{\mathbf{a}}$ to this yields back $S[-\mathbf{b}]$, so $P_{\mathbf{a}}B_{\mathbf{a}}(S[-\mathbf{b}]) \cong S[-\mathbf{b}]$ if $\mathbf{b} \preceq \mathbf{a}$.
3. If $F \subseteq \{1,\ldots,n\}$, then $\check{C}(S/\mathfrak{m}^F) \cong \underline{E}(S/\mathfrak{m}^F)$, the *injective hull* of $S/\mathfrak{m}^F$ in $\mathcal{M}$; see Example 1.1 and [GW, Section 1.3].
4. If $\mathbf{b} \preceq \mathbf{a}$ and $J = S[-\mathbf{b}]^{[\mathbf{a}]} = \langle x_i^{a_i+1-b_i} \mid b_i \geq 1\rangle$ is the ideal from Example 2.4, then $B_{\mathbf{a}}(S/J)$ is the artinian quotient $S/\langle x_1^{a_1+1-b_1}, \ldots, x_n^{a_n+1-b_n}\rangle$ of S which is nonzero precisely in degrees from $[\mathbf{0}, \mathbf{a} - \mathbf{b}]$. Compare this to $B_{\mathbf{a}}(S[-\mathbf{b}])$ from part 2. □

Lemma 2.9 *The functors* $B_{\mathbf{a}}$*,* $P_{\mathbf{a}}$*, and* $\check{C}$ *from* $\mathcal{M}$ *to itself are exact.*

Proof: Straightforward from the definitions, since a sequence of modules in $\mathcal{M}$ is exact if and only if it is exact in each $\mathbb{Z}^n$-graded degree. □

The functors in Definition 2.7 can be restricted to each of $\mathcal{M}^{\mathbf{a}}$, $\mathcal{M}^{\mathbf{a}}_{+}$, $\mathcal{M}^{\mathbf{a}}_{-}$, and $\overline{\mathcal{M}}^{\mathbf{a}}$. Their effects are summarized in Table 1, where the restriction of a functor is assumed to have the indicated target, and is denoted by the same symbol as the functor itself. The $\equiv$ symbol means that the target equals the source, and the restricted functor replaces each object by an isomorphic one. Theorem 2.11 states that all of the restricted functors in the table (with their indicated sources and targets) are actually equivalences of categories.

The following lemma will be used in the proof of Theorem 2.11.

Table 1: The functors of Definition 2.7 on the categories of Definition 2.1 and Theorem 2.11.

	$\mathcal{M}^{\mathbf{a}}$	$\overline{\mathcal{M}}^{\mathbf{a}}$	$\mathcal{M}_{+}^{\mathbf{a}}$	$\mathcal{M}_{-}^{\mathbf{a}}$
$B_{\mathbf{a}}$	$\overline{\mathcal{M}}^{\mathbf{a}}$	$\equiv$	$\overline{\mathcal{M}}^{\mathbf{a}}$	$\overline{\mathcal{M}}^{\mathbf{a}}$
$P_{\mathbf{a}}$	$\equiv$	$\mathcal{M}_{+}^{\mathbf{a}}$	$\equiv$	$\mathcal{M}^{\mathbf{a}}$
$\check{C}$	$\equiv$	$\mathcal{M}_{-}^{\mathbf{a}}$	$\mathcal{M}^{\mathbf{a}}$	$\equiv$

Lemma 2.10 *Morphisms in $\mathcal{M}^{\mathbf{a}}$, $\mathcal{M}_{+}^{\mathbf{a}}$, and $\mathcal{M}_{-}^{\mathbf{a}}$ are uniquely determined by their components in degrees $\mathbf{b}$ with $\mathbf{0} \preceq \mathbf{b} \preceq \mathbf{a}$.*

Proof: Only the case $\mathcal{M}^{\mathbf{a}}$ is demonstrated here; the other two cases involve the same arguments. Suppose that $\varphi : M \to M'$ in $\mathcal{M}^{\mathbf{a}}$, and let $\varphi_{\mathbf{b}} : M_{\mathbf{b}} \to M'_{\mathbf{b}}$ be its component in degree $\mathbf{b} \in \mathbb{Z}^n$. Setting $y = \mathbf{x}^{\mathbf{b}\vee\mathbf{0}-\mathbf{b}}$ and $z = \mathbf{x}^{\mathbf{b}-\mathbf{a}\wedge\mathbf{b}} = \mathbf{x}^{\mathbf{b}\vee\mathbf{0}-\mathbf{a}\wedge\mathbf{b}\vee\mathbf{0}}$ (where the parentheses on $\mathbf{a} \wedge \mathbf{b} \vee \mathbf{0}$ can be left off without ambiguity since $\mathbf{0} \preceq \mathbf{a}$), the multiplication maps

$$\cdot y : M_{\mathbf{b}} \to M_{\mathbf{b}\vee\mathbf{0}} \quad \text{and} \quad \cdot z : M_{\mathbf{a}\wedge\mathbf{b}\vee\mathbf{0}} \to M_{\mathbf{b}\vee\mathbf{0}}$$

are isomorphisms (and also with M replaced by M'). Since φ is a module homomorphism, we have in any case

$$(\varphi_{\mathbf{b}\vee\mathbf{0}})(\cdot y) = (\cdot y)(\varphi_{\mathbf{b}}) \quad \text{and} \quad (\varphi_{\mathbf{b}\vee\mathbf{0}})(\cdot z) = (\cdot z)(\varphi_{\mathbf{a}\wedge\mathbf{b}\vee\mathbf{0}}).$$

Thus $\varphi_{\mathbf{b}} = (\cdot y)^{-1}(\varphi_{\mathbf{b}\vee\mathbf{0}})(\cdot y) = (\cdot y)^{-1}(\cdot z)(\varphi_{\mathbf{a}\wedge\mathbf{b}\vee\mathbf{0}})(\cdot z)^{-1}(\cdot y)$. □

Theorem 2.11 *The categories $\mathcal{M}^{\mathbf{a}}$, $\mathcal{M}_{+}^{\mathbf{a}}$, $\mathcal{M}_{-}^{\mathbf{a}}$, and $\overline{\mathcal{M}}^{\mathbf{a}}$ are all equivalent.*

Proof: By [MaL, IV.4, Theorem 1] it is enough to show that, for each $\mathcal{N} \in \{\mathcal{M}^{\mathbf{a}}, \mathcal{M}_{+}^{\mathbf{a}}, \mathcal{M}_{-}^{\mathbf{a}}\}$,

i. the functor $B_{\mathbf{a}} : \mathcal{N} \to \overline{\mathcal{M}}^{\mathbf{a}}$ is fully faithful, and
ii. every object in $\overline{\mathcal{M}}^{\mathbf{a}}$ is isomorphic to $B_{\mathbf{a}}M$ for some $M \in \mathcal{N}$.

The faithfulness of $B_{\mathbf{a}}$ in (i) is the content of Lemma 2.10. Furthermore, given $\overline{\varphi} : B_{\mathbf{a}}M \to B_{\mathbf{a}}M'$ for $M, M' \in \mathcal{N}$, we can (with y and z as in the proof of Lemma 2.10) *define* $\varphi_{\mathbf{b}} : M_{\mathbf{b}} \to M'_{\mathbf{b}}$ by

$$\varphi_{\mathbf{b}} \quad = \quad \begin{cases} (\cdot y)^{-1}(\cdot z)(\overline{\varphi}_{\mathbf{a}\wedge\mathbf{b}\vee\mathbf{0}})(\cdot z)^{-1}(\cdot y) & \text{if } \mathcal{N} = \mathcal{M}^{\mathbf{a}} \\ (\cdot z)(\overline{\varphi}_{\mathbf{a}\wedge\mathbf{b}})(\cdot z)^{-1} & \text{if } \mathcal{N} = \mathcal{M}_{+}^{\mathbf{a}} \\ (\cdot y)^{-1}(\overline{\varphi}_{\mathbf{b}\vee\mathbf{0}})(\cdot y) & \text{if } \mathcal{N} = \mathcal{M}_{-}^{\mathbf{a}} \end{cases},$$

whence $B_{\mathbf{a}}$ is full when restricted to $\mathcal{N}$. To show (ii) note that, by definition, $B_{\mathbf{a}}\check{C}P_{\mathbf{a}}$, $B_{\mathbf{a}}P_{\mathbf{a}}$, and $B_{\mathbf{a}}\check{C}$ (viewed as functors $\overline{\mathcal{M}}^{\mathbf{a}} \to \overline{\mathcal{M}}^{\mathbf{a}}$) are all isomorphic to the identity of $\overline{\mathcal{M}}^{\mathbf{a}}$. □

2.4 Existence of Alexander duality functors

The point of proving Theorem 2.11 is that $\overline{\mathcal{M}}^{\mathbf{a}}$ consists of finite-length modules over S (or even over the ring $B_{\mathbf{a}}S$), and duality for these modules is familiar. Using the equivalences above, Matlis duality $\underline{\mathrm{Hom}}_k(-, k[-\mathbf{a}]) = (-)^\vee[-\mathbf{a}]$ from $\overline{\mathcal{M}}^{\mathbf{a}}$ to itself becomes Alexander duality on $\mathcal{M}_+^{\mathbf{a}}$. Thus, we get from $M \in \mathcal{M}_+^{\mathbf{a}}$ to its Alexander dual via the following steps:

$$M \quad \overset{\mathcal{M}_+^{\mathbf{a}} \equiv \overline{\mathcal{M}}^{\mathbf{a}}}{\longmapsto} \quad B_{\mathbf{a}}M \quad \overset{\substack{\text{Matlis duality} \\ \overline{\mathcal{M}}^{\mathbf{a}} \to \overline{\mathcal{M}}^{\mathbf{a}}}}{\longmapsto} \quad (B_{\mathbf{a}}M)^\vee[-\mathbf{a}] \quad \overset{\overline{\mathcal{M}}^{\mathbf{a}} \equiv \mathcal{M}_+^{\mathbf{a}}}{\longmapsto} \quad P_{\mathbf{a}}\Big((B_{\mathbf{a}}M)^\vee[-\mathbf{a}]\Big).$$

Alternatively, one can use Matlis duality $\mathcal{M}_-^{\mathbf{a}} \to \mathcal{M}_+^{\mathbf{a}}$:

$$M \quad \overset{\mathcal{M}_+^{\mathbf{a}} \equiv \mathcal{M}_-^{\mathbf{a}}}{\longmapsto} \quad \check{C}B_{\mathbf{a}}M \quad \overset{\substack{\text{Matlis duality} \\ \mathcal{M}_-^{\mathbf{a}} \to \mathcal{M}_+^{\mathbf{a}}}}{\longmapsto} \quad (\check{C}B_{\mathbf{a}}M)^\vee[-\mathbf{a}].$$

The two modules at which we have just arrived are isomorphic by Theorem 2.11 because their restrictions to the interval $[\mathbf{0}, \mathbf{a}]$ (i.e. their images in $\overline{\mathcal{M}}^{\mathbf{a}}$ under $B_{\mathbf{a}}$) are isomorphic.

Definition 2.12 (Alexander duality) *Given $M \in \mathcal{M}_+^{\mathbf{a}}$, define the* Alexander dual

$$M^{\mathbf{a}} \quad = \quad P_{\mathbf{a}}\Big((B_{\mathbf{a}}M)^\vee[-\mathbf{a}]\Big) \quad \cong \quad (\check{C}B_{\mathbf{a}}M)^\vee[-\mathbf{a}]$$

of M with respect to $\mathbf{a}$*, where $(-)^\vee$ is the Matlis dual in $\mathcal{M}$, as in Section 1.2. Equivalently, $M^{\mathbf{a}}$ is the positively* $\mathbf{a}$*-determined module which satisfies $B_{\mathbf{a}}(M^{\mathbf{a}}) \cong (B_{\mathbf{a}}M)^\vee[-\mathbf{a}]$.*

We can now complete the proof of Theorem 2.6 by verifying that the functor in Definition 2.12 really deserves to be called an Alexander duality functor.

Proposition 2.13 *The functor $(-)^{\mathbf{a}} : \mathcal{M}_+^{\mathbf{a}} \to \mathcal{M}_+^{\mathbf{a}}$ satisfies the conditions of Theorem 2.6.*

Proof: The exactness, k-linearity, and contravariance are obvious. The action on principal ideals and $A_{\mathbf{a}}A_{\mathbf{a}} \cong \mathrm{id}_{\mathcal{M}_+^{\mathbf{a}}}$ follow from the last statement of the definition along with Examples 2.8.4 and 2.8.2. For other monomial ideals, use [Mil, Theorem 2.1], which says that $J^{[\mathbf{a}]} + \langle x_1^{a_1+1}, \ldots, x_n^{a_n+1}\rangle = (\langle x_1^{a_1+1}, \ldots, x_n^{a_n+1}\rangle : J)$ as follows. Quotienting both sides by $\langle x_1^{a_1+1}, \ldots, x_n^{a_n+1}\rangle$ yields $B_{\mathbf{a}}(J^{[\mathbf{a}]}) \cong (0 :_{B_{\mathbf{a}}S} B_{\mathbf{a}}J) = \mathrm{ann}_{B_{\mathbf{a}}S}(B_{\mathbf{a}}J)$. But the annihilator in question is $\underline{\mathrm{Hom}}_k(B_{\mathbf{a}}S/B_{\mathbf{a}}J, k[-\mathbf{a}]) \cong B_{\mathbf{a}}(S/J)^\vee[-\mathbf{a}]$ by local duality for the zero-dimensional Gorenstein ring $B_{\mathbf{a}}S$. Thus $B_{\mathbf{a}}(J^{[\mathbf{a}]}) \cong B_{\mathbf{a}}(S/J)^\vee[-\mathbf{a}]$, as required. □

Remark 2.14 It is likely that the exactness property in Theorem 2.6 follows from the others, since $\mathcal{M}_+^{\mathbf{a}}$ is equivalent to the category $\overline{\mathcal{M}}^{\mathbf{a}}$ consisting of finite-length modules (Theorem 2.11). In any case, Theorem 2.6 essentially says that there is an equivalence of $\mathcal{M}_+^{\mathbf{a}}$ with its opposite category $(\mathcal{M}_+^{\mathbf{a}})^{\mathrm{op}}$ whose square is the identity, and it may seem that specifying $A_{\mathbf{a}}$ on inclusions of principal ideals is unnecessary. This is not the case, however, because there are equivalences $\mathcal{M}_+^{\mathbf{a}}$ to itself which are not isomorphic to the identity functor. For example, any permutation of the indices $\{1, \ldots, n\}$ has this property. Thus, the Alexander duality of Definition 2.12 followed by a permutation of order two of $\{1, \ldots, n\}$ satisfies every part of Theorem 2.6 except for the action on inclusions. (Are there others?) The action on inclusions works as substitute for the S-linearity in the usual definition of dualizing functor. This is necessary because $\underline{\mathrm{Hom}}_S(-, -)$ doesn't make sense in any of the four equivalent categories of Theorem 2.11: it tends to take values outside the desired categories.

In Section 4 we will want to know the consequences of Alexander duality for free and injective resolutions. This will necessarily involve Alexander duality between pairs of the equivalent categories of Theorem 2.11 other than $(\mathcal{M}_+^{\mathbf{a}}, \mathcal{M}_+^{\mathbf{a}})$. As is the case with $(-)^{\mathbf{a}}$, the functors which define these dualities are essentially Matlis duality with a shift $[-\mathbf{a}]$, composed additionally on either side with one or two of the equivalences of Theorem 2.11. Also as with $(-)^{\mathbf{a}}$, these *generalized Alexander duality functors* are defined on all of $\mathcal{M}$, and become anti-equivalences only when the source and target are restricted to the appropriate subcategories. Although it is not difficult to write down explicitly all of the 16 generalized Alexander duality functors, only the ones that are used in the rest of this paper are given notations here.

Definition 2.15 *Define the functors* $(-)_{\succeq \mathbf{0}}$, $A_{\mathbf{a}}^{+,0}$, *and* $A_{\mathbf{a}}^{0,+}$ *from* $\mathcal{M} \to \mathcal{M}$ *as follows. Given a module* $M \in \mathcal{M}$,

1. *Let* $M_{\succeq \mathbf{0}} = \bigoplus_{\mathbf{b} \in \mathbb{N}^n} M_{\mathbf{b}}$, *which is the* $\mathbb{N}^n$-graded *submodule of* M.
2. *Let* $A_{\mathbf{a}}^{+,0} M = (\check{C}M)^\vee[-\mathbf{a}]$. *When restricted to the appropriate subcategories,* $A_{\mathbf{a}}^{+,0}$ *gives rise to the generalized Alexander duality functors*
$$A_{\mathbf{a}}^{+,0} : \mathcal{M}_+^{\mathbf{a}} \to \mathcal{M}^{\mathbf{a}} \qquad \text{and} \qquad A_{\mathbf{a}}^{+,0} : \overline{\mathcal{M}}^{\mathbf{a}} \to \mathcal{M}_+^{\mathbf{a}}.$$
3. *Let* $A_{\mathbf{a}}^{0,+} M = M^\vee[-\mathbf{a}]_{\succeq \mathbf{0}}$. *When restricted to the appropriate subcategories,* $A_{\mathbf{a}}^{0,+}$ *gives rise to the generalized Alexander duality functors*
$$A_{\mathbf{a}}^{0,+} : \mathcal{M}^{\mathbf{a}} \to \mathcal{M}_+^{\mathbf{a}} \qquad \text{and} \qquad A_{\mathbf{a}}^{0,+} : \mathcal{M}_+^{\mathbf{a}} \to \overline{\mathcal{M}}^{\mathbf{a}}.$$

The functor $(-)_{\succeq \mathbf{0}}$ induces an equivalence $\mathcal{M}^{\mathbf{a}} \to \mathcal{M}_+^{\mathbf{a}}$ for every $\mathbf{a}$, so it is in some sense a gentle version of $P_{\mathbf{a}} B_{\mathbf{a}}$. It should be thought of essentially as an inverse to the Čech hull.

Remark 2.16 As pointed out in [Mil, Proposition 3.5 and Corollary 3.6], Alexander duality is much simpler if one has no preference for finitely generated modules over other kinds (such as artinian ones). Indeed, Alexander duality for all of $\mathcal{M}$ should really be just Matlis duality. However, the interest in finitely generated modules requires an interaction of Matlis duality with the boundary of the positive orthant $\mathbb{N}^n$. In particular, free modules become preferable to arbitrary flat modules. The extension functors of Definition 2.7 are simply the means for dealing with this boundary problem. Of course, there is nothing too special about $\mathbb{N}^n$ as opposed to any of its shifts $\mathbf{b} + \mathbb{N}^n$; requiring that M be $\mathbb{N}^n$-graded as opposed to just finitely generated is merely a choice of "basepoint" for the reflection $\mathbb{N}^n \to \mathbf{a} - \mathbb{N}^n$. This choice is natural in the sense that it induces the right duality for ideals of S.

Remark 2.17 Yanagawa [Yan2] correctly predicted that if there is a definition of Alexander duality for squarefree modules, then the dual of the Stanley-Reisner ring S/I_Δ for a simplicial complex Δ would be the Stanley-Reisner ideal $I_{\Delta^\vee}$, and not $S/I_{\Delta^\vee}$, where $\Delta^\vee$ is the simplicial complex Alexander dual to Δ.

When thinking about $\mathbb{Z}^n$ or $\mathbb{N}^n$-gradings, the author had in mind monomial ideals and the modules (such as local cohomology) associated to them. It is unclear, however, precisely what role $\mathbb{N}^n$ plays in the duality. Perhaps it is the existence of the extension functors in Definition 2.7, or the self-duality of $\mathbb{N}^n$ as a submonoid of $\mathbb{Z}^n$. In any case, thinking of S as the coordinate ring of an affine toric variety, one could ask whether there exists an analog of Alexander duality for affine semigroups Q other than $\mathbb{N}^n$. One would like such analog to satisfy all of the conditions of Theorem 2.6, but the duals of the Q-graded ideals would have to be defined first, as would the notion of positively $\mathbf{a}$-determined.

