A KLEIMAN–BERTINI THEOREM FOR SHEAF TENSOR PRODUCTS

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Abstract. Fix a variety X with a transitive (left) action by an algebraic group G. Let $\mathcal{E}$ and $\mathcal{F}$ be coherent sheaves on X. We prove that for elements g in a dense open subset of G, the sheaf $\mathcal{T}or_i^X(\mathcal{E}, g\mathcal{F})$ vanishes for all $i > 0$. When $\mathcal{E}$ and $\mathcal{F}$ are structure sheaves of smooth subschemes of X in characteristic zero, this follows from the Kleiman–Bertini theorem; our result has no smoothness hypotheses on the supports of $\mathcal{E}$ or $\mathcal{F}$, or hypotheses on the characteristic of the ground field.

All schemes in this note are of finite type over an arbitrary base field k. We make no restrictions on the characteristic of k. All groups G are assumed to be smooth over k. In particular, the local rings of $G \times_k k'$ are regular, for all field extensions k' of k. By a *transitive* action of G on X, we mean one such that $G \times X \to X \times X$ is scheme-theoretically surjective (this implies that X is reduced; see the Remark).

The aim of this note is to prove the following result.

Theorem. *Let X be a variety with a transitive left action of an algebraic group G. Let $\mathcal{E}$ and $\mathcal{F}$ be coherent sheaves on X and, for all k-rational points $g \in G$, let $g\mathcal{F}$ denote the pushforward of $\mathcal{F}$ along multiplication by g. Then there is a dense Zariski open subset U of G such that, for $g \in U$, the sheaf $\mathcal{T}or_i(\mathcal{E}, g\mathcal{F})$ is 0 for all $i > 0$.*

Remark. There are a number of subtleties that can be ignored for those working over the field of complex numbers.

1. Transitivity implies that the action is *geometrically transitive*, in the sense that, after base extending to an algebraic closure of k, any choice of closed point in X results in a scheme-theoretically surjective map $G \to X$. Hence X is geometrically reduced, and therefore reduced.
2. If k is infinite, and G is connected as well as affine, and either (i) G is reductive or (ii) k is perfect, then U always contains a k-rational point of G [Bor91, Corollary 18.3].
3. For the open set U that we construct, if k' is any extension of k and g is a k'-rational point of U, then $\mathcal{T}or_i^{X'}(\mathcal{E}, g\mathcal{F})$ will vanish as well, where $X' = X \times_k k'$ is the base extension to k'.

The theorem should be thought of as an analogue of the Kleiman–Bertini theorem [Kle74], which concerns the characteristic zero case in which $\mathcal{E} = \mathcal{O}_Y$ and $\mathcal{F} = \mathcal{O}_Z$ are structure sheaves of smooth subvarieties: Y meets gZ transversally for generic $g \in G$ in that situation, immediately implying that $\mathcal{E}$ and $g\mathcal{F}$ have no higher $\mathcal{T}or$. The conclusion that Y and