3 Monomial matrices

Injective resolutions contain much more information than free resolutions. In fact, an injective resolution can be thought of as combining the information from the free resolutions of every localization of a module; see Example 4.8 for a geometric interpretation of this statement, Corollary 4.12 for a combinatorial interpretation, and Corollary 4.15 for a general statement. The extra information provided by injective resolutions will be indispensable for the $\mathbb{N}^n$-graded analog of the Eagon-Reiner and Terai theorems in Section 4.5. On the other hand, complexes of flat modules appear naturally in Section 6 as generalizations of the Čech complex. This section provides the foundations for working with maps and complexes of injective and flat modules in $\mathcal{M}$, including what it means to be minimal; interactions with localization; and a handy new matrix notation for maps between injective or flat modules. The important class of *cellular* monomial matrices is introduced in Section 3.3.

3.1 Motivation and definition

First, let's examine the standard notion of matrix for maps of $\mathbb{Z}^n$-graded free modules. Of course, any rectangular array whose (p, q)-entry is of the form $\lambda_{pq}\mathbf{x}^{\mathbf{b}_{pq}}$ denotes such a map; but the array does not determine the map uniquely. We need also to keep track of the degrees of the generators of the summands in the source and the target. To do this, we use instead a bordermatrix (the TEX command used to produce the arrays below) with each column labelled by the degree of the corresponding source summand, and each row labelled by the degree of the corresponding target summand. Of course, now that we are keeping track of the degree shifts in the source and target, we can replace the monomial entry $\lambda_{pq}\mathbf{x}^{\mathbf{b}_{pq}}$ by the scalar λ_{pq}. For instance, the right-hand 1×1 bordermatrix in Equation (4) represents

$$\begin{array}{c} \\ \mathbf{b}_{1\cdot} \\ \vdots \\ \mathbf{b}_{p\cdot} \\ \vdots \end{array} \begin{array}{c} \begin{array}{cccc} \mathbf{b}_{\cdot 1} & \cdots & \mathbf{b}_{\cdot q} & \cdots \end{array} \\ \left(\begin{array}{ccc} & & \\ & & \\ & \lambda_{pq} & \\ & & \\ & & \end{array}\right) \end{array} \qquad \begin{array}{c} \\ (1,1,8) \end{array} \begin{array}{c} (3,2,8) \\ \left(\ 2\ \right) \end{array} \tag{4}$$

the map $S[-(3,2,8)] \to S[-(1,1,8)]$ which is 2 times the canonical inclusion. Of course, in order for λ_{pq} to be nonzero, it must be that $\mathbf{b}_{p\cdot} \preceq \mathbf{b}_{\cdot q}$.

In trying to make this notation work for maps between $\mathbb{Z}^n$-graded flat or injective modules, the first question is how to represent a $\mathbb{Z}^n$-shift of a localization $S[\mathbf{x}^{-\overline{F}}]$ in a standard way. The problem is that the multiplication map $\cdot x_i : S[\mathbf{x}^{-\overline{F}}] \to S[\mathbf{x}^{-\overline{F}}][\mathbf{e}_i]$ is an isomorphism *of degree* $\mathbf{0}$ whenever $i \notin F$. The 1×1 bordermatrix for such a map should really have the same label on its row and column, with the single scalar entry being $1 \in k$. This problem is aggravated for maps between shifts of different localizations. To alleviate it, shifts of flat modules are denoted by vectors $\mathbf{b} \in (\mathbb{Z} \cup \{*\})^n$, as follows.

The $*$ is meant to represent an arbitrary integer. For purposes of order and addition, the rules governing $*$ treat it like $-\infty$, except that multiplication by -1 leaves $*$ unchanged:

$$* + b \;=\; * \quad \text{and} \quad * \;<\; b \quad \text{for all } b \in \mathbb{Z}\,, \quad \text{while} \quad -1 \cdot * \;=\; *\,. \tag{5}$$

Thus $\mathbf{b} \in (\mathbb{Z} \cup \{*\})^n$ may be expressed as $\mathbf{b} = \mathbf{b}_{\mathbb{Z}} + \mathbf{b}_*$, where $\mathbf{b}_{\mathbb{Z}} \in \mathbb{Z}^n$ is obtained from $\mathbf{b}$ by setting $*$ to zero, and $\mathbf{b}_*$ is obtained by setting all integers to zero. If the set of indices where $\mathbf{b}_*$ has a $*$ is $F \subseteq \{1, \ldots, n\}$, then we write $\mathbf{b}_* = *F$ and $\overline{\mathbf{b}}_* = *\overline{F}$, and consider them also as subsets of $\{1, \ldots, n\}$. This enables our conventions for shifts of localizations:

1. If $S[\mathbf{x}^{-\overline{F}}][\mathbf{b}]$ is written, it is required that $\mathbf{b}_* = *\overline{F}$. In other words, $\mathbf{b}$ has a $*$ in the i^{th} place if and only if $\cdot x_i$ is an automorphism (of degree $\mathbf{e}_i$) of $S[\mathbf{x}^{-\overline{F}}]$. If no shift is explicitly written, then it is assumed $\mathbf{b} = \mathbf{b}_* = *\overline{F}$.
2. If $\mathbf{c} \in \mathbb{Z}^n$, then $S[\mathbf{x}^{-\overline{F}}][\mathbf{b}][\mathbf{c}] = S[\mathbf{x}^{-\overline{F}}][\mathbf{b} + \mathbf{c}]$. Observe that $(\mathbf{b} + \mathbf{c})_* = \mathbf{b}_*$ here.

The reason for choosing these conventions is because:

Lemma 3.1 *There exists a nonzero map* $\varphi : S[\mathbf{x}^{-\overline{F}'}][-\mathbf{b}'] \to S[\mathbf{x}^{-\overline{F}}][-\mathbf{b}]$ *in* $\mathcal{M}$ *between shifts of localizations of* S *if and only if* $\mathbf{b} \preceq \mathbf{b}'$. *If there is a nonzero map, then*

$$\operatorname{Hom}_{\mathcal{M}}(\, S[\mathbf{x}^{-\overline{F}'}][-\mathbf{b}']\,,\, S[\mathbf{x}^{-\overline{F}}][-\mathbf{b}]\,) \quad \cong \quad k\,.$$

Proof: Each variable which has been inverted in the source must also be inverted in the target; i.e. $\mathbf{b}_* \supseteq \mathbf{b}'_*$. In addition, since every such nonzero map must be injective, every integer entry of $\mathbf{b}$ must be $\leq$ the corresponding integer entry of $\mathbf{b}'$, or else there will be no place in the target to send the element 1 of degree $\mathbf{b}'_{\mathbb{Z}}$ from the source. On the other hand, there may be an index i such that $b_i = *$ and $b'_i \in \mathbb{Z}$. Using the rules in (5) and putting these conditions together amounts to simply $\mathbf{b} \preceq \mathbf{b}'$. The last statement holds because any nonzero map is determined by the image of 1. $\square$

Definition 3.2 *A* monomial matrix *is a bordermatrix* Λ *as in the left-hand side of Equation 4 such that* $\mathbf{b}_{p\cdot} \preceq \mathbf{b}_{\cdot q}$ *whenever* $\lambda_{pq} \neq 0$.

The vectors $\mathbf{b}_{p\cdot}$ and $\mathbf{b}_{\cdot q}$ are called the *row* and *column shifts of* Λ, respectively, while $(\mathbf{b}_{p\cdot})_*$ and $(\mathbf{b}_{\cdot q})_*$ are the *row* and *column* $*$*-vectors of* Λ. The λ_{pq} are called *scalar entries.* Λ is $\mathbb{Z}^n$*-finite* if for each $\mathbf{c} \in \mathbb{Z}^n$ there are only finitely many shifts of Λ which are $\preceq \mathbf{c}$.

Of course, there is the usual notion of *submatrix*, obtained by choosing some of the p and some of the q. The 1×1 monomial submatrix Λ_{pq} is called the (p, q)-component of Λ. A *homologically graded matrix* Λ is a sequence $\{\Lambda^d\}$ with $\mathbf{b}^{d-1}_{\cdot q} = \mathbf{b}^d_{p\cdot}$ if $p = q$. Matrices below may be homologically graded, although the homological indexing is usually suppressed.

Example 3.3 The left bordermatrix in Equation (6) is a monomial matrix over $k[x,y,z]$, while the right bordermatrix is not; given the shifts on the right bordermatrix, all of the

$$\begin{array}{c} \\ (*,-4,*) \\ (*,*,-3) \\ (-2,*,9) \end{array}\!\!\begin{array}{c} \begin{array}{ccc} (-7,*,-3) & (0,*,17) & (*,3,-1) \end{array} \\ \left(\begin{array}{ccc} 0 & 0 & 1 \\ 1 & 2 & 5 \\ 0 & -1 & 0 \end{array}\right) \end{array} \qquad \begin{array}{c} \\ (*,*,5) \\ (8,*,0) \end{array}\!\!\begin{array}{c} \begin{array}{ccc} (*,*,*) & (-2,*,4) & (9,*,*) \end{array} \\ \left(\begin{array}{ccc} 1 & -1 & 3 \\ -2 & -1 & 1 \end{array}\right) \end{array} \tag{6}$$

scalar entries should be zero. Given the shifts on the left bordermatrix, all of the zeros are forced by the condition for a monomial matrix. □

Example 3.4 Here is a more substantive example. The Koszul complex on $x, y \in k[x,y]$ is

$$0 \leftarrow S \xleftarrow{\begin{array}{c} \quad (1,0)\ (0,1) \\ (0,0)\left(\begin{array}{cc} 1 & 1 \end{array}\right) \end{array}} S[(-1,0)] \oplus S[(0,-1)] \xleftarrow{\begin{array}{c} (1,1) \\ \begin{array}{c} (1,0) \\ (0,1) \end{array}\!\left(\begin{array}{c} -1 \\ 1 \end{array}\right) \end{array}} S[(-1,-1)] \leftarrow 0\,.$$

If all of the zeros in the shifts of the monomial matrix are changed to $*$ and subsequently $(-1,-1)$ is added to every shift, the result is the Čech complex on x, y:

$$0 \leftarrow S[x^{-1}, y^{-1}] \xleftarrow{\begin{array}{c} \quad (0,*)\ (*,0) \\ (*,*)\left(\begin{array}{cc} 1 & 1 \end{array}\right) \end{array}} S[y^{-1}] \oplus S[x^{-1}] \xleftarrow{\begin{array}{c} (0,0) \\ \begin{array}{c} (0,*) \\ (*,0) \end{array}\!\left(\begin{array}{c} -1 \\ 1 \end{array}\right) \end{array}} S \leftarrow 0\,.$$

It will be seen in Lemma 3.23 that changing a zero to a $*$ is taking the Čech hull. Thus, up to a shift by $\mathbf{1}$, *the Čech complex is the Čech hull of the Koszul complex.* This transition is generalized to squarefree monomial ideals other than $\mathfrak{m}$ in Theorem 6.2. □

3.2 Flat and injective modules

Monomial matrices can be used to represent any map of $\mathbb{Z}^n$-finite injective or flat modules. Before proving this, here is the description of all injective and $\mathbb{Z}^n$-finite flat modules in $\mathcal{M}$.

Lemma 3.5 *Every injective module in $\mathcal{M}$ is a direct sum of $\mathbb{Z}^n$-graded shifts of* indecomposable injectives $\underline{E}(S/\mathfrak{m}^F)$. *Every $\mathbb{Z}^n$-finite flat module in $\mathcal{M}$ is a direct sum of shifts of* indecomposable flat modules $S[\mathbf{x}^{-\overline{F}}]$.

Proof: The statement about injectives is [GW, Theorem 1.3.3]. It has already been seen in Section 1 that $J \in \mathcal{M}$ is flat if and only if the Matlis dual $J^\vee = \underline{\mathrm{Hom}}_k(J,k)$ of J is injective; therefore the second statement will follow from the first if Matlis duality commutes with direct sums of modules $M_\alpha \in \mathcal{M}$ whenever the direct sum is $\mathbb{Z}^n$-finite. Precisely,

Claim 3.6 *If $M = \bigoplus_\alpha M_\alpha$ is $\mathbb{Z}^n$-finite, then the map $\bigoplus_\alpha (M_\alpha)^\vee \to M^\vee$ is an isomorphism.*

Proof: The map is obviously injective. Furthermore, given $\mathbf{b} \in \mathbb{Z}^n$, the direct sum $M_{[-\mathbf{b}]}$ of all M_α which are nonzero in degree $-\mathbf{b}$ is finite. Taking the Matlis dual of the isomorphism $M \cong M_{[-\mathbf{b}]} \oplus M/M_{[-\mathbf{b}]}$ shows that the summand $(M_{[-\mathbf{b}]})^\vee$ of $\bigoplus_\alpha (M_\alpha)^\vee$ surjects onto $(M^\vee)_\mathbf{b}$. Since $\mathbf{b}$ is arbitrary, the proof of the Claim (and the Lemma) is complete. □

Up until now, the notation in Section 3 has been geared only towards flat modules, and not injectives. However, $\cdot x_i$ is an automorphism (of degree $\mathbf{e}_i$) of $\underline{E}(S/\mathfrak{m}^F)$ if and only if it is an automorphism of $S[\mathbf{x}^{-\overline{F}}] \cong \underline{E}(S/\mathfrak{m}^F)^\vee$, so conventions 1 and 2 before Lemma 3.1 work just as well with $\underline{E}(S/\mathfrak{m}^F)$ in place of $S[\mathbf{x}^{-\overline{F}}]$.

The next theorem is the main result of the section. For notation, $S[\mathbf{x}^{-\overline{F}}][-\mathbf{b}]$ is said to be *generated in degree* $\mathbf{b}$ by the *generator* 1. Similarly, $\underline{E}(S/\mathfrak{m}^F)[\mathbf{b}]$ is *cogenerated in degree* $-\mathbf{b}$ by the *cogenerator* 1. A generator of an irreducible flat S-module is really an honest basis element for the same module over the appropriate localization of S, just as a cogenerator for an irreducible injective module is an honest socle element over the appropriate localization.

Theorem 3.7 *To give a monomial matrix Λ is equivalent to giving either*

1. *a map $\lambda : N' \to N$ of flat modules along with decompositions into irreducibles*

$$N' = \bigoplus_q S[\mathbf{x}^{-\overline{F}_{\cdot q}}][-\mathbf{b}_{\cdot q}] \quad \text{and} \quad N = \bigoplus_p S[\mathbf{x}^{-\overline{F}_{p\cdot}}][-\mathbf{b}_{p\cdot}]$$

   as well as generators for the summands; or

2. *a map $\lambda : N \to N'$ of injective modules along with decompositions into irreducibles*

$$N = \bigoplus_p \underline{E}(S/\mathfrak{m}^{F_{p\cdot}})[\mathbf{b}_{p\cdot}] \quad \text{and} \quad N' = \bigoplus_q \underline{E}(S/\mathfrak{m}^{F_{\cdot q}})[\mathbf{b}_{\cdot q}]$$

   as well as cogenerators for the summands.

In either case, Λ is said to represent λ. *Every map of injective modules can be represented by some monomial matrix Λ, as can every map of $\mathbb{Z}^n$-finite flat modules. A homologically graded matrix can be construed as a homological complex of flat modules or a cohomological complex of injective modules.*

Proof: Lemma 3.1 and its Matlis dual, along with Lemma 3.5. □

If $\{\Lambda^d\}$ is a matrix representing a homological complex $\mathbb{F}$ of flat modules with differential $\{\lambda_d\}$ and a cohomological complex $\mathbb{I}$ of injective modules with differential $\{\lambda^d\}$, the conventions for homologically graded monomial matrices force the conventions

$$\cdots \leftarrow F_{d-1} \stackrel{\lambda_d}{\leftarrow} F_d \leftarrow \cdots \qquad \text{and} \qquad \cdots \to I^{d-1} \stackrel{\lambda^d}{\to} I^d \to \cdots$$

because $\mathbf{b}^d_{q\cdot}$ represents a shift in F_d and I^d.

Remark 3.8 A map of modules represented by a monomial matrix is also a map of ungraded S-modules. In order to get the resulting matrix for the ungraded map, each scalar entry λ_{pq} needs to be multiplied by $\mathbf{x}^{\mathbf{b}_{pq}}$, where $\mathbf{b}_{pq} = (\mathbf{b}_{\cdot q})_\mathbb{Z} - (\mathbf{b}_{p\cdot})_\mathbb{Z}$.

3.3 Cellular monomial matrices

This subsection outlines a class of monomial matrices which generalize the construction in [BS, Section 1] of *cellular free complexes* from labelled cellular complexes. Specifically, we define the analogous notions of *cellular* and *cocellular flat complexes* as well as *cellular* and *cocellular injective complexes*. Examples of these include the Koszul and Čech complexes in Example 3.4 as well as their more general relatives in Example 6.6, below. Other interesting examples are furnished by the Scarf complexes, which will be used systematically as test cases for the duality theorems of Section 4. More generally, the geometric interpretation of resolutions afforded by cellular and cocellular monomial matrices interacts nicely with the duality theorems in Section 4, providing a connection with duality in algebraic topology. First we recall the relevant topological notions, extracting the exposition from [BS, Section 1]. See also [BH, Section 6.2] for more details on regular cell complexes.

Let X be a *regular cell complex* with vertex set V; in the examples here, X will be either a triangulation of a simplex or the set of faces of a polytope. X is assumed to come equipped with an *incidence function* $\varepsilon(G, G')$, defined on pairs of cells in X and taking values in $\{-1, 0, 1\}$. The value of $\varepsilon(G, G')$ is nonzero if and only if G' is a facet of G. The empty set $\emptyset$ is always a face of X, and is a facet of each vertex. The incidence function ε defines a differential ∂ on the *augmented oriented chain complex* $\widetilde{\mathcal{C}}.(X; k) = \bigoplus_{G\in X} kG$ of X with coefficients in k. Precisely, ∂ is defined on a face $G \in X$ by $\partial\, G = \sum_{G'\in X} \varepsilon(G, G')\, G'$, and is homologically graded by dimension, with the empty set in homological degree -1. The k-dual $\widetilde{\mathcal{C}}^{\cdot}(X; k) := \mathrm{Hom}_k(\widetilde{\mathcal{C}}.(X; k), k)$ is called the *augmented oriented cochain complex* of X, and is to be regarded as a homologically graded complex. The homological grading of $\widetilde{\mathcal{C}}^{\cdot}(X; k)$ has the empty set in homological degree 1, vertices in homological degree 0, edges in homological degree -1, …, and facets in homological degree $-\dim X$.

Definition 3.9 *A monomial matrix Λ whose scalar entries constitute $\widetilde{\mathcal{C}}.(X; k)$ up to a homological shift for some regular cell complex X is called a* cellular monomial matrix supported on X. *Similarly, if the scalar entries are a homological shift of $\widetilde{\mathcal{C}}^{\cdot}(X; k)$ then Λ is called* cocellular.

In this section and Section 4, the usual homological shift is (-1) for cellular monomial matrices (putting the empty set in homological degree 0), and $(-\dim X)$ for cocellular monomial matrices (putting the empty set in homological degree $1 + \dim X$ and the facets in homological degree 0).

One should think of the shifts as being (not necessarily unique) labels on the faces of X (see Figure 1). Given such a *labelled cell complex* X, we say that X *determines* a cellular (or cocellular) monomial matrix if the corresponding bordermatrix for $\widetilde{\mathcal{C}}.(X; k)$ (or $\widetilde{\mathcal{C}}^{\cdot}(X; k)$) is a monomial matrix. The next proposition says what it means to determine a (co)cellular monomial matrix, in terms of the labelling. For notation, observe that the meet $\wedge$ and join $\vee$ from Section 2.3 work just as well on the poset $(\mathbb{Z} \cup \{*\})^n$. For a face G in a labelled regular cell complex X, let $\mathbf{a}_G$ denote the label on G.

Proposition 3.10 *Let X be a labelled regular cell complex. Then X determines a cellular monomial matrix if and only if $\mathbf{a}_{G'} \preceq \mathbf{a}_G$ for all faces $G' \subseteq G \in X$; and X determines a cocellular monomial matrix if and only if $\mathbf{a}_{G'} \succeq \mathbf{a}_G$ for all faces $G' \subseteq G \in X$.*

Proof: By transitivity of $\preceq$ it is enough to check when G' is a facet of G. In this case, the nonvanishing of $\varepsilon(G, G')$ is enough. $\square$

Example 3.11 In Figure 1, the same simplicial complex determines two different cocellular monomial matrices. For clarity, the notation $\bar{1} * 1$ is used as a shorthand for the vector $(-1, *, 1) \in (\mathbb{Z} \cup \{*\})^3$. $\square$

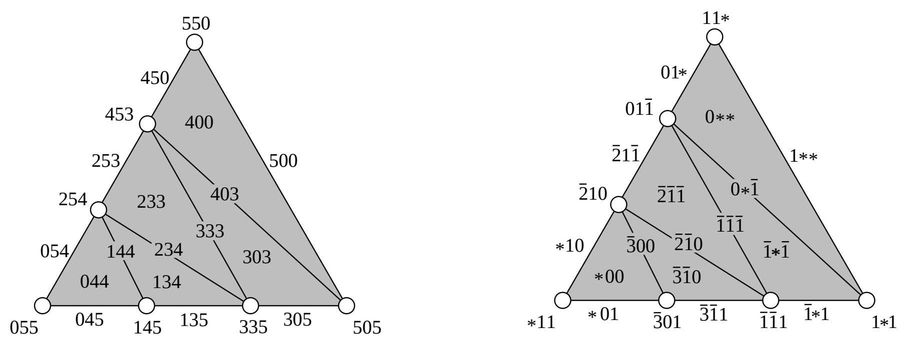

Figure 1: Two cocellular monomial matrices supported on the same simplicial complex.

By Theorem 3.7, a cellular or cocellular monomial matrix can represent a complex of either flat or injective modules, and we say that such a complex is a (co)cellular flat or (co)cellular injective complex, as the case may be. With shifts in homological gradings and Matlis duality involved, it can often be confusing to determine when a flat or injective complex is cellular rather than cocellular. The rule of thumb is: the flat version goes the right way. For instance, the differential of a cellular flat complex goes in the same direction as $\widetilde{\mathcal{C}}.(X; k)$, while the differential of a cellular injective complex goes in the same direction as $\widetilde{\mathcal{C}}^{\cdot}(X; k)$. In Figure 2, for example, the scalar matrices give the boundary complex of the (unlabelled) simplicial complex depicted there. Both labellings of the simplicial complex determine cellular monomial matrices, with the homological grading increasing to the right: the empty set is labelled by the row shift on the leftmost bordermatrix, and the facets are labelled by the column shifts of the rightmost bordermatrix. (For clarity, the two sets of column shifts are depicted above and below the scalar matrices. The row shifts have been left off of all but the first bordermatrix, but the definition of homologically graded monomial matrix says that the row shifts equal the column shifts on the next bordermatrix to the left.)

The reason why (co)cellular flat and injective complexes are interesting is because sometimes they are resolutions of interesting modules; that is, they are acyclic. (See Example 6.6, though, for an important non-acyclic cocellular flat complex). Historically, the first algebraic complexes to admit descriptions by cellular means were undoubtedly the Koszul and Čech complexes on $x_1, \ldots, x_n$, which are resolutions of k and $\underline{E}(k)$, respectively. Both are supported on a simplex with vertices $x_1, \ldots, x_n$. The *Taylor complex* [Tay] of a monomial ideal $J = \langle m_1, \ldots, m_r \rangle$ generalizes the Koszul complex for free resolutions, and is supported on a simplex spanned by $m_1, \ldots, m_r$, although still the cellular structure was not explicit at the time of its invention. The insight of [BPS] was to examine a subcomplex of the Taylor complex, called the *Scarf complex* Δ_J, which is defined as follows. Let $m_\sigma = \mathrm{lcm}(m \in \sigma)$

for any subset $\sigma \subseteq \{m_1, \ldots, m_r\}$. Then

$$\Delta_J := \{\sigma \subseteq \{m_1, \ldots, m_r\} \mid \text{if } m_\sigma = m_\tau \text{ for some } \tau \subseteq \{m_1, \ldots, m_r\} \text{ then } \tau = \sigma\}\,.$$

Each face $\sigma \in \Delta_J$ is labelled by the exponent on m_σ. The point of this definition is that for "most" monomial ideals, the monomial matrix determined by Δ_J is acyclic.

Definition 3.12 ([MSY, BPS]) *The monomial ideal J from above is* generic *if, whenever $\tau = \{m_i, m_j\}$ is* not *a face of Δ_J, we have $\mathbf{x}^{\mathbf{a}_\tau - \mathrm{supp}(\mathbf{a}_\tau)} \in J$.*

This definition agrees with that in [MSY], where a number of equivalent conditions are given characterizing the condition of being generic (some of which are implicitly conditions on the injective resolution of S/J). Ideals called generic in [BPS] are called *strongly generic* in [MSY]. The following theorem was proved first in [BPS] for strongly generic monomial ideals and later extended in [MSY] to generic ones. Here, the ramifications of this theorem will be explored in the context of duality for free and injective resolutions.

Theorem 3.13 ([BPS, MSY]) *If J is generic, then the labelled complex Δ_J determines a cellular free resolution of S/J.*

Example 3.14 The Scarf complex of the generic monomial ideal $J = \langle xz^2, x^3z, x^4y, x^2y^2\rangle$ is the thickened subdivided quadrilateral (including the two triangles) in the bottom simplicial complex of Figure 2. The monomial matrix it determines is the submatrix of the big matrix whose shifts are labelled with an "x", and it represents a free resolution of S/J by Theorem 3.13. In fact, the entire bottom simplicial complex is the Scarf complex of $J + \langle x^5, y^5, z^5\rangle$, so it represents a free resolution of $S/(J + \langle x^5, y^5, z^5\rangle) = B_{(4,4,4)}(S/J)$. □

3.4 Minimality

Definition 3.15 *The matrix Λ is called* minimal *if no component Λ_{pq} represents an isomorphism. Equivalently, Λ is minimal if $\lambda_{pq} = 0$ whenever $\mathbf{b}_{p\cdot} = \mathbf{b}_{\cdot q}$. A map λ of flat or injective modules is called* minimal *if there is a minimal matrix for λ.*

Example 3.16 The Scarf complex of any monomial ideal is minimal because its face labels are distinct by definition. In particular, the bottom simplicial complex of Figure 2 determines a minimal cellular monomial matrix by Example 3.14. The reader may check that, in fact, all of the face labels in all four of the labelled triangulations of Figures 1 and 2 have distinct face labels and thus determine minimal injective and flat complexes. This (and much more) will be explained by Theorem 4.5, Corollary 4.9 and Example 4.8. □

Proposition 3.17 *Let λ be a morphism of $\mathbb{Z}^n$-finite flat or injective modules in $\mathcal{M}$. Minimality of λ is independent of the matrix chosen to verify it. Furthermore, λ is minimal if and only if the Matlis dual $\lambda^\vee$ is minimal.*

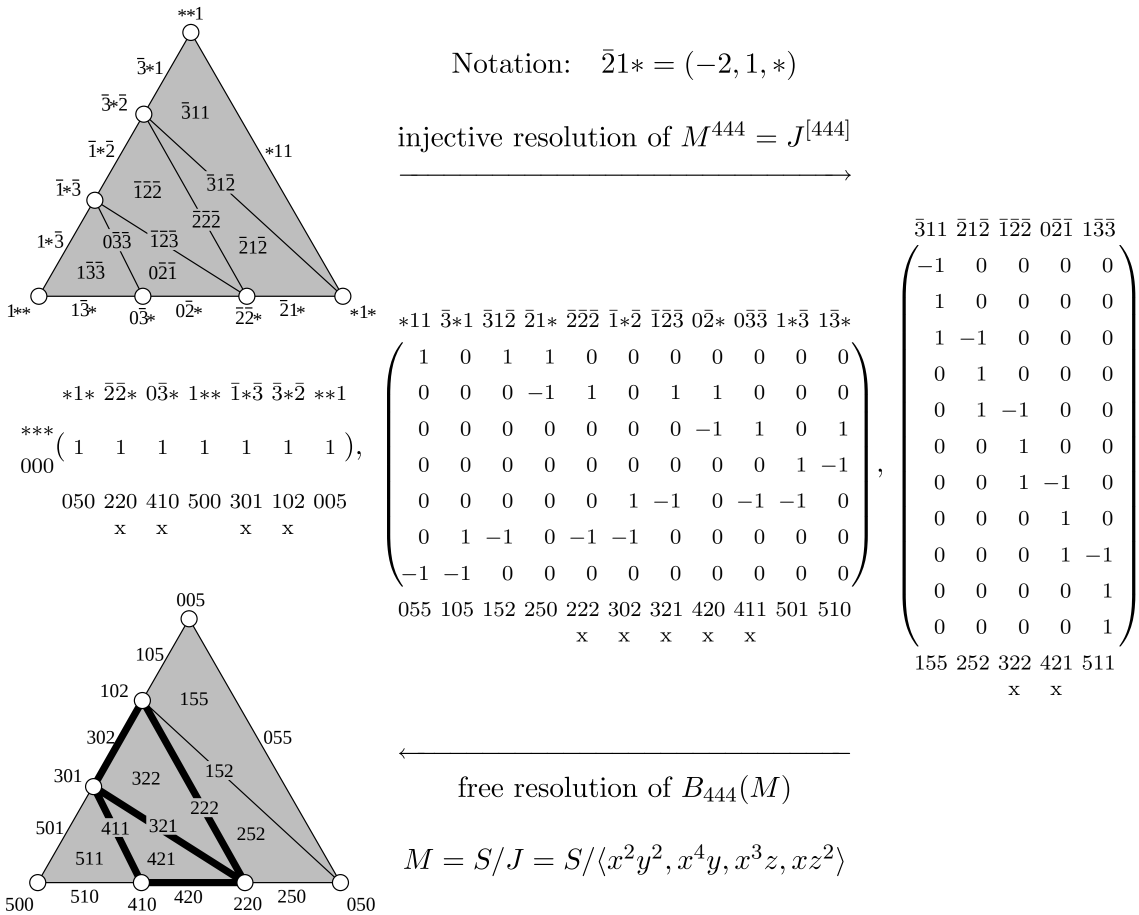

Figure 2: Two cellular monomial matrices supported on the cell complex $X = \Delta_{J+\langle x^5,y^5,z^5\rangle}$.

Proof: It is enough to prove the first statement when the modules are injective and the second statement for any particular matrix Λ. Assuming, then, that Λ represents the map $\lambda : N \to N'$ of injective modules, we show that Λ is minimal if and only if for every $F \subseteq \{1, \ldots, n\}$ the induced map

$$\lambda^F : \underline{\mathrm{Hom}}_S(S/\mathfrak{m}^F, N) \otimes S[\mathbf{x}^{-\overline{F}}] \quad \to \quad \underline{\mathrm{Hom}}_S(S/\mathfrak{m}^F, N') \otimes S[\mathbf{x}^{-\overline{F}}]$$

is zero. Given F, the localization kills any summands of N and N' which are shifts of $\underline{E}(S/\mathfrak{m}^{F'})$ for $F' \not\subseteq F$; and $\underline{\mathrm{Hom}}_S(S/\mathfrak{m}^F, -)$ depends only on the summands of N and N' which are shifts of $S/\mathfrak{m}^{F'}$ for $F' \supseteq F$. Therefore we are reduced to the case where all summands of both N and N' are shifts of $\underline{E}(S/\mathfrak{m}^F)$—that is, the case where all of the $*$-vectors in Λ are equal to $*\overline{F}$. With this assumption, λ^F is just the map between the socles of $N \otimes S[\mathbf{x}^{-\overline{F}}]$ and $N' \otimes S[\mathbf{x}^{-\overline{F}}]$. Now a homomorphism of indecomposable injectives of the same support is nonzero on the socle if and only if it is an isomorphism (injective hull is an essential extension and every injection splits). Applying this to each component of Λ proves the first statement. The second statement follows from the next lemma. □

Lemma 3.18 *Let λ be any map of $\mathbb{Z}^n$-finite flat or injective modules. Given a matrix Λ for λ, the matrix for $\lambda^\vee$ induced by Matlis duality is the same matrix Λ. [N.B. The homological grading is left unchanged.]*

Proof: Use the isomorphism $\underline{E}(S/\mathfrak{m}^F)[\mathbf{b}]^\vee = S[\mathbf{x}^{-\overline{F}}][-\mathbf{b}]$ and Theorem 3.7. □

Example 3.19 The homologically graded monomial matrix for the Čech complex in Example 3.4 represents a minimal injective resolution of the canonical module $S[-(1,1)]$. □

Heuristically, if elements in flat modules act like column vectors, then elements of injective modules act like row vectors. This phenomenon can be seen explicitly in Figure 2, where the shifts on top of the scalar matrix determine an injective resolution of $J^{[444]}$, while the shifts on the bottom determined a free resolution of $B_{444}(S/J)$ going the other way.

In the case of a minimal cellular flat or injective resolution, Proposition 3.10 depends only on the vertex labels. It should be noted that the conclusion of the following proposition is used in [BS] as the *definition* of cellular complex (although only for free modules). Many nonminimal cellular resolutions, including the *hull resolution* [BS] satisfy this condition.

Proposition 3.20 *If a minimal acyclic complex of flat or injective modules is represented by a cellular monomial matrix, then the label $\mathbf{a}_G$ on each face G equals the join $\bigvee_{v\in G}\mathbf{a}_v$ of the labels on all vertices $v \in G$.*

Proof: By Matlis duality it is enough to prove this for the cellular flat complex $\mathbb{F}$ supported on a labelled cell complex X. For any $\mathbf{c} \in \mathbb{Z}^n$, let $\mathbf{c} \vee X$ be obtained from X by replacing $\mathbf{a}_G$ with $\mathbf{c} \vee \mathbf{a}_G$ for all faces $G \in X$. Then $\mathbf{c} \vee X$ still determines a cellular monomial matrix. Furthermore, if $\mathbf{c}_0 \in \mathbb{Z}^n$ is any integer vector $\preceq (\mathbf{a}_G)_\mathbb{Z}$ for all $G \in X$, then one verifies easily that X satisfies the conclusion of the Proposition if and only if $\mathbf{c} \vee X$ does for all integer vectors $\mathbf{c} \preceq \mathbf{c}_0$. Denote by $\mathbf{c} \vee \mathbb{F}$ the complex of flat modules determined by $\mathbf{c} \vee X$.

Fixing $\mathbf{c} \preceq \mathbf{c}_0$, it is enough to show that the conclusion holds for $\mathbf{c} \vee X$. For $\mathbf{b} \in \mathbb{Z}^n$, the complex $\mathbb{F}_\mathbf{b}$ of k-vector spaces of $\mathbb{F}$ in degree $\mathbf{b}$ is the same as the complex $(\mathbf{c} \vee \mathbb{F})_\mathbf{b}$. Indeed, the complexes in question are both given by the *scalar* submatrix matrix whose shifts are $\preceq \mathbf{b}$. In particular, $\mathbb{F}$ is acyclic implies that $\mathbf{c} \vee \mathbb{F}$ is acyclic. Shifting everything by $\mathbf{c}$, we may assume that $\mathbf{c} = \mathbf{0}$, and in this case, the result is [Mil, Proposition 5.20]. □

3.5 Functors and monomial matrices

The motivation for defining monomial matrices is the way they separate and keep track of the different aspects of a complex of $\mathbb{Z}^n$-graded flat or injective modules. Monomial matrices make it easy to take submatrices determined by summands (i.e. shifts) with certain properties, or to alter specific properties of the summands represented by the rows and columns simply by applying some numerical operation to the shifts. The purpose of this subsection (and the next) is two show how these numerical operations reflect the functors which arise in Alexander duality.

For ease of organization and reference, Sections 3.5 and 3.6 contain numerous technical lemmas and propositions. The reader is advised to skim these briefly before proceeding to Section 4, returning back as necessary.

The following definition describes the operations that will come up so often in Section 4, and gives them a notation making them visible at a glance.

Definition 3.21 (Operations on monomial matrices) *Let Λ be a homologically graded monomial matrix.*

1. *$\Lambda(-n)$ denotes the homological shift of Λ up by n.*
2. *Λ^* denotes the matrix obtained from Λ by switching the rows and columns (taking the transpose) and multiplying the shifts as well as the homological degrees by -1. In symbols, $(\lambda^*)^{-d}_{qp} = \lambda^d_{pq}$, $(\mathbf{b}^*)^{-d}_{\cdot p} = -\mathbf{b}^d_{p\cdot}$, and $(\mathbf{b}^*)^{-d}_{q\cdot} = -\mathbf{b}^d_{\cdot q}$.*
3. *A* relabelling function *is a map $(\mathbb{Z} \cup \{*\})^n \to (\mathbb{Z} \cup \{*\})^n$. Most of the relabelling functions in this paper are products of functions $\mathbb{Z} \cup \{*\} \to \mathbb{Z} \cup \{*\}$ which can be described as follows. Let $t \in \mathbb{Z}$, and $u_1, u_2, u_3 \in \mathbb{Z} \cup \{*\}$. The array*

$$\begin{bmatrix} b: & < & t & < \\ \mapsto & u_1 & u_2 & b+u_3 \end{bmatrix}$$

denotes a function $f : \mathbb{Z} \cup \{*\} \to \mathbb{Z} \cup \{*\}$ which takes the values

$$f(b) \quad = \quad \begin{cases} u_1 & \text{if} \quad b < t \\ u_2 & \text{if} \quad b = t \\ b+u_3 & \text{if} \quad b > t \end{cases} .$$

The first $<$ may be replaced by $*$, meaning that $* < b < t$ doesn't occur in the input.

Example 3.22 Each component of the relabelling function going from the bottom labelling in Figure 2 to the top one can be given by the first array below. The same array gives the relabelling to go from left to right in Figure 1.

$$\begin{bmatrix} b: & < & 1 & < \\ \mapsto & * & -3 & b-4 \end{bmatrix} \qquad\qquad \begin{bmatrix} b: & * & -3 & < \\ \mapsto & 0 & 1 & b+4 \end{bmatrix}$$

On the other hand, each component of the relabelling function to go from right to left in Figure 1 and top to bottom in Figure 2 can be given by the right array above. □

The next three lemmas will be crucial to understanding the duality theorem 4.5 for resolutions in terms of matrices.

Lemma 3.23 *Let $\lambda : N' \to N$ be a map of $\mathbb{Z}^n$-finite flat modules.*

1. *The Čech hull replaces the nonpositive coordinates in the shifts by $*$.*
2. *The functor $(-)_{\succeq \mathbf{0}}$ of Definition 2.15.1 replaces the nonpositive coordinates (including all coordinates equal to $*$) in the shifts by zero.*

Proof: If N is an irreducible flat module generated in degree $\mathbf{b} \in (\mathbb{Z} \cup \{*\})^n$, then it follows easily from the definitions that $N_{\succeq \mathbf{0}}$ is a free module generated in degree $\mathbf{b} \vee \mathbf{0}$ (remember that $* < 0$). This proves part 2. Using part 2 and the fact that $\check{C}$ factors through $(-)_{\succeq \mathbf{0}}$, part 1 is a matrix realization of Example 2.8.1, where the shift there on $S[\mathbf{x}^{-\overline{F}}][-\mathbf{b}]$ should be $*\overline{F} + \mathbf{b}$ to agree with the conventions from before Lemma 3.1. □

Lemma 3.24 *Let λ be a map $N' \to N$ of flat modules or $N \to N'$ of injective modules. Given a matrix for λ, the matrix for $\lambda[-\mathbf{a}]$ is gotten by replacing each shift $\mathbf{b}$ with*

$$\begin{cases} \mathbf{b}+\mathbf{a} & \text{if the modules are flat, or} \\ \mathbf{b}-\mathbf{a} & \text{if the modules are injective.} \end{cases}$$

Proof: Follows from conventions for shifts of flat and injective modules. □

Lemma 3.25 *If Λ represents a map λ of finitely generated free modules in $\mathcal{M}$, then Λ induces on $\underline{\mathrm{Hom}}_S(\lambda, S)$ the* free transpose *matrix Λ^*.* □

3.6 Localization and restriction

It will be necessary to work with localizations of $\mathbb{Z}^n$-graded modules, especially for Corollary 4.18. However, $M[\mathbf{x}^{-\overline{F}}]$ is a $\mathbb{Z}^n$-graded module over $S[\mathbf{x}^{-\overline{F}}]$, which is not a polynomial ring because of its homogeneous units. To remove these, we use the following definition, which amounts to setting the variables x_i with $i \in \overline{F}$ equal to 1, or "taking the degree zero part of the homogeneous localization of N at $\mathfrak{m}^F$" as in algebraic geometry. See [Mil, Definition 4.2] and the comments following it for more on this localization.

Definition 3.26 *Let $\mathbb{Z}^F \subseteq \mathbb{Z}^n$ be the coordinate space spanned by $\{\mathbf{e}_i \mid i \in F\}$. Define, for $N \in \mathcal{M}$,*

1. $S_{[F]} := k[x_i \mid i \in F]$ *a $\mathbb{Z}^F$-graded k-subalgebra of S*
2. $N_{[F]} := \bigoplus_{\mathbf{b}\in\mathbb{Z}^F} N_{\mathbf{b}}$ *a $\mathbb{Z}^F$-graded $S_{[F]}$-module*
3. $N_{(F)} := N[\mathbf{x}^{-\overline{F}}]_{[F]}$ *a $\mathbb{Z}^F$-graded $S_{(F)} = S_{[F]}$-module*

and let $\mathcal{M}_{[F]}$ be the category of $\mathbb{Z}^F$-graded $S_{[F]}$-modules. If $\mathbf{b} \in (\mathbb{Z} \cup \{*\})^n$, then define the element $\mathbf{b} \cdot F \in (\mathbb{Z} \cup \{*\})^F$ by forgetting the coordinates not in F. If $\mathbf{b} \in (\mathbb{Z} \cup \{*\})^n$ and the context requires that $\mathbf{b} \in (\mathbb{Z} \cup \{*\})^F$, e.g. if $\mathbf{b}$ appears in a monomial matrix over $S_{[F]}$, then it will always be the case that $b_i \in \{0, *\}$ for all $i \notin F$. In this situation, it is assumed that $\mathbf{b}$ is replaced by $\mathbf{b} \cdot F$.

Of course, the restriction functor $N \mapsto N_{[F]}$ is exact, as is homogeneous localization $N \mapsto N_{(F)}$. It will be important in Theorem 4.16, Theorem 4.20, and much of Section 5 to know what happens to injective and flat modules under restriction and localization. This is the purpose of Propositions 3.27, 3.29, and 3.31.

Proposition 3.27 *1. Suppose Λ is a matrix representing a map $\lambda : N' \to N$ of free modules. Then the induced matrix representing the map $\lambda_{[F]} : N'_{[F]} \to N_{[F]}$ of free $S_{[F]}$-modules is the submatrix $\Lambda_{[F]}$ whose shifts $\mathbf{b}$ satisfy $\mathbf{b} \cdot \overline{F} \preceq \mathbf{0} \cdot \overline{F}$.*

2. If $\mathbb{F} = \mathbb{F}.$ is a free resolution in $\mathcal{M}$ of a module N, then $\mathbb{F}_{[F]}$ is a free resolution in $\mathcal{M}_{[F]}$ of $N_{[F]}$. If $\mathbb{F}$ is minimal then $\mathbb{F}_{[F]}$ is minimal.

Proof: The restriction $S[-\mathbf{b}]_{[F]}$ is $S_{[F]}[-\mathbf{b} \cdot F]$ if $\mathbf{b} \cdot \overline{F} \preceq \mathbf{0} \cdot \overline{F}$ and 0 otherwise. This proves part 1. Exactness of restriction together with the fact that minimality is always preserved under taking submatrices proves part 2 from part 1. □

Lemma 3.28 *Restriction of any module $N \in \mathcal{M}$ commutes with Čech hull. In symbols, $\check{C}(N)_{[F]} \cong \check{C}_{[F]}(N_{[F]})$, where $\check{C}_{[F]}$ is the Čech hull over $S_{[F]}$.*

Proof: Follows from the definitions (2.7.3 and 3.26.2). □

Proposition 3.29 *1. Suppose Λ is a matrix representing a map $\lambda : N \to N'$ of injective modules. Then the induced matrix representing the map $\lambda_{(F)} : N_{(F)} \to N'_{(F)}$ of injectives in $\mathcal{M}_{[F]}$ is the submatrix $\Lambda_{(F)}$ with $*$-vectors containing $*\overline{F}$ (that is, the submatrix where the shifts $\mathbf{b}$ satisfy $\operatorname{supp}(\mathbf{b}_{\mathbb{Z}}) \subseteq F$). In particular, $\Lambda_{(F)}$ is minimal if Λ is.*

2. If $\mathbb{I}^{\cdot}$ is an injective resolution in $\mathcal{M}$ of a module N, then $\mathbb{I}^{\cdot}_{(F)}$ is an injective resolution of $N_{(F)}$ in $\mathcal{M}_{[F]}$. If $\mathbb{I}^{\cdot}$ is minimal then so is $\mathbb{I}^{\cdot}_{(F)}$.

Proof: It is an easy fact that $\underline{E}(S/\mathfrak{m}^{F'}) \otimes_S S[\mathbf{x}^{-\overline{F}}]$ is isomorphic to $\underline{E}(S/\mathfrak{m}^{F'})$ if $F' \subseteq F$ and is otherwise zero. Therefore, a component Λ_{pq} survives the localization if and only if both $*$-vectors contain $*\overline{F}$. Furthermore, it is a consequence of Lemma 3.28 and Example 2.8.3 that

$$\Big(\underline{E}(S/\mathfrak{m}^{F'})\Big)_{[F]} \quad\cong\quad \underline{E}_{[F]}\Big((S/\mathfrak{m}^{F'})_{[F]}\Big), \tag{7}$$

where $\underline{E}_{[F]}$ is the injective hull in $\mathcal{M}_{[F]}$. For the first statement it remains only to show that (7) still holds if the $\underline{E}(S/\mathfrak{m}^{F'})$ on the left is shifted by $\mathbf{b}$ and the right-hand side is shifted by the vector $\mathbf{b} \cdot F \in (\mathbb{Z} \cup \{*\})^F$ (Definition 3.26). This follows from the next lemma, below. The exactness of homogeneous localization proves the second statement from the first. □

Lemma 3.30 *Suppose that the variables $\{x_i \mid i \notin F\}$ act as units on $N \in \mathcal{M}$. Then restriction $(-)_{[F]}$ commutes with arbitrary $\mathbb{Z}^n$-shifts of N; i.e. $N[\mathbf{b}]_{[F]} \cong N_{[F]}[\mathbf{b} \cdot F]$ for $\mathbf{b} \in \mathbb{Z}^n$.*

Proof: Any shift of such an N may be accomplished (up to isomorphism in $\mathcal{M}$) by a vector $\mathbf{b} \in \mathbb{Z}^F$, and $(-)_{[F]}$ obviously commutes with such a shift. □

Proposition 3.31 *1. Suppose that Λ is a matrix representing a map $\lambda : N' \to N$ of free modules. Then the induced matrix $\Lambda_{(F)}$ representing $\lambda_{(F)} : N'_{(F)} \to N_{(F)}$ of free $S_{[F]}$-modules is gotten by replacing each shift $\mathbf{b}$ with $\mathbf{b} \cdot F \in \mathbb{Z}^F$.*

2. If Λ represents a free resolution of M, then $\Lambda_{(F)}$ represents a free resolution of $M_{(F)}$.

Proof: After tensoring with $S[\mathbf{x}^{-\overline{F}}]$, Lemma 3.30 and the exactness of homogeneous localization imply the result. □

4 Resolutions and duality

It has been seen in the recent work of many authors (see the Introduction) that Alexander duality manifests itself in the numerical data associated to monomial ideals. The purpose of this section is to demonstrate the duality not just for monomial ideals, but for all $\mathbb{N}^n$-graded modules; and not just at the level of Betti and Bass numbers, but at the level of free and injective *resolutions*. That is, the *maps* in a free resolution of M come from the maps in an injective resolution of M and are dual to maps in the injective resolution of $M^{\mathbf{a}}$. This is a powerful notion, yielding as immediate corollaries (in Section 5) the menagerie of equalities and inequalities between Betti and Bass numbers of a positively $\mathbf{a}$-determined module and its Alexander dual. Section 4.3 interprets duality for resolutions in terms of lcm-lattices of monomial ideals [GPW], showing how they determine minimal injective resolutions. Section 4.4 investigates the interaction of duality for resolutions with localization, in the process proving a structure theorem for injective resolutions. Finally, Section 4.5 contains the generalization to $\mathbb{N}^n$-graded modules of the theorems of Eagon-Reiner [ER] and Terai [Ter] relating the projective dimension of a squarefree monomial ideal to the shifts in the minimal free resolution of its Alexander dual. The proof uses duality for resolutions (Theorem 4.5) and properties of injective resolutions in an essential way, casting new light on the principles underlying the results in [ER, Ter].

4.1 Preliminaries

Given $\mathbf{b} \in \mathbb{Z}^n$ and $M \in \mathcal{M}$, let $M_{\succeq \mathbf{b}} \subseteq M$ be the submodule generated by all homogeneous elements in degrees $\succeq \mathbf{b}$ (the special case $\mathbf{b} = \mathbf{0}$ is Definition 2.15.1). The following lemma describes how the functors $P_{\mathbf{a}}$, $\check{C}$, and $(-)_{\succeq \mathbf{b}}$ act on free, flat, and injective modules in $\mathcal{M}$. Its proof is immediate from Lemma 3.5, Lemma 2.9, and the definitions.

Lemma 4.1 *The functors in the first column of the following table*

	injective	free	flat
$P_{\mathbf{a}}$	injective	free	flat
$\check{C}$	injective	flat	flat
$(-)_{\succeq \mathbf{b}}$	finite length	free	free

are exact and alter $\mathbb{Z}^n$-finite free, flat, and injective modules in the indicated manner. In particular, $P_{\mathbf{a}}$ is the identity on an indecomposable flat module L if and only if $L_{\mathbf{a}} \cong k$; otherwise, $P_{\mathbf{a}}L = 0$. □

Remark 4.2 One could work with projective and injective objects, and even resolutions, in the categories $\mathcal{M}^{\mathbf{a}}$, $\overline{\mathcal{M}}^{\mathbf{a}}$, $\mathcal{M}^{\mathbf{a}}_{+}$, and $\mathcal{M}^{\mathbf{a}}_{-}$. However, this ignores the boundary effects (Remark 2.16) which give the theory its richness. Nevertheless, this point of view is exploited to great advantage in [Yan3] to show that the Bass numbers of $\mathbf{1}$-determined modules (which he calls *straight*) are finite.

The next lemma will be used in the proof of Proposition 4.4, which gives a connection between Alexander and local duality. The proposition will be applied in the duality theorem for resolutions as a crucial step in going from an injective resolution of M to a free resolution of M. The proof of the proposition shows how the functors of Section 2.3 can replace limits (as in [Mil, Section 6]) in the world of finitely determined $\mathbb{Z}^n$-graded modules.

Lemma 4.3 *Every $\mathbb{Z}^n$-finite flat module $L \in \mathcal{M}$ has a unique submodule L' which is minimal among the submodules N such that L/N is free. In addition, L' is flat.*

Proof: By Lemma 3.5, it is enough to prove the Matlis dual statement, which reads: The $\mathbb{Z}^n$-finite injective module $L^\vee$ has a unique maximal submodule which is the Matlis dual of a free module, and the quotient of $L^\vee$ by this maximal submodule is injective. [The submodule will be $(L/L')^\vee$.] But the Matlis duals of $\mathbb{Z}^n$-finite free modules are precisely the $\mathbb{Z}^n$-finite injective modules with support on $\mathfrak{m}$, so the desired maximal submodule of $L^\vee$ is thus $H^0_\mathfrak{m}(L^\vee)$. The quotient $L^\vee/H^0_\mathfrak{m}(L^\vee)$ is injective by Lemma 3.5. $\square$

Proposition 4.4 *Let M be positively $\mathbf{a}$-determined. If $\mathbb{L} = \mathbb{L}.$ is a $\mathbb{Z}^n$-finite flat resolution in $\mathcal{M}$ of $M^\vee$, then $\mathbb{F} = \underline{\mathrm{Hom}}_S(\mathbb{L}, S[-\mathbf{1}])$ is a complex of free modules. Furthermore, $\mathbb{F}$ has zero homology except in homological degree $-n$, where $H_{-n}\mathbb{F} \cong M$. In particular, the homological shift $\mathbb{F}(-n)$ of $\mathbb{F}$ up by n is a free resolution of M whenever $\mathbb{L}.$ is zero in homological degrees $> n$.*

Proof: Let $\mathbb{L}' \subseteq \mathbb{L}$ be as in Lemma 4.3, so that $\mathbb{F}' = \mathbb{L}/\mathbb{L}'$ is a complex of free modules in $\mathcal{M}$. Then $\underline{\mathrm{Hom}}_S(\mathbb{L}', S) = 0$ by minimality of $\mathbb{L}'$ as a submodule of $\mathbb{L}$. Therefore,

$$\mathbb{F} \;=\; \underline{\mathrm{Hom}}_S(\mathbb{L}, S[-\mathbf{1}]) \;=\; \underline{\mathrm{Hom}}_S(\mathbb{F}', S[-\mathbf{1}]) \tag{8}$$

because $\mathbb{L}'$ is a split submodule of $\mathbb{L}$. It remains only to calculate the homology of $\mathbb{F}$.

For $t \in \mathbb{N}$, denote by $\mathbf{t}$ the vector $t \cdot \mathbf{1} = (t, \ldots, t) \in \mathbb{N}^n$. Now choose $t \in \mathbb{N}$ large enough so that $\mathbb{F}'$ is generated in degrees $\succeq -(\mathbf{t} - \mathbf{1})$ and $\mathbf{t} \succeq \mathbf{a}$. Then $\mathbb{L}_{\succeq -\mathbf{t}}$ is a complex of free modules by Lemma 4.1, and is therefore a free resolution of the artinian module $(M^\vee)_{\succeq -\mathbf{t}}$. It follows from local duality [GW, Theorem 2.2.2] that $(\mathbb{L}_{\succeq -\mathbf{t}})^* := \underline{\mathrm{Hom}}_S(\mathbb{L}_{\succeq -\mathbf{t}}, S[-\mathbf{1}])$ is a homological free complex which has as its only nonzero homology the module

$$H_{-n}((\mathbb{L}_{\succeq -\mathbf{t}})^*) \;\cong\; ((M^\vee)_{\succeq -\mathbf{t}})^\vee \;\cong\; ((M^\vee)^\vee)_{\preceq \mathbf{t}} \;\cong\; M_{\preceq \mathbf{t}} \;=\; B_\mathbf{t} M\,,$$

where $M_{\preceq \mathbf{t}}$ denotes the quotient module $\bigoplus_{\mathbf{b} \preceq \mathbf{t}} M_\mathbf{b}$ of M.

The complex $\mathbb{L}'$ is a direct sum of flat modules none of which is free, so $\mathbb{L}'_{\succeq -\mathbf{t}}$ is generated in degrees $\not\succeq -(\mathbf{t} - \mathbf{1})$. Therefore, the quotient complex $\underline{\mathrm{Hom}}_S(\mathbb{L}'_{\succeq -\mathbf{t}}, S[-\mathbf{1}])$ of $(\mathbb{L}_{\succeq -\mathbf{t}})^*$ is a free module generated in degrees $\not\preceq \mathbf{t}$. Meanwhile, the assumption on t means that the subcomplex $\mathbb{F} \subseteq (\mathbb{L}_{\succeq -\mathbf{t}})^*$ is generated in degrees $\preceq \mathbf{t}$. By Lemma 4.1, applying the positive extension functor $P_\mathbf{t}$ to the exact sequence of free modules

$$0 \;\to\; \mathbb{F} \;\to\; (\mathbb{L}_{\succeq -\mathbf{t}})^* \;\to\; \underline{\mathrm{Hom}}_S(\mathbb{L}'_{\succeq -\mathbf{t}}, S[-\mathbf{1}]) \;\to\; 0$$

yields an isomorphism $\mathbb{F} \cong P_\mathbf{t}((\mathbb{L}_{\succeq -\mathbf{t}})^*)$. But exactness of $P_\mathbf{t}$ also implies that $H_r\mathbb{F} \cong P_\mathbf{t} H_r((\mathbb{L}_{\succeq -\mathbf{t}})^*)$ is zero except when $r = -n$, in which case $H_{-n}\mathbb{F} \cong P_\mathbf{t}(B_\mathbf{t} M)$. Since M is positively $\mathbf{a}$-determined, it is also positively $\mathbf{t}$-determined, because $\mathbf{t} \succeq \mathbf{a}$. We conclude that $P_\mathbf{t} B_\mathbf{t} M \cong M$ by Theorem 2.11 and Table 1 (before Lemma 2.10). $\square$

4.2 Duality for resolutions

Now we present the duality theorem for resolutions, the culmination of our study of Alexander duality. It can be thought of as a simultaneous generalization of [Mil, Theorem 4.10] and [Mil, Theorem 5.8] to the case of arbitrary positively $\mathbf{a}$-determined modules. Although the generalized Alexander duality functors (Definition 2.15) appear all over the place in Theorem 4.5, caution is warranted since the resolutions to which they are applied are rarely $\mathbf{a}$-determined (positively or negatively, or neither). Indeed, see Lemma 4.7 and its proof: the free resolution $\mathbb{F}$ in Theorem 4.5 almost always has summands in degrees $\not\preceq \mathbf{a}$ because applying $B_\mathbf{a}$ to a positively $\mathbf{a}$-determined module usually introduces a syzygy (even a first syzygy) in some degree $\not\preceq \mathbf{a}$. An instructive example is the module $B_\mathbf{1} S = S/\langle x_1^2, \ldots, x_n^2 \rangle$, which is nonzero only in degrees $\preceq \mathbf{1}$, but whose free resolution has a summand in every degree from the set $\{0, 2\}^n$. Nevertheless, if the resolutions in Theorem 4.5 are minimal, no information is lost in taking the generalized Alexander duals (Corollary 4.9).

In Theorem 4.5, as in Proposition 4.4, the notation $(-n)$ attached to a homologically graded complex indicates that the homological grading is to be shifted up by n. Recall from Definition 2.15 that $A_\mathbf{a}^{0,+}(N) = (N^\vee[-\mathbf{a}])_{\succeq \mathbf{0}}$ and $A_\mathbf{a}^{+,0}(N) = (\check{C}N)^\vee[-\mathbf{a}]$ for any $N \in \mathcal{M}$.

Theorem 4.5 (Duality for resolutions) *Let $M \in \mathcal{M}_+^\mathbf{a}$ be positively $\mathbf{a}$-determined. Suppose that $\mathbb{I} = \mathbb{I}^\cdot$ is a $\mathbb{Z}^n$-finite injective resolution of M. Suppose also that $\mathbb{F} := \mathbb{F}_\cdot$ is a $\mathbb{Z}^n$-finite free resolution of $B_\mathbf{a}(M^\mathbf{a})$. Then $\mathbb{I}$ and $\mathbb{F}$ both have length $\geq n$, and whenever either is minimal it has length exactly n. We can construct the following resolutions from $\mathbb{I}$ and $\mathbb{F}$:*

1. *The homological complex $A_\mathbf{a}^{0,+}\,\mathbb{I}$ is a free resolution of $B_\mathbf{a}(M^\mathbf{a})$.*
2. *The cohomological complex $A_\mathbf{a}^{+,0}\,\mathbb{F}$ is an injective resolution of M.*
3. *The homological complex $\mathbb{F}^*[-\mathbf{a}-\mathbf{1}] := \underline{\mathrm{Hom}}_S(\mathbb{F}, S[-\mathbf{a}-\mathbf{1}])(-n)$ is a free resolution of $B_\mathbf{a} M$ whenever $\mathbb{F}$ has length n.*
4. *The homological complex $P_\mathbf{a}\mathbb{F}$ is a free resolution of $M^\mathbf{a}$.*
5. *The four homological complexes $\underline{\mathrm{Hom}}_S(S^\vee, H_\mathfrak{m}^0\,\mathbb{I})$, $\underline{\mathrm{Hom}}_S(S^\vee, \mathbb{I})$, $\underline{\mathrm{Hom}}_S(\mathbb{I}^\vee, S)$, and $\underline{\mathrm{Hom}}_S((H_\mathfrak{m}^0\,\mathbb{I})^\vee, S)$ are isomorphic. When shifted homologically up by n, each is a free resolution of $M[\mathbf{1}]$ as long as $\mathbb{I}$ has length n.*

In terms of monomial matrices, we have the following (recall the conventions in Definition 3.21). Let Λ be a monomial matrix for $\mathbb{I}$ and Φ a monomial matrix for $\mathbb{F}$.

1′. *Applying $\begin{bmatrix} b_i: & < & -a_i & < \\ \mapsto & 0 & 0 & b_i + a_i \end{bmatrix}$ to Λ yields a matrix for a free resolution of $B_\mathbf{a}(M^\mathbf{a})$.*

2′. *Applying $\begin{bmatrix} b_i: & < & 0 & < \\ \mapsto & * & * & b_i - a_i \end{bmatrix}$ to Λ yields a matrix for an injective resolution of M.*

3′. *If $\mathbb{F}$ has length n then adding $\mathbf{a}+\mathbf{1}$ to each shift of $\Phi^*(-n)$ yields a matrix representing a free resolution of $B_\mathbf{a} M$.*

4′. *The submatrix $\Phi^{\preceq \mathbf{a}}$ whose shifts are $\preceq \mathbf{a}$ represents a free resolution of $M^\mathbf{a}$.*

5′. Let $\Lambda_{\mathbb{Z}}$ be the submatrix of Λ whose shifts are in $\mathbb{Z}^n$. Then $\Lambda_{\mathbb{Z}}^*(-n)$ represents a free resolution of $M[\mathbf{1}]$ whenever $\mathbb{I}$ has length n.

Of course, M and $M^{\mathbf{a}}$ can be switched throughout. All of the operations preserve minimality.

Proof: The statements about lengths of resolutions are standard.

(1): The exactness of $A_{\mathbf{a}}^{0,+}$ and the fact that it takes injective modules to free modules both follow from Lemma 4.1. Definition 2.15.3 implies that $A_{\mathbf{a}}^{0,+}\,\mathbb{I}$ resolves $B_{\mathbf{a}}(M_{\mathbf{a}})$. To get part 1′, use Lemmas 3.18, 3.24, and 3.23: the monomial matrix in question is obtained from Λ by first adding $\mathbf{a}$ to all shifts and then replacing all negative coordinates (including $*$) by zero.

(2): The exactness of $A_{\mathbf{a}}^{+,0}$ and the fact that it takes injective modules to free modules both follow from Lemma 4.1. Definition 2.15.2 implies that $A_{\mathbf{a}}^{+,0}\,\mathbb{F}$ resolves M. To get part 2′, use the same lemmas used for part 1′.

(3): By definition we have an isomorphism $(B_{\mathbf{a}}M^{\mathbf{a}})^\vee[-\mathbf{a}] \cong B_{\mathbf{a}}M$, so this is just local duality [GW, Theorem 2.2.2]. For part 3′ use Lemmas 3.25 and 3.24.

(4): Use Lemma 4.1 and the fact that $P_{\mathbf{a}}B_{\mathbf{a}}(M^{\mathbf{a}}) \cong M^{\mathbf{a}}$ (Theorem 2.11 and Table 1). For part 4′, apply Lemma 4.1.

(5): The first complex is isomorphic to the second because $S^\vee$ is artinian, so any homomorphism $S^\vee \to \mathbb{I}$ lands inside $H^0_{\mathfrak{m}}(\mathbb{I})$. The second and third complexes are isomorphic for the same reason that the first and fourth are: because $\underline{\mathrm{Hom}}_S(-^\vee, S) \cong \underline{\mathrm{Hom}}_k(-^\vee \otimes_S S^\vee, k) \cong \underline{\mathrm{Hom}}_S(S^\vee, -)$ by Equation (2) in Section 1.2. If the third complex has length n, Proposition 4.4 implies that it is a free resolution of $M[\mathbf{1}]$ because $\mathbb{L} := \mathbb{I}^\vee$ is a flat resolution of $M^\vee$ by Matlis duality. To prove part 5′, first take $H^0_{\mathfrak{m}}\,\mathbb{I}$, producing the submatrix with shifts in $\mathbb{Z}^n$, and then apply Lemmas 3.18 and 3.25 to the fourth complex.

The statement about minimality requires an intermediate result, Lemma 4.7. For use in the lemma, the following result [BH2, Theorem 3.1(a)] is derived as a corollary to part 4′.

Corollary 4.6 (Bruns-Herzog) *Suppose M is positively $\mathbf{a}$-determined (equivalently, all nonzero Betti numbers in homological degrees 0 and 1 of the finitely generated $\mathbb{N}^n$-graded module $M \in \mathcal{M}$ occur in degrees $\preceq \mathbf{a}$). Then all nonzero Betti numbers of M occur in nonnegative degrees $\preceq \mathbf{a}$; that is, the minimal free resolution of M is positively $\mathbf{a}$-determined.*

Proof: The condition on the zeroth and first Betti numbers means that M is positively $\mathbf{a}$-determined by Proposition 2.5. The minimal free resolution of M is $\mathbb{N}^n$-graded, so the result follows from part 4′ above. □

Lemma 4.7 *If $\mathbb{F}$ is minimal then $\mathbb{F}$ is positively $(\mathbf{a}+\mathbf{1})$-determined (equivalently, all shifts $\mathbf{b}$ in Φ satisfy $\mathbf{0} \preceq \mathbf{b} \preceq \mathbf{a}+\mathbf{1}$). If $\mathbb{I}$ is minimal, then $\mathbb{I}[-\mathbf{1}]$ is $(\mathbf{a}+\mathbf{1})$-determined (equivalently, the shifts $\mathbf{b}$ in Λ satisfy $\mathbf{1}-\mathbf{a} \preceq \mathbf{b}_{\mathbb{Z}} \preceq \mathbf{1}$).*

Proof: The parenthesized statements come from Example 2.2.2 (and Lemma 3.24 for $\mathbb{I}[-\mathbf{1}]$). The module $B_{\mathbf{a}}(M^{\mathbf{a}}) = P_{\mathbf{a}+\mathbf{1}}(B_{\mathbf{a}}(M^{\mathbf{a}}))$ is positively $(\mathbf{a}+\mathbf{1})$-determined by definition, so

$\mathbb{F}$ is positively $(\mathbf{a}+\mathbf{1})$-determined by Corollary 4.6. It is now enough to show that there exists *some* injective resolution of $M \in \mathcal{M}_+^{\mathbf{a}}$ whose shift by $[-\mathbf{1}]$ is $(\mathbf{a}+\mathbf{1})$-determined, since the minimal injective resolution will then be forced to satisfy the condition, as well. But the complex $A_{\mathbf{a}}^{+,0}\mathbb{F}$ is an injective resolution of M by part 2 above, so the equality $(A_{\mathbf{a}}^{+,0}\mathbb{F})[-\mathbf{1}] = A_{\mathbf{a}+\mathbf{1}}^{+,0}\mathbb{F}$ implies the proposition because $\mathbb{F}$ started out in $\mathcal{M}_+^{\mathbf{a}+\mathbf{1}}$. □

To finish the proof of minimality preservation, check that inequalities $\mathbf{b}_{p\cdot} \neq \mathbf{b}_{\cdot q}$ persist under the operations in parts $1'$–$5'$ whenever the vectors satisfy the conditions of the Lemma. Note that taking submatrices always has this property. The proof of the theorem is complete. □

Example 4.8 Each part of the theorem can be illustrated by the cocellular and cellular monomial matrices represented in Figures 1 and 2. The numbering in this example corresponds to the numbering in Theorem 4.5. Here, $\mathbf{a} = (4,4,4) =: 444$.

1. The top labelling in Figure 2 determines a cellular minimal injective resolution of $M^{444} = J^{[444]}$. The bottom labelling determines a cellular minimal free resolution of $B_{444}(M)$. See the relabelling functions in Example 3.22. Similarly, the right labelling in Figure 1 determines a cocellular minimal injective resolution of $M = S/J$, while the left labelling determines a cocellular free resolution of $B_{444}(M^{444})$.
2. This is just part 1 in reverse. Again, see the arrays in Example 3.22.
3. This part has not been mentioned yet in the examples; it connects Figures 1 and 2. The bottom labels of Figure 2 give a minimal free resolution of $B_{444}(S/J)$, and are obtained by subtracting from $555 = \mathbf{a}+\mathbf{1}$ the labels of Figure 1, which give a minimal free resolution of

   $$B_{444}((S/J)^{444}) \quad = \quad (J^{[444]} + \langle x^5, y^5, z^5\rangle)/\langle x^5, y^5, z^5\rangle \,.$$

   Note that this transition is the one that goes from cellular to cocellular (and back).
4. This is why the thickened subdivided quadrilateral in Figure 2 gives the minimal free resolution of $M = S/J$: the shifts labelled with an "x" are precisely those that are $\preceq 444$ in the free resolution of $B_{444}(M)$. (Switch M and $M^{\mathbf{a}}$ in Theorem 4.5.4 here.)
5. This says that the *interior* faces in the top picture of Figure 2 give a *free* resolution of $J^{[444]}$, since the interior faces are precisely those with no $*$. More precisely, if one takes the negatives of the interior face labels and adds $\mathbf{1} = 111$, then the coboundary complex $\widetilde{\mathcal{C}}^{\cdot}(X;k)$ induces a free resolution of $J^{[444]}$. This free resolution is called the *coScarf resolution* of $J^{[444]}$. Its origin in the context of injective resolutions explains, finally, why the "unbounded faces" had to be left out in [Stu, Mil]: by Theorem 4.5.5 they correspond to injective summands which don't show up in the free resolution. More generally, coScarf resolutions are defined for *cogeneric monomial ideals* [Stu, Mil, MSY], which are by definition the Alexander duals to generic ideals, and the (labelled) triangulation is called the *coScarf triangulation.* □

There are analogs of coScarf complexes for cellular resolutions of arbitrary monomial ideals; see Example 4.10.

There is a fundamental difference between parts 1, 2, and 3 of Theorem 4.5 and parts 4 and 5: the latter parts involve a certain loss of information (by taking submatrices), whereas the former do not. This is evident from the examples above: one can get all the way from the right of Figure 1 to the top of Figure 2 by going through the left of Figure 1 and the bottom of Figure 2. Corollary 4.9 says that this is phenomenon is completely general.

Corollary 4.9 (Duality for minimal resolutions) *Assume notation from Theorem 4.5. The following resolutions are equivalent, in the sense that the functors in Theorem 4.5 (and their compositions) translate between them, and there are matrices for them which are the same up to relabelling the shifts and taking a transpose:*

1. *A minimal injective resolution of M.*
2. *A minimal free resolution of $B_{\mathbf{a}}(M^{\mathbf{a}})$.*
3. *A minimal free resolution of $B_{\mathbf{a}}M$.*
4. *A minimal injective resolution of $M^{\mathbf{a}}$.*

A matrix for the minimal injective resolution of $M^{\mathbf{a}}$ is obtained by applying the relabelling function

$$\left[\begin{array}{cccc} b_i: & * & -1 & < \\ \mapsto & 1 & * & b_i+1-a_i \end{array}\right]$$

to $\Lambda^*(-n)$.

Proof: Straightforward from Theorem 4.5. (Notice that the shifts satisfy the conditions of Lemma 4.7, so the relabelling function need not be specified outside those ranges.) □

Example 4.10 Here is a synopsis of what duality for resolutions says about cellular and cocellular resolutions in general. Suppose the free resolution $\mathbb{F}$ in Theorem 4.5 is a cellular resolution $\mathbb{F}_X$ of $B_{\mathbf{a}}(S/J)$ for some monomial ideal J, supported on a labelled cell complex X of dimension $n-1$. Theorem 4.5.4 specializes to [Mil, Theorem 5.8] in this case, and yields in particular the construction of the *cohull resolution* of a monomial ideal [Mil, Definition 5.15] from the *hull resolution* [BS, Theorem 2.5]. Heuristically, the resolutions obtained from $\mathbb{F}_X$ in Theorem 4.5 and Corollary 4.9 are described as follows (even when they aren't minimal):

free resolution of S/J	$\leftrightarrow$	boundary complex of $X\|_{\text{vertices} \neq x_i^{a_i+1}}$
free resolution of $B_{\mathbf{a}}(S/J)$	$\leftrightarrow$	boundary complex of X
injective resolution of $J^{[\mathbf{a}]}$	$\leftrightarrow$	coboundary complex of X
injective resolution of S/J	$\leftrightarrow$	boundary complex of X
free resolution of $B_{\mathbf{a}}(J^{[\mathbf{a}]})$	$\leftrightarrow$	coboundary complex of X
free resolution of $J^{[\mathbf{a}]}$	$\leftrightarrow$	coboundary complex on interior faces of X

where, of course, the labellings vary. The first three are cellular, the fourth and fifth are cocellular, and the last is *relative cocellular* [Mil, Section 5], generalizing the coScarf complex. These can all be matched with Examples 3.11, 3.14, and 4.8, along with Figures 1 and 2. □

4.3 LCM-lattices

In [GPW], the *lcm-lattice* of a monomial ideal $J = \langle m_1, \dots, m_r\rangle$ is introduced:

$$L_J \quad := \quad \{\mathrm{lcm}(m_i \in A) \mid A \subseteq \{m_1, \dots, m_r\}\}.$$

L_J is a join lattice (with atoms $m_1, \dots, m_r$) inside of $\mathbb{N}^n$. Gasharov, Peeva, and Welker show that L_J carries homological information about J in much the same way that the intersection lattice of a subspace arrangement carries homological information about the complement of the arrangement. In particular, one of their main results [GPW, Theorem 3.3] is that L_J determines the minimal free resolution of S/J, in the following sense.

Theorem 4.11 (Gasharov-Peeva-Welker) *Let J and J' be two monomial ideals in polynomial rings S and S'. All of the nonzero Betti numbers of S/J occur in degrees appearing in L_J. Let $g : L_J \to L_{J'}$ be a map which is a bijection on the atoms and preserves joins. If each shift $\mathbf{b}$ in a monomial matrix for the minimal free resolution of S/J is replaced by $g(\mathbf{b})$, then the result is a monomial matrix for a free resolution of S'/J'. If g is an isomorphism, then the relabelled monomial matrix is minimal.*

A natural question to ask is, "what determines the minimal injective resolution?" Duality for resolutions produces the following response. Let $\mathfrak{m}^{\mathbf{a}+\mathbf{1}} = \langle x_1^{a_1+1}, \dots, x_n^{a_n+1}\rangle$.

Corollary 4.12 *Let J be a monomial ideal generated in degrees $\preceq \mathbf{a}$. Then $L_{J+\mathfrak{m}^{\mathbf{a}+\mathbf{1}}}$ determines the minimal injective resolution of S/J and the minimal injective resolution of the Alexander dual ideal $J^{[\mathbf{a}]} = (S/J)^{\mathbf{a}}$. In particular, $L_{J+\mathfrak{m}^{\mathbf{a}+\mathbf{1}}}$ determines the minimal free resolutions of both S/J and $S/J^{[\mathbf{a}]}$.*

Proof: The lcm-lattice of $J+\mathfrak{m}^{\mathbf{a}+\mathbf{1}}$ determines the minimal free resolution of $S/(J+\mathfrak{m}^{\mathbf{a}+\mathbf{1}}) = B_{\mathbf{a}}(S/J)$ by Theorem 4.11. The rest follows from Corollary 4.9 and Theorem 4.5. □

Remark 4.13 The injective resolution of J looks pretty much like that of S/J. Indeed, the minimal injective resolution of S is the generalized Alexander dual of the Koszul complex (Example 3.19) and has one summand $\underline{E}(S/\mathfrak{m}^F)[F]$ for each $F \subseteq \{1, \dots, n\}$. Using the exact sequence $0 \to J \to S \to S/J \to 0$ and the definition of Bass numbers (Section 5) in terms of $\underline{\mathrm{Ext}}$ functors [Mil, Definition 4.8], one checks (with the long exact sequence for $\underline{\mathrm{Ext}}$) that the only differences between the minimal injective resolutions of J and S/J come from the indecomposable summands in the injective resolution of S. These get attached either to S/J or J according to whether $\mathfrak{m}^F$ does or does not contain J.

If $S = S'$, the replacement g in Theorem 4.11 can be viewed as a relabelling function in the sense of Definition 3.21 by letting it take arbitrary values off of L_J. But even if $S = S'$, the map $g : L_J \to \mathbb{Z}^n$ is not necessarily induced by a product of maps $\mathbb{Z} \to \mathbb{Z}$ on $\mathbb{Z}^n \to \mathbb{Z}^n$.

The condition of $J + \mathfrak{m}^{\mathbf{a}+\mathbf{1}}$ and $J' + (\mathfrak{m}')^{\mathbf{a}'+\mathbf{1}'}$ having isomorphic lcm-lattices is much stronger than J and J' having isomorphic lcm-lattices. This again comes back to the wealth of information carried by minimal injective resolutions, such as irreducible decompositions and minimal free resolutions over all localizations (Corollary 4.15). A reflection of this is given in [MSY, Theorem 1.5], where the condition of being a *generic monomial ideal* is shown to be equivalent to having the minimal injective resolution stay constant under deformation of the exponent vectors in the minimal generators of J. In essence, the information missing from the minimal free resolution allows it to stay constant under more lax conditions, whereas the minimal injective resolution is more rigid.

Example 4.14 Let $n = 3$, with $J = \langle x^2z^2, xyz, y^2z^2\rangle$ and $J' = \langle x^2, xyz, y^2z^2\rangle$. It is easily checked that the lcm-lattices L_J and $L_{J'}$ are isomorphic, although $L_{J+\langle x^3,y^3,z^3\rangle} \not\cong L_{J'+\langle x^3,y^3,z^3\rangle}$. In fact, S/J has 5 irreducible components, while S/J' has only 4. It follows that the injective hulls of S/J and S/J' have different numbers of indecomposable summands (see, for instance [Mil, Proposition 4.12]). □

4.4 Restriction, localization, and duality

We turn our attention to the interaction of localization with duality for resolutions. The first result (Corollary 4.15) describes the structure of a minimal injective resolution of a finitely generated module M by determining where all of the summands come from: the shifts of $\underline{E}(S/\mathfrak{m}^F)$ correspond to the summands in a minimal free resolution of $M_{(F)}$.

Given an injective module $I \in \mathcal{M}$ and $F \subseteq \{1,\ldots,n\}$, the localization $I_{(F)}$ kills all of the summands in I whose support is not contained in $\mathfrak{m}^F$ (by Proposition 3.29). On the other hand, the functor $H^0_{\mathfrak{m}^F}$ picks out those summands whose supports contain $\mathfrak{m}^F$ (by definition). Combining these two functors, we find that $(H^0_{\mathfrak{m}^F} I)_{(F)} = H^0_{\mathfrak{m}^F}(I_{(F)})$ is the restriction to $S_{[F]}$ of the subquotient of I consisting of all summands which are shifts of $\underline{E}(S/\mathfrak{m}^F)$. If $I = \mathbb{I}$ is a minimal injective resolution of $M \in \mathcal{M}^{\mathbf{a}}_+$, then Theorem 4.5.5 implies the next corollary; if M is just finitely generated, we can shift everything back by $[\mathbf{b}\cdot F]$ after applying the argument to $M[-\mathbf{b}]$.

Corollary 4.15 *Let $M \in \mathcal{M}$ be finitely generated, with minimal injective resolution $\mathbb{I}$. If $|F| = i$, then the complex*

$$\underline{\mathrm{Hom}}_{S_{[F]}}\Big(S^{\vee}_{[F]}\,,\,(H^0_{\mathfrak{m}^F}\mathbb{I})_{(F)}\Big)(-i)$$

of $S_{[F]}$-modules is a minimal free resolution of $M_{(F)}[F]$. In terms of matrices, if Λ is a monomial matrix representing $\mathbb{I}$ and $\Lambda_{(F)}$ is the submatrix with $*$-vectors equal to $*\overline{F}$, then $\Lambda^*_{(F)}(-i)$ represents a minimal free resolution of $M_{(F)}[F]$. □

For more comments on the meaning of this corollary, see Question 4.28.

The next theorem generalizes [Mil, Proposition 4.6] to $\mathbb{N}^n$-graded modules. The subscript $\mathbf{a}\cdot F$ in $A^{+,0}_{\mathbf{a}\cdot F}$ and $A^{0,+}_{\mathbf{a}\cdot F}$ indicates that the duality is taking place over $S_{[F]}$, that is, in $\mathcal{M}_{[F]}$. For the matrices over $S_{[F]}$, recall the convention that all coordinates in $\overline{F}$ will either be 0 or $*$, and these are to be ignored (Definition 3.26).

Theorem 4.16 (Restriction-localization) *Let $M \in \mathcal{M}^{\mathbf{a}}_+$. Suppose that $\mathbb{I}$ is a $\mathbb{Z}^n$-finite injective resolution of M, $\mathbb{F}$ is a $\mathbb{Z}^n$-finite free resolution of $B_{\mathbf{a}}(M^{\mathbf{a}})$, and $F \subseteq \{1,\ldots,n\}$.*

1. $A^{0,+}_{\mathbf{a}\cdot F}(\mathbb{I}_{(F)})$ *and* $(A^{0,+}_{\mathbf{a}}\mathbb{I})_{[F]}$ *are both free resolutions in* $\mathcal{M}_{[F]}$ *of* $B_{\mathbf{a}}(M^{\mathbf{a}})_{[F]}$. *If* $\mathbb{I}$ *is minimal, then* $A^{0,+}_{\mathbf{a}\cdot F}(\mathbb{I}_{(F)}) \cong (A^{0,+}_{\mathbf{a}}\mathbb{I})_{[F]}$ *are minimal.*
2. $A^{+,0}_{\mathbf{a}\cdot F}(\mathbb{F}_{[F]}) \cong (A^{+,0}_{\mathbf{a}}\mathbb{F})_{(F)}$ *is an injective resolution in* $\mathcal{M}_{[F]}$ *of* $M_{(F)}$ *which is minimal if* $\mathbb{F}$ *is minimal.*

The functors in parts 1 and 2 preserve minimality. In terms of monomial matrices, we have the following. Suppose Λ *is a monomial matrix for* $\mathbb{I}$ *and* Φ *is a monomial matrix for* $\mathbb{F}$.

1′. *Let* $\Lambda_{(F)}$ *be the submatrix of* Λ *whose* $*$*-vectors contain* $*\overline{F}$. *Applying the relabelling function* $\left[\begin{array}{cccc} b_i: & < & -a_i & < \\ \mapsto & 0 & 0 & b_i+a_i \end{array}\right]$ *to* $\Lambda_{(F)}$ *yields a matrix for* $A^{0,+}_{\mathbf{a}\cdot F}(\mathbb{I}_{(F)})$ *of part 1.*

2′. *Let* $\Phi_{[F]}$ *be the submatrix of* Φ *whose shifts* $\mathbf{b}$ *satisfy* $\mathbf{b}\cdot\overline{F} \preceq \mathbf{0}\cdot\overline{F}$. *Applying the relabelling function* $\left[\begin{array}{cccc} b_i: & < & 0 & < \\ \mapsto & * & * & b_i-a_i \end{array}\right]$ *to* $\Phi_{[F]}$ *yields a matrix for part 2.*

If Λ *is minimal and the matrix output by part 1′ is* $\Phi_{[F]}$, *then the matrix output by part 2′ is* $\Lambda_{(F)}$. *Similarly, if* Φ *is minimal and the matrix output by part 2′ is* $\Lambda_{(F)}$, *then the matrix output by part 1′ is* $\Phi_{[F]}$.

Proof: (1): That $(A^{0,+}_{\mathbf{a}}\mathbb{I})_{[F]}$ is a free resolution in $\mathcal{M}_{[F]}$ of $B_{\mathbf{a}}(M^{\mathbf{a}})_{[F]}$ and preserves minimality is Proposition 3.27 applied to Theorem 4.5.1. That $A^{0,+}_{\mathbf{a}\cdot F}(\mathbb{I}_{(F)})$ is a free resolution of some module N and preserves minimality is Theorem 4.5.1 applied (over $S_{[F]}$) to Proposition 3.29. Furthermore, the module $N \cong A^{0,+}_{\mathbf{a}\cdot F}(M_{(F)})$ is independent of whether $\mathbb{I}$ is minimal. Thus we need only verify the isomorphism of complexes when $\mathbb{I}$ is minimal.

In fact, the proof only uses $\mathbb{I}[-\mathbf{1}] \in \mathcal{M}^{\mathbf{a}+\mathbf{1}}$ (see Lemma 4.7). Indeed, with this hypothesis we show that the matrix produced in part 1′ represents both complexes of part 1. For $A^{0,+}_{\mathbf{a}\cdot F}(\mathbb{I}_{(F)})$ this is Theorem 4.5.2′ applied (over $S_{[F]}$) after Proposition 3.29. For $(A^{0,+}_{\mathbf{a}}\mathbb{I})_{[F]}$, first apply Theorem 4.5.1′: the hypothesis on $\mathbb{I}$ means that that $b_i \mapsto 0$ if and only if $b_i = *$, and all of the other coordinates are strictly positive. The result now follows from Proposition 3.27.

(2): That $(A^{+,0}_{\mathbf{a}}\mathbb{F})_{(F)}$ is an injective resolution in $\mathcal{M}_{[F]}$ of $M_{(F)}$ and preserves minimality is Proposition 3.29 applied to Theorem 4.5.2. Therefore, we need only show that the matrix produced by part 2′ represents both complexes of part 2. That it represents $A^{+,0}_{\mathbf{a}\cdot F}(\mathbb{F}_{[F]})$ is Theorem 4.5 applied (over $S_{[F]}$) to Proposition 3.27. For $(A^{+,0}_{\mathbf{a}}\mathbb{F})_{(F)}$, first apply Theorem 4.5.2′, whose relabelling array is the same. The $*$-vector of a relabelled shift $\mathbf{b}$ ends up containing $*\overline{F}$ if and only if $\mathbf{b}\cdot\overline{F} \preceq \mathbf{0}\cdot\overline{F}$. Therefore, the result follows from Proposition 3.29.

The final claims depend on the conclusion of Lemma 4.7, which is weaker than minimality. □

Example 4.17 The restriction-localization theorem is easiest to see in Figure 2. If we invert y in the injective resolution of the cogeneric ideal $J^{[444]}$ represented by the top labelling, we simply restrict to the faces which have a $*$ in their second slot. Applying Theorem 4.16, this gives the same subcomplex as picking the faces in the bottom labelling whose second entries are 0. In general, localization of cogeneric monomial ideals acts on the coScarf triangulation by restriction to a face of the simplex. This can be useful for keeping track of numerical information related to resolutions, as in [MSY, Theorem 4.9].

In Figure 1, on the other hand, inverting the variable y in the Scarf injective resolution of S/J in the right of Figure 1 yields the set of faces Y containing the vertex v labelled $1 * 1$. These 2 triangles, 3 edges, and vertex are, by Theorem 4.16, the same faces which in the left labelling have second coordinate 0. In general, the localization $J_{(F)}$ is still a generic monomial ideal, so its minimal injective resolution should still be a triangulation of a simplex. To see how Y is a triangulation of a simplex, define the *contrastar* and *link* of v in the simplicial complex X to be

$$\operatorname{cost}_X v := \{G \in X \mid v \notin G\} \quad \text{and} \quad \operatorname{lk}_X v := \{G \in X \mid v \notin G \text{ and } v \cup G \in X\}\,. \qquad (9)$$

Then the *relative chain complex* $\widetilde{\mathcal{C}}.(X, \operatorname{cost}_X v; k) := \widetilde{\mathcal{C}}.(X; k)/\widetilde{\mathcal{C}}.(\operatorname{cost}_X v; k)$ is isomorphic to $\widetilde{\mathcal{C}}.(\operatorname{lk}_X v)$, essentially because adding v to every face in $\operatorname{lk}_X v$ gives $Y = X \setminus \operatorname{cost}_X v$ [Grä, Lemma 1.3]. In general, localization of generic ideals takes links in the Scarf triangulation.

The reader familiar with Alexander duality for squarefree monomial ideals will notice that the Alexander dual to restriction (which is what we got in the earlier part of this example) is taking links (which is what we got in the latter part). Only this time, it happens on Scarf and coScarf complexes instead of on Stanley-Reisner complexes—that is, on resolutions instead of on modules. □

Corollary 4.18 $(M_{(F)})^{\mathbf{a}\cdot F} \cong (M^{\mathbf{a}})_{[F]}$ *and* $(M^{\mathbf{a}})_{(F)} \cong (M_{[F]})^{\mathbf{a}\cdot F}$ *for* $M \in \mathcal{M}^{\mathbf{a}}_{+}$. *In words, localizing and then dualizing is the same as dualizing and then restricting; while dualizing and then localizing is the same as restricting and then dualizing.*

Proof: It is enough to show the first isomorphism, switching M and $M^{\mathbf{a}}$ for the second. In Theorem 4.16.2, $\mathbb{F}_{[F]}$ resolves $B_{\mathbf{a}}(M^{\mathbf{a}})_{[F]} = B_{\mathbf{a}\cdot F}(M^{\mathbf{a}}{}_{[F]})$, so applying Theorem 4.5.2 (over $S_{[F]}$) we find that $M_{(F)}$ is the Alexander dual of $(M^{\mathbf{a}})_{[F]}$ with respect to $\mathbf{a} \cdot F$. □

4.5 Projective dimension and support-regularity

The duality between Cohen-Macaulayness and linear resolutions for squarefree monomial ideals [ER], or more generally the duality between projective dimension and regularity [Ter] cannot hold verbatim in the $\mathbb{N}^n$-graded case because the projective dimension is bounded while the regularity is not (see [Mil, p. 6–7]). Nonetheless, it will be shown here that this duality has an explanation in terms of injective resolutions, and can therefore be directly generalized to the $\mathbb{N}^n$-graded case. Heuristically, the projective dimension of a finitely generated module M depends only the codimensions of the irreducible summands in a minimal injective resolution of M, and in which cohomological degrees they occur. If M is positively $\mathbf{a}$-determined, Alexander duality changes these codimensions into the sizes of the supports of the Betti degrees of $M^{\mathbf{a}}$.

Definition 4.19 *The* support-regularity *of an* $\mathbb{N}^n$*-graded module is*

$$\operatorname{supp.reg}(M) \quad := \quad \max\{|\operatorname{supp}(\mathbf{b})| - i \mid \beta_{i,\mathbf{b}}(M) \neq 0\}\,,$$

and M *is said to have a* support-linear *free resolution if there is a nonnegative integer* d *such that* $|\operatorname{supp}(\deg m)| = d$ *for all minimal generators* m *of* M *and* $\operatorname{supp.reg}(M) = d$.

Observe that in case M is a squarefree module (i.e. $M \in \mathcal{M}_+^{\mathbf{1}}$), all of the nonzero Betti numbers of M occur in degrees $\mathbf{b} \in \{0,1\}^n$, so that $\operatorname{supp.reg}(M)$ agrees with the *regularity* $\operatorname{reg}(M) := \max_i\{\,|\mathbf{b}| - i \mid \beta_{i,\mathbf{b}}(M) \neq 0\}$ and support-linearity is the same thing as linearity for a minimal free resolution of M. The next theorem answers [HRW, Question 10] by giving the most general form of [ER, Theorem 3] and [Ter, Theorem 0.2 and Corollary 0.3].

Theorem 4.20 *Let M be positively $\mathbf{a}$-determined. Then* $\operatorname{proj.dim}(M) = \operatorname{supp.reg}(M^{\mathbf{a}})$. *In particular, M is Cohen-Macaulay if and only if $M^{\mathbf{a}}$ has a support-linear free resolution.*

Corollary 4.21 *Let $M \in \mathcal{M}_+^{\mathbf{1}}$ be squarefree. Then* $\operatorname{proj.dim}(M) = \operatorname{reg}(M^{\mathbf{1}})$. *In particular, M is Cohen-Macaulay if and only if $M^{\mathbf{1}}$ has a linear free resolution.*

The proof of the theorem will follow after some analysis of the codimensions of indecomposable summands in a minimal injective resolution.

Definition 4.22 *The* elevation *of an injective module I is* $\operatorname{elev}(I) := \max\{c \mid I$ has a summand of codimension $c\}$. *The* rise *of a complex $\mathbb{I} : 0 \to I^0 \to I^{\mathbf{1}} \to \cdots$ of injectives is* $\operatorname{rise}(\mathbb{I}) := \max_i\{\operatorname{elev}(I^i) - i\}$.

Proposition 4.23 *Let $N \in \mathcal{M}$ be finitely generated and $\mathbb{I}$ its minimal injective resolution. Then* $\operatorname{proj.dim}(N) = \operatorname{rise}(\mathbb{I})$, *and N is Cohen-Macaulay if and only if* $\operatorname{rise}(\mathbb{I}) = \operatorname{codim}(I^0)$.

Proof: Taking a $\mathbb{Z}^n$-shift of N if necessary (and choosing a large $\mathbf{a}$), we may assume $N \in \mathcal{M}_+^{\mathbf{a}}$. $\mathbb{I}$ has at least one summand isomorphic to $\underline{E}(k)$, and the lowest cohomological degree in which this occurs is $n - \operatorname{proj.dim}(N)$ by Theorem 4.5.5. Since $\underline{E}(k)$ has codimension n, we have $\operatorname{rise}(\mathbb{I}) \geq n - (n - \operatorname{proj.dim}(N)) = \operatorname{proj.dim}(N)$. Choose now a summand $\underline{E}(S/\mathfrak{m}^F)[\mathbf{b}]$ of $\mathbb{I}$ in cohomological degree i such that $\operatorname{rise}(\mathbb{I}) = |F| - i$. Applying Proposition 3.29 and subsequently Theorem 4.5.5 over $S_{[F]}$, we find that $|F| - i \leq \operatorname{proj.dim}_{S_{[F]}}(N_{(F)})$, and this is obviously $\leq \operatorname{proj.dim}(N)$. Thus $\operatorname{rise}(\mathbb{I}) = \operatorname{proj.dim}(N)$. The statement about Cohen-Macaulayness follows because $\operatorname{codim}(N) = \operatorname{codim}(I^0)$. □

Example 4.24 Let J be a generic monomial ideal. The injective resolution $\mathbb{I}$ of S/J is given by the boundary map on a triangulation Γ of a simplex on $x_1, \ldots, x_n$ (Figure 1, right). The dimension of the indecomposable summand corresponding to a face $G \in \Gamma$ is the number of $x_i \in G$. Since the boundary map on Γ removes one vertex at a time, $\operatorname{elev}(I^{j+1}) = \min\{n, \operatorname{elev}(I^j) + 1\}$. Thus $\operatorname{rise}(\mathbb{I}) = \operatorname{elev}(I^0)$, and $\operatorname{proj.dim}(S/J)$ is the largest codimension of an associated prime of S/J; see [Yan1] or [MSY] for stronger results of this type. In particular, S/J is Cohen-Macaulay if and only if it is equidimensional [BPS]. □

In view of Proposition 4.23, why introduce the notion of rise? In the proof of Theorem 4.20, it will be seen that if $\mathbb{I}$ is a minimal injective resolution of $M \in \mathcal{M}_+^{\mathbf{a}}$, the minimal free resolution of $M^{\mathbf{a}}$ comes not directly from $\mathbb{I}$, but from $\check{C}\,\mathbb{I}$. To use this, we first need to know that $\check{C}\,\mathbb{I}$ is a quotient of $\mathbb{I}$ which is still a complex of injective modules.

Lemma 4.25 *If N is an indecomposable injective, then $\check{C}N \cong N$ if the $\mathbb{Z}^n$-graded component $N_{\mathbf{0}}$ is nonzero; otherwise, $\check{C}N = 0$. Taking Čech hull of an injective module commutes with localization: $(\check{C}N)_{(F)} \cong \check{C}(N_{(F)})$, where the second Čech hull is over $S_{[F]}$.*

Proof: The first statement is immediate from the definitions (and is Matlis dual to the last statement of Lemma 4.1). The second statement is by Lemma 3.28 and Proposition 3.29. □

Proposition 4.26 $\mathrm{rise}(\mathbb{I}) = \mathrm{rise}(\check{C}\,\mathbb{I})$ *if $\mathbb{I}$ is a minimal injective resolution of $M \in \mathcal{M}_{+}^{\mathbf{a}}$.*

Proof: Choose a summand $\underline{E}(S/\mathfrak{m}^F)[-\mathbf{b}]$ of $\mathbb{I}$ in minimal cohomological degree i (and therefore with minimal F) such that $\mathrm{rise}(\mathbb{I}) = |F| - i$. By Lemma 4.25 and Proposition 3.29 $\mathrm{rise}(\mathbb{I}_{(F)}) = \mathrm{rise}(\mathbb{I}) \geq \mathrm{rise}(\check{C}\,\mathbb{I}) \geq \mathrm{rise}(\check{C}\,\mathbb{I}_{(F)})$. Therefore, by localizing at $\mathfrak{m}^F$ if necessary, we assume $F = \mathbf{1}$ and $\mathbf{b} \in \mathbb{Z}^n$. It is enough to show that $\mathbf{b}$ can be chosen $\succeq \mathbf{0}$, for then $\underline{E}(k)[-\mathbf{b}]$ survives the Čech hull. This is, by Theorem 4.5.5′ and the minimality of i, equivalent to showing that the minimal free resolution of M has a summand $S[-\mathbf{c}]$ in homological degree $\mathrm{proj.dim}(M)$ with $\mathbf{1} \preceq \mathbf{c} := \mathbf{b} + \mathbf{1}$.

Let $d = \mathrm{proj.dim}(M)$. The assumptions above have reduced to the case where $\mathrm{elev}(I^j) < j + d$ whenever $\mathrm{elev}(I^j) < n$. Using Propositions 3.29 and 4.23, this implies that $d > \mathrm{proj.dim}_{S_{[F]}}(M_{[F]})$ whenever $\mathbf{0} \preceq F \prec \mathbf{1}$. It follows $\underline{\mathrm{Ext}}_S^d(M, S)$ has support on $\mathfrak{m}$, because its localization at every nonmaximal associated prime is zero (every associated prime is monomial). Suppose e is a generator for a summand $S[-\mathbf{c}]$ in homological degree d of the minimal free resolution $\mathbb{F}$ of M. Then the image in $\underline{\mathrm{Ext}}_S^d(M, S)$ of the dual basis vector e^* (for any basis) is nonzero and annihilated by some power of $\mathfrak{m}$. If $(m_1, \ldots, m_r)$ is the image of e (in some basis) under the differential of $\mathbb{F}$, then the annihilator of e^* is contained in the ideal generated by the m_i. In particular, every variable appears in at least one of the monomials $m_1, \ldots, m_r$. But M is $\mathbb{N}^n$-graded, so $\mathbb{F}$ is, as well, whence $\mathbf{x}^{\mathbf{c}}$ must be divisible by the least common multiple of the m_i. □

Proof of Theorem 4.20: Let $\mathbb{I}$ be a minimal injective resolution of $M \in \mathcal{M}_{+}^{\mathbf{a}}$ and $\mathbb{F} = P_{\mathbf{a}}(A_{\mathbf{a}}^{0,+}\mathbb{I})$ the minimal free resolution of $M^{\mathbf{a}}$ induced by Alexander duality (Theorem 4.5.1 and 4.5.4). Suppose Λ is a monomial matrix for $\mathbb{I}$ and let $\Phi^{\preceq\mathbf{a}}$ be the induced matrix for $\mathbb{F}$ (by applying Theorem 4.5.1′ and 4.5.4′). The submatrix $\check{\Lambda}$ of Λ whose shifts do not have any coordinates equal to 1 is the induced matrix for $\check{C}\,\mathbb{I}$ by Lemma 4.25. Therefore, applying the relabelling function in Theorem 4.5.1′ to $\check{\Lambda}$ yields the same monomial matrix $\Phi^{\preceq\mathbf{a}}$. It follows that the support of the shift of a summand in $\mathbb{F}$ is the support of the integer part of the shift of the corresponding summand in $\check{C}\,\mathbb{I}$. Since the support size of the integer part is the codimension, the theorem follows from Propositions 4.23 and 4.26. □

Proposition 4.26 says that the elevations in a minimal injective resolution $\mathbb{I}$ of a well-behaved (e.g. Cohen-Macaulay) module do not rise too quickly. However, it does *not* rule out the possibility of some nonzero component λ_{pq} in $\mathbb{I}$ mapping a low-codimension summand to a high-codimension summand. Let's make this precise.

Definition 4.27 *The* jump *of a homomorphism $\lambda : I \to I'$ of indecomposable injectives is* $\mathrm{jump}(\lambda) = \mathrm{codim}(I') - \mathrm{codim}(I)$ *if $\lambda \neq 0$. If $\lambda = 0$ then set* $\mathrm{jump}(\lambda) = -1$.

For instance, Corollary 4.15 says that the components of $\mathbb{I}$ having jump zero are reasonably well-understood, inasmuch as free resolutions have been studied thoroughly in the literature. But what about the components of jump > 0?

Question 4.28 What determines the jumps in components of a minimal injective resolution of a finitely generated $\mathbb{Z}^n$-graded module?

The question makes sense also for $\mathbb{Z}$-graded or non-graded modules. But for $\mathbb{Z}^n$-graded modules we already have ways of analyzing some of the jumps in the minimal injective resolution $\mathbb{I}$. For example, suppose that $M \in \mathcal{M}_+^{\mathbf{1}}$ is squarefree with matrix Λ representing $\mathbb{I}$. Lemma 4.7 says that all of the shifts in Λ are in $\{1, 0, *\}^n$. Furthermore, Lemma 4.25 says that $\check{C}\,\mathbb{I}$ is the $\mathbf{1}$-determined part of $\mathbb{I}$, taking the submatrix with shifts in $\{0, *\}^n$. Examining the proof of Theorem 4.20, the jump between any two such summands is the ($\mathbb{Z}$-graded) degree of the corresponding map in the dual free resolution of $M^{\mathbf{1}}$ from Theorem 4.5.2. Therefore, if M is Cohen-Macaulay, the jumps between $\mathbf{1}$-determined summands in Λ are all ≤ 1 by Corollary 4.21. It is not true, however, that all of the jumps in Λ are ≤ 1, even if M is Cohen-Macaulay.

Example 4.29 A counterexample is given by $M = k[t, x, y, z]/\langle xyz, tx, ty\rangle$, which is Cohen-Macaulay, but whose minimal injective resolution has a component Λ_{pq} in a matrix for $I^1 \to I^2$ with $\mathbf{b}_{p\cdot} = (0, *, *, 1)$ and $\mathbf{b}_{\cdot q} = (0, 0, 0, 1)$. We have $\text{jump}(\Lambda_{pq}) = 4 - 2 = 2$. Of course, this calculation can be (and was) done by computer [CoCoA], using Theorem 4.5. □

5 Duality for Bass and Betti numbers

One of the features in the recent literature on Alexander duality for Stanley-Reisner rings is a collection of equalities and inequalities between Betti numbers of an ideal its dual; e.g. see [BCP]. The proofs always involved calculating the Betti numbers as Betti numbers of certain simplicial complexes and getting relations among them using topological methods. Thus these proofs worked by comparing some homology or cohomology. More recently, Bass numbers have been brought into play [Mil] to express the duality, in the process eliminating the dependence on topological data to compute the numbers. But the methods relied on $\underline{\text{Tor}}$ and $\underline{\text{Ext}}$ modules, i.e. still on some kind of homology. The aim here is to show how all of the relations between Betti and Bass numbers are really residual data from duality for resolutions. In particular, each of the equalities below follows from the equivalence of some pair of *complexes*, so that, of course, the numerical information must be the same.

The following definition of Bass numbers at monomial primes agrees with the definition [Mil, Definition 4.8] in terms of certain $\underline{\text{Ext}}$ functors. These Bass numbers also agree with the usual (ungraded) Bass numbers at $\mathfrak{m}^F$ and determine the ungraded Bass numbers at all primes; see [GW, Theorems 1.2.3 and 1.3.4].

Definition 5.1 *Suppose* $0 \to I^0 \to I^1 \to \cdots$ *is a minimal injective resolution of* $M \in \mathcal{M}$. *The* i^{th} Bass number at F (or $\mathfrak{m}^F$) in degree $\mathbf{b} \cdot F$ *is the number* $\mu_{i,\mathbf{b}\cdot F}(F, M)$ *of summands of* I^i *that are isomorphic to* $\underline{E}(S/\mathfrak{m}^F)[-\mathbf{b}]$. *Set* $\mu_{i,\mathbf{b}}(M) = \mu_{i,\mathbf{b}\cdot\mathbf{1}}(\mathbf{1}, M)$, *the Bass number of* M *at* $\mathfrak{m}$ *in degree* $\mathbf{b}$.

Note that $\underline{E}(S/\mathfrak{m}^F)[-\mathbf{b}]$ corresponds to a shift $-\mathbf{b}$ (not $\mathbf{b}$) in a monomial matrix.

5.1 $\mathbb{N}^n$-graded modules

The next corollary is a first indication of how duality for resolutions "explains" equalities between Bass and Betti numbers. Though it is not new, it will be needed in Theorem 5.3.

Corollary 5.2 *For any finitely generated $M \in \mathcal{M}$, $\beta_{n-i,\mathbf{b}}(M) = \mu_{i,\mathbf{b}-\mathbf{1}}(M)$ for all $i \in \mathbb{Z}$ and $\mathbf{b} \in \mathbb{Z}^n$.*

Proof: Shifting degrees and choosing $\mathbf{a}$ large enough, we can assume that M is positively $\mathbf{a}$-determined. The result now follows from Theorem 4.5.5 and 4.5.5′. □

The next theorem is the main result of the section, and generalizes [Mil, Theorem 4.10], which was stated for ideals, to the case of arbitrary positively $\mathbf{a}$-determined modules. Note that $F = \mathbf{0}$ is allowed here; it was barred from [Mil, Theorem 4.10] because S/I is on the left hand side of part (ii) there. If $\mathrm{supp}(\mathbf{b}) = F$, the notation $\mathbf{b}^{\mathbf{a}} := (\mathbf{a} \cdot F) - \mathbf{b} + F$ is chosen so that $\langle \mathbf{x}^{\mathbf{b}} \rangle^{[\mathbf{a}]} = \langle x_i^{(\mathbf{b}^{\mathbf{a}})_i} \mid b_i \neq 0 \rangle$ as in Example 2.4. It may be easier to parse the definition of $\mathbf{b}^{\mathbf{a}}$ by observing that $\mathbf{b}^{\mathbf{a}} - F = \mathbf{a} \cdot F - \mathbf{b}$.

Theorem 5.3 *Suppose that $\mathbf{0} \preceq F \preceq \mathbf{b} \preceq \mathbf{a} \cdot F$, i.e. $F = \mathrm{supp}(\mathbf{b})$. Then:*

1. $\beta_{i,\mathbf{b}}(M^{\mathbf{a}}) = \mu_{i,\mathbf{b}^{\mathbf{a}}-F}(F, M)$,
2. $\mu_{n-i,\mathbf{b}-\mathbf{1}}(M^{\mathbf{a}}) = \mu_{i,\mathbf{b}^{\mathbf{a}}-F}(F, M)$, and
3. $\beta_{i,\mathbf{b}}(M^{\mathbf{a}}) = \beta_{|F|-i,\,\mathbf{b}^{\mathbf{a}}}(M_{(F)})$.

Of course, the pair $(M, M^{\mathbf{a}})$ can be switched, as can $(\mathbf{b}, \mathbf{b}^{\mathbf{a}})$, $(i, n-i)$, and $(i, |F|-i)$.

Proof: Suppose Λ represents a minimal injective resolution of M. Let Φ be the matrix output from Theorem 4.5.1′. Each shift $\mathbf{b}$ in the matrix $\Phi^{\preceq\mathbf{a}}$ from Theorem 4.5.4′ which satisfies $F \preceq \mathbf{b} \preceq \mathbf{a} \cdot F$ comes from a shift $*\overline{F} + (\mathbf{a} \cdot F) - \mathbf{b}$ in Λ by Theorem 4.5.2′, proving part 1. Part 2 follows from part 1 by Corollary 5.2. Finally, the right-hand sides of parts 1 and 3 are equal by the matrix part of Corollary 4.15, so part 3 follows from part 1. □

Remark 5.4 There is an even more comprehensive equality like Theorem 5.3.2 involving *every* Bass number of both M and $M^{\mathbf{a}}$. It comes from the equivalence of Corollary 4.9, and is best read off the relabelling function given there.

Example 5.5 The previous theorem and remark are so transparent for generic and cogeneric monomial ideals that almost nothing is going on. The only thing we're doing is taking the same faces of a simplicial complex (Figures 1 and 2) and putting different $(\mathbb{Z} \cup \{*\})^n$ labels on them with different homological degrees, sometimes with β, and sometimes with μ. A similar observation holds for any minimal cellular or cocellular injective resolution. □

Theorem 5.3 can be used to prove the following Betti-Betti inequality theorem. The version [Mil, Theorem 4.13] for monomial ideals was the key ingredient for the first proof of [MSY, Theorem 4.12].

Theorem 5.6 *If* $\mathbf{0} \preceq F \preceq \mathbf{b} \preceq \mathbf{a} \cdot F$ *then*

$$\beta_{i,\mathbf{b}}(M^{\mathbf{a}}) \quad \leq \quad \sum_{\mathbf{c} \cdot F = \mathbf{b}^{\mathbf{a}}} \beta_{|F|-i,\mathbf{c}}(M).$$

Of course, M *and* $M^{\mathbf{a}}$ *can be switched, as can* $\mathbf{b}$ *and* $\mathbf{b}^{\mathbf{a}}$*, or* i *and* $|F| - i$*.*

Proof: Since $\mathbf{b}^{\mathbf{a}} \cdot F = \mathbf{b}^{\mathbf{a}}$, the sum on the right is $\geq \beta_{|F|-i,\mathbf{b}^{\mathbf{a}}}(M_{(F)})$ by Proposition 3.31, with equality if and only if the matrix $\Lambda_{(F)}$ there is minimal. Now use Theorem 5.3.3. □

5.2 Squarefree modules

As in [Mil, Corollary 4.14], the case $\mathbf{a} = \mathbf{1}$ of Theorem 5.6 is interesting. This time, however, the result is new, generalizing [BCP, Corollary 2.6] to arbitrary squarefree modules.

Corollary 5.7 *Let* $M \in \mathcal{M}_+^{\mathbf{1}}$ *be a squarefree module. Then*

$$\beta_{i,\mathbf{b}}(M^{\mathbf{1}}) \quad \leq \quad \sum_{\mathbf{b} \preceq \mathbf{c} \preceq \mathbf{1}} \beta_{|\mathbf{b}|-i,\mathbf{c}}(M).$$

Of course, M *and* $M^{\mathbf{1}}$ *can be switched, as can* i *and* $|\mathbf{b}| - i$*.*

Proof: For $\mathbf{0} \preceq \mathbf{b} \preceq \mathbf{1}$, $\mathbf{b}^{\mathbf{1}} = \mathbf{b} = \text{supp}(\mathbf{b})$. Furthermore, $\mathbf{c} \cdot \mathbf{b} = \mathbf{b}^{\mathbf{1}}$ implies that $\mathbf{c} \succeq \mathbf{b}$. Finally, $\beta_{|\mathbf{b}|-i,\mathbf{c}}(M) = 0$ unless $\mathbf{c} \preceq \mathbf{1}$ by Corollary 4.6. □

Following [BCP], again, Corollary 5.7 implies that certain Betti numbers of a squarefree module M are equal to those of its Alexander dual $M^{\mathbf{1}}$. To state the result, the following definition is required.

Definition 5.8 *Let* $M \in \mathcal{M}_+^{\mathbf{1}}$ *be a squarefree module. A Betti number* $\beta_{i,\mathbf{b}}(M)$ *is called* i-extremal *if* $\beta_{i,\mathbf{c}}(M) = 0$ *for* $\mathbf{c} \succ \mathbf{b}$*. Define* $\beta_{i,\mathbf{b}}(M)$ *to be* extremal *if* $\beta_{j,\mathbf{c}}(M) = 0$ *whenever* $j \geq i$*,* $\mathbf{c} \succ \mathbf{b}$*, and* $|\mathbf{c}| - |\mathbf{b}| \geq j - i$*, where* $|\mathbf{b}| := \sum_{k=1}^{n} b_k$ *for* $\mathbf{b} \in \mathbb{N}^n$*.*

Proposition 5.9 *Let* $M \in \mathcal{M}_+^{\mathbf{1}}$ *be squarefree. If* $\beta_{i,\mathbf{b}}(M^{\mathbf{1}})$ *is i-extremal, then* $\beta_{i,\mathbf{b}}(M^{\mathbf{1}}) \geq \beta_{|\mathbf{b}|-i,\mathbf{b}}(M)$*. If, furthermore,* $\beta_{i,\mathbf{b}}(M^{\mathbf{1}})$ *is extremal, then* $\beta_{i,\mathbf{b}}(M^{\mathbf{1}}) = \beta_{|\mathbf{b}|-i,\mathbf{b}}(M)$*.*

Proof: The proof of [BCP, Theorem 2.8] works here, since they use only the special case [BCP, Corollary 2.6] of Corollary 5.7. □

This allows for another proof of Corollary 4.21. Indeed, the proof of [BCP, Corollary 2.9] again works here since they use only the special case [BCP, Theorem 2.8] of Proposition 5.9. Römer [Röm, Theorem 4.6] also proved the equality of extremal Betti numbers.

6 The canonical Čech complex and local duality

Alexander duality contains and is much stronger than local duality for $\mathbb{Z}^n$-graded modules. This is because local duality works on the level of cohomology, whereas the Alexander duality functors work on the level of resolutions. This strength is demonstrated by first extracting local duality from Theorem 4.5:

Corollary 6.1 (Local duality at the maximal ideal) *For any finitely generated module $M \in \mathcal{M}$, $H^i_{\mathfrak{m}}(M)^\vee \cong \underline{\mathrm{Ext}}_S^{n-i}(M, \omega_S)$, where $\omega_S = S[-\mathbf{1}]$ is the canonical module.*

Proof: Shifting $\mathbb{Z}^n$-degrees and choosing $\mathbf{a}$ large enough, we may assume that $M \in \mathcal{M}_+^{\mathbf{a}}$. Using the notation of Theorem 4.5, the matrix $\Lambda_{\mathbb{Z}}$ in Theorem 4.5.5′ represents simultaneously the cohomological complexes $H^0_{\mathfrak{m}}(\mathbb{I})$ and $\underline{\mathrm{Hom}}_S(\mathbb{F}', S[-\mathbf{1}])(n)$, where $\mathbb{F}' = \underline{\mathrm{Hom}}_S(\mathbb{I}^\vee, S)[\mathbf{1}]$ is a minimal free resolution of M. Lemma 3.18 says that the complexes $\underline{\mathrm{Hom}}_S(\mathbb{F}', S[-\mathbf{1}])(n)$ and $H^0_{\mathfrak{m}}(\mathbb{I})$ are therefore Matlis dual, so their homology is, as well. □

It should be noted that although local duality was used in the proofs of Proposition 4.4 and Theorem 4.5.3, it was only required for finite length modules, a case which is much simpler to prove than for arbitrary finitely generated modules.

The ultimate aim of this section is to prove, in Corollary 6.8, the generalization of local duality to the case of H^i_I for any squarefree monomial ideal I. Perhaps more importantly, though, is the *canonical Čech complex* of I introduced along the way in Theorem 6.2. This complex of flat modules bears the same relation to I as the usual Čech complex does to $\mathfrak{m}$; see Example 6.4. As an application of Theorem 6.2, the module structure of $H^r_I(S)$ and Terai's formula [Ter2] for its Hilbert series are obtained in Corollary 6.7 directly from Gräbe's module structure for $H^r_{\mathfrak{m}}(S/I)$ [Grä] and Hochster's formula for its Hilbert series, thus avoiding the calculation of simplicial cohomology as in [Ter2] and [Mus].

6.1 Canonical Čech complex

Throughout this section, $I \subset S$ will denote a fixed squarefree monomial ideal of height d.

Theorem 6.2 *Suppose that $\mathbb{F}$ is a positively $\mathbf{1}$-determined free resolution of S/I and $\mathbb{F}^* = \underline{\mathrm{Hom}}_S(\mathbb{F}, S)$ is the free transpose of $\mathbb{F}$ (with the cohomological grading induced by the homological grading of $\mathbb{F}$). If $M \in \mathcal{M}$ then*

$$H^i_I(M) \quad \cong \quad H^i(M \otimes_S \check{C}(\mathbb{F}^* \otimes \omega_S)[\mathbf{1}]).$$

In particular, if S/I is Cohen-Macaulay then

$$H^i_I(M) \cong \underline{\mathrm{Tor}}^S_{d-i}(M, \check{C}(\omega_{S/I}))[\mathbf{1}]\,.$$

When $\mathbb{F}$ is minimal, the complex $\check{C}(\mathbb{F}^* \otimes \omega_S)[\mathbf{1}]$ is called the canonical Čech complex *of I.*

Proof: Let Λ be a matrix for $\mathbb{F}$ with row shifts $\{\mathbf{b}_{p\cdot}\}$ and column shifts $\{\mathbf{b}_{\cdot q}\}$. Lemmas 3.24 and 3.25 say that the induced matrix for $\mathbb{F}^* \otimes \omega_S$ has row shifts $\{\overline{\mathbf{b}_{\cdot q}}\}$ and column shifts $\{\overline{\mathbf{b}_{p\cdot}}\}$ (recall that $\overline{\mathbf{b}} = \mathbf{1} - \mathbf{b}$ if $\mathbf{0} \preceq \mathbf{b} \preceq \mathbf{1}$). The complex $\check{C}(\mathbb{F}^* \otimes \omega_S)$ is the direct limit (union) over all $t \in \mathbb{N}$ of the free complexes $\mathbb{L}_t := \check{C}(\mathbb{F}^* \otimes \omega_S)_{\succeq -t\cdot\mathbf{1}}$ (recall the definition of $(-)_{\succeq \mathbf{b}}$ from Section 4.1). Now $\check{C}(S[-\overline{F}]) \cong S[\mathbf{x}^{-F}][-\overline{F} + *F]$ whenever $F \subseteq \{1, \ldots, n\}$ by Example 2.8.1, and there is an easy isomorphism $S[\mathbf{x}^{-F}][-\overline{F} + *F]_{\succeq -t\cdot\mathbf{1}} \cong S[-\overline{F} + t \cdot F]$. It follows that the induced matrix for $\mathbb{L}_t$ has row shifts $\{\overline{\mathbf{b}_{\cdot q}} - t\,\mathbf{b}_{\cdot q}\}$ and column shifts $\{\overline{\mathbf{b}_{p\cdot}} - t\,\mathbf{b}_{p\cdot}\}$.

If I is has generators $m_1, \ldots, m_r$, then the t^{th} *Frobenius power* of I for $t \in \mathbb{N}$ is the ideal $I^{[t]} = \langle m_1^t, \ldots, m_r^t \rangle$. The resolution $\mathbb{F}$ of S/I induces a resolution $\mathbb{F}_{[t]}$ of $S/I^{[t]}$ which is obtained from $\mathbb{F}$ by substituting x_s^t for every occurrence of the variable x_s. The effect on the matrix Λ for $\mathbb{F}$ is to multiply all of the shifts of Λ by t. This, in turn, induces a matrix on $\mathbb{F}^*_{[t]}[-\mathbf{1}] := \underline{\mathrm{Hom}}_S(\mathbb{F}_{[t]}, S[-\mathbf{1}])$ whose row shifts are $\{\mathbf{1} - t\,\mathbf{b}_{\cdot q}\}$ and whose column shifts are $\{\mathbf{1} - t\,\mathbf{b}_{p\cdot}\}$ by Lemmas 3.24 and 3.25. Comparing this with the matrix for $\mathbb{L}_t$ above, we conclude that $\mathbb{F}^*_{[t]}[-\mathbf{1}] \cong \mathbb{L}_{t-1}$.

The free complex $\mathbb{F}^*_{[t]}[-\mathbf{1}]$ has homology $\underline{\mathrm{Ext}}^{\cdot}_S(S/I^{[t]}, S[-\mathbf{1}])$, so we can calculate

$$
\begin{aligned}
H^i_I(M)[-\mathbf{1}] &\cong \varinjlim_t \underline{\mathrm{Ext}}^i_S(S/I^{[t]}, M[-\mathbf{1}]) \\
&\cong \varinjlim_t H^i(\mathbb{F}^*_{[t]}[-\mathbf{1}] \otimes_S M) \\
&\cong \varinjlim_t H^i(\mathbb{L}_{t-1} \otimes_S M) \\
&\cong H^i \varinjlim_t (\mathbb{L}_{t-1} \otimes_S M) \\
&\cong H^i((\varinjlim_t \mathbb{L}_{t-1}) \otimes_S M) \\
&\cong H^i(\check{C}(\mathbb{F}^* \otimes \omega_S) \otimes_S M)\,,
\end{aligned}
$$

proving the first equation. The remaining statement holds because $\mathbb{F}^* \otimes \omega_S$ is a free resolution of $\omega_{S/I}$ when S/I is Cohen-Macaulay, and because the Čech hull, which takes free modules to flat modules by Lemma 4.1, is exact. In particular, $\check{C}(\mathbb{F}^* \otimes \omega_S)$ is a flat resolution of $\check{C}(\omega_{S/I})$, and can be tensored with M to calculate the $\underline{\mathrm{Tor}}$ module in question. □

The canonical Čech complex of I depends (up to isomorphism of $\mathbb{Z}^n$-graded complexes) only on I, not on any system of generators of I. Caution is warranted, however, because the canonical Čech complex does *not* depend only on the Stanley-Reisner complex of I. For instance, the minimal free resolution of S/I depends on the characteristic of S.

Remark 6.3 The theorem is still true without assuming $\mathbb{F}$ is positively $\mathbf{1}$-determined, but the notation in the proof becomes worse if Λ has shifts which are not in $\{0,1\}^n$.

Example 6.4 The canonical Čech complex of $\mathfrak{m}$ is the usual Čech complex. Since the Koszul complex is self-dual the Cech hull of the Koszul complex is the Čech complex shifted by $[-\mathbf{1}]$ (see Example 3.4). More generally, the case where $\mathbb{F}.$ is the *Taylor complex* of I is the result of applying direct limits to [Mus, Theorem 1.1]. There, limits were applied to the homology of $\mathbb{F}^*$, but not to the complex itself.

This example is the reason for the term "Čech hull". The Čech hull appeared in [Mil] before Theorem 6.2 was known, although the use of the term there was to indicate that it transformed merely the homology of the Koszul complex into the homology of the Čech complex, as opposed to transforming the complexes themselves. □

Remark 6.5 The canonical Čech complex calculates local cohomology in any grading (including the non-grading) for which the variables are homogeneous. This is because no properties of M were used in the proof of Theorem 6.2 and the limit works just as well in any grading. For instance, gradings by quotients of $\mathbb{Z}^n$ appear in calculations of sheaf cohomology on toric varieties [EMS]. See Example 6.6 for the canonical Čech complex in the case of a smooth projective toric variety.

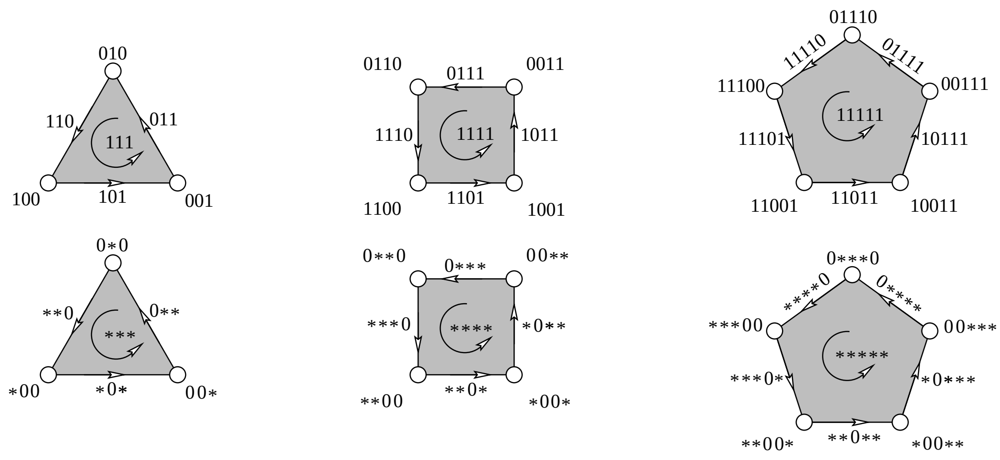

Figure 3: Some cellular resolutions and cocellular canonical Čech complexes.

Example 6.6 The triangle, square, and pentagon in Figure 3 can be regarded as polyhedral cell complexes with orientations as shown. When labelled as in the top diagrams, these respectively determine cellular minimal free resolutions of

$$S/\langle a,b,c\rangle, \quad S/\langle ab,bc,cd,ad\rangle, \quad \text{and} \quad S/\langle abc,bcd,cde,ade,abe\rangle$$

for appropriate S. The associated canonical Čech complexes are depicted in the bottom diagrams, which determine *cocellular* monomial matrices (Proposition 3.10). The empty set is labelled by $0\cdots 0$ in all six pictures. Therefore, if $I = \langle m_1,\ldots,m_r\rangle$, the first map in the canonical Čech complex is $S \to \bigoplus_{i=1}^r S[m_i^{-1}]$, just like the usual Čech complex. Observe that the bottom triangle actually is the usual Čech complex, whereas the top triangle is the Koszul complex (see Examples 3.4 and 6.4).

This example works more generally for the *irrelevant ideal* I of the *Cox homogeneous coordinate ring* of a smooth projective toric variety V (see [Ful] and [Cox] for the relevant background). One is particularly interested in computing local cohomology with support on I in this case, because it is important for sheaf cohomology on V [EMS]. The minimal free resolution of I [MPS] is a linear cellular resolution supported on the moment polytope P of V, which is simple. Interestingly, this resolution can also be viewed as being cocellular, supported on the simplicial polytope Q polar to P. Therefore, the canonical Čech complex can be regarded either as a cellular or a cocellular flat complex. This is particularly evident for the cases above, where the polar polytope can be inscribed. □

6.2 Applications of the canonical Čech complex

Recall the famous theorem of Hochster [Sta, Theorem II.4.1] which gives the $\mathbb{Z}^n$-graded Hilbert series $F(H^r_{\mathfrak{m}}(S/I),\lambda)$ for the local cohomology of S/I at $\mathfrak{m}$ in terms of the homology of the links (Equation (9)) in the Stanley-Reisner complex Δ:

$$F(H^r_{\mathfrak{m}}(S/I),\lambda) \quad = \quad \sum_{F\in\Delta} \dim_k \widetilde{H}_{r-|F|-1}(\mathrm{lk}_\Delta F;k) \prod_{j\in F} \frac{\lambda_j^{-1}}{1-\lambda_j^{-1}}\,, \qquad (10)$$

where $\widetilde{H}$ is reduced homology. Recently, Terai [Ter2] wrote down the Hilbert series of the modules $H^i_I(S)$, inspired by the above formula. His proof used methods similar to those used in proving Hochster's formula, exploiting simplicial complexes appearing in various $\mathbb{Z}^n$-graded degrees of the usual Čech complex. Terai used his formula to answer, in the special case of squarefree monomial ideals, some questions of Huneke [Hun] concerning when local cohomology with support on an ideal is zero, finitely generated, or artinian. In Corollary 6.7, Terai's formula is shown to be equivalent to Equation (10), each being derived from the other via Theorem 6.2.

Hochster's formula gives only the dimensions of the $\mathbb{Z}^n$-graded degrees of $H^r_{\mathfrak{m}}(S/I)$. But a paper of Gräbe [Grä] (which seems to have been overlooked by many recent studies) gives the module structure of $H^r_{\mathfrak{m}}(S/I)$ in terms of simplicial homology. On the other hand, [Mus] and [Yan2] (which appeared more or less simultaneously with [Ter2]) used simplicial methods to obtain results on the $\mathbb{Z}^n$-graded degrees and module structure of $H^r_I(S)$, including the fact that $H^r_I(S)[-\mathbf{1}]$ is $\mathbf{1}$-determined. These structure results, which include much of [Mus, Section 2] and [Yan2, Section 3], follow from the next corollary applied to the structure results of [Grä, Section 2]. The moral of the story is that there is really only one set of simplicial (co)homology floating around, and all of the objects whose structure can be expressed in terms of it are equivalent in one way or another via a generalized Alexander duality.

Corollary 6.7 *The module $H^{n-r}_{\mathfrak{m}}(S/I)[-\mathbf{1}]$ is negatively $\mathbf{1}$-determined, and its Alexander dual in $\mathcal{M}^{\mathbf{1}}$ is $H^r_I(S)[-\mathbf{1}]$. Equivalently,*

$$H^r_I(S) \quad\cong\quad \check{C}(H^{n-r}_{\mathfrak{m}}(S/I)^\vee)[\mathbf{1}] \quad\cong\quad \check{C}(\underline{\mathrm{Ext}}^r_S(S/I,\omega_S))[\mathbf{1}]\,.$$

It follows that the $\mathbb{Z}^n$-graded Hilbert series of $H^r_I(S)$ is given by [Ter2]:

$$F(H^r_I(S),\lambda) \quad=\quad \sum_{F\in\Delta} \dim_k \widetilde{H}_{n-r-|F|-1}(\mathrm{lk}_\Delta F;k) \prod_{i\in\overline{F}} \frac{\lambda_i^{-1}}{1-\lambda_i^{-1}} \prod_{j\in F} \frac{1}{1-\lambda_j}\,.$$

Proof: The cohomology of the complex $\mathbb{F}^*\otimes\omega_S$ of Theorem 6.2 is $\underline{\mathrm{Ext}}^r_S(S/I,\omega_S)$. Exactness of $\check{C}(-)[\mathbf{1}]$ implies that it commutes with taking cohomology, so setting $M=S$ in Theorem 6.2 yields $H^r_I(S) \cong \check{C}(\underline{\mathrm{Ext}}^r_S(S/I,\omega_S))[\mathbf{1}]$. The remaining isomorphism in the first equation follows from local duality (Corollary 6.1). The equivalence of the first statement with the first equation is by definition of the appropriate generalized Alexander duality.

To prove the formula for the Hilbert series, start with Equation (10). Now replace r by $n-r$ and λ_j by λ_j^{-1}, yielding the Hilbert series of the $\mathbb{N}^n$-graded module $H^{n-r}_{\mathfrak{m}}(S/I)^\vee$:

$$F(H^{n-r}_{\mathfrak{m}}(S/I)^\vee,\lambda) \quad=\quad \sum_{F\in\Delta} \dim_k \widetilde{H}_{n-r-|F|-1}(\mathrm{lk}_\Delta F;k) \prod_{j\in F} \frac{\lambda_j}{1-\lambda_j}\,.$$

As a $\mathbb{Z}^n$-graded k-vector space, the Čech hull of an $\mathbb{N}^n$-graded module is obtained by tensoring the graded piece in degree $\mathbf{b}$ with the inverse polynomial ring $S[x_i^{-1} \mid i \notin \mathrm{supp}(\mathbf{b})]$. This yields

$$F(\check{C}(H^{n-r}_{\mathfrak{m}}(S/I)^\vee),\lambda) \quad=\quad \sum_{F\in\Delta} \dim_k \widetilde{H}_{n-r-|F|-1}(\mathrm{lk}_\Delta F;k) \prod_{j\in F} \frac{\lambda_j}{1-\lambda_j} \prod_{i\in\overline{F}} \frac{1}{1-\lambda_i^{-1}}\,.$$

Shifting the input module by $[\mathbf{1}]$ multiplies this whole expression by $\lambda_1^{-1}\cdots\lambda_n^{-1} = \lambda^{-F}\lambda^{-\overline{F}}$, and gives the Hilbert series of $H_I^r(S)$ by the first equation in the corollary. □

The duality part of the corollary can also be read as

$$H_I^r(S)^\vee[\mathbf{1}] \quad\cong\quad \check{C}(H_{\mathfrak{m}}^{n-r}(S/I)^\vee)^\vee \quad\cong\quad P_{\mathbf{1}}(H_{\mathfrak{m}}^{n-r}(S/I)[-\mathbf{1}]).$$

In the case of a local ring $(R,\mathfrak{p})$, the Matlis dual of $H_I^r(R)$ agrees with the local cohomology at $\mathfrak{p}$ of the formal scheme obtained by completion of $\mathrm{Spec}(R)$ along $\mathrm{Spec}(R/I)$ [Ogus, Proposition 2.2]. Thus the positive extension functor $P_{\mathbf{1}}$ mimics, in the $\mathbb{Z}^n$-graded setting, the transition from local cohomology of $\mathrm{Spec}(R/I)$ to local cohomology of the formal completion, and the Čech hull is Matlis dual to this operation.

The final main result is the generalization of local duality to $H_I^i(-)$. Recall from Definition 2.15.2 that $A_{\mathbf{1}}^{+,0} : \mathcal{M}_+^{\mathbf{1}} \to \mathcal{M}^{\mathbf{1}}$ is the generalized Alexander duality functor $\check{C}(-)^\vee[-\mathbf{1}] : \mathcal{M}_+^{\mathbf{a}} \to \mathcal{M}^{\mathbf{a}}$.

Corollary 6.8 (Local duality with monomial support) *Let $\mathbb{F}$ be a positively $\mathbf{1}$-determined free resolution of S/I (for example, a minimal free resolution) and $\mathbb{F}^* = \underline{\mathrm{Hom}}_S(\mathbb{F}, S)$. For $M \in \mathcal{M}$,*

$$H_I^i(M)^\vee \quad\cong\quad H^i\underline{\mathrm{Hom}}_S(M, A_{\mathbf{1}}^{+,0}(\mathbb{F}^*\otimes\omega_S))\,.$$

If S/I is Cohen-Macaulay, then

$$H_I^i(M)^\vee \quad\cong\quad \underline{\mathrm{Ext}}_S^{d-i}(M, \check{C}(\omega_{S/I}^{\mathbf{1}})).$$

Proof: This is the Matlis dual of Theorem 6.2. □

To compare the statement of Corollary 6.8 with that of Corollary 6.1 for $I = \mathfrak{m}$, observe that $\omega_{S/\mathfrak{m}}$ is just k. Therefore, $\check{C}(\omega_{S/\mathfrak{m}}^{\mathbf{1}}) \cong \check{C}(k^{\mathbf{1}}) \cong \check{C}(S[-\mathbf{1}]) \cong S[-\mathbf{1}] \cong \omega_S$.

The first equation in Corollary 6.8 says that the complex $A_{\mathbf{1}}^{+,0}(\mathbb{F}^*\otimes\omega_S)$ of $\mathbb{Z}^n$-graded injective modules plays for $H_I^{\cdot}$ a role similar to that of a dualizing complex for $H_{\mathfrak{m}}^{\cdot}$. It would be very interesting to know whether there exists an analogous "Alexander dualizing complex" for more general ideals J. In particular, one would like to have an "Alexander dualizing module" analogous to $\check{C}(\omega_{S/I}^{\mathbf{1}})$ when S/J is Cohen-Macaulay. These existence questions are probably most natural in the context of $\mathbb{Z}$-graded rings, complete local rings, or proper schemes over a field.

The possible uses of the material in this section toward effective computation of local cohomology at monomial ideals have not been seriously explored, although Proposition 6.9, below, is one observation along these lines. Effective computation of local cohomology with monomial support is important for calculations of sheaf cohomology on toric varieties [EMS]. In particular, the interaction of local duality at monomial ideals with the finiteness conditions there should be investigated: what happens to local duality at squarefree monomial ideals when the grading is coarser than the $\mathbb{Z}^n$-grading?

The canonical Čech complex is usually a much smaller complex than the usual Čech complex; it is certainly shorter. The point is that it still consists of direct sums of localizations of (squarefree shifts of) S, so that:

Proposition 6.9 *If char$(k) = 0$, the canonical Čech complex is a complex of holonomic modules over the Weyl algebra.*

Proof: [Yan3, Remark 2.13]. □

Thus, the methods of Walther [Wal] for algorithmic computation of $H_I^{\cdot}(M)$ (even in the nongraded case, cf. Remark 6.5) might be useful.

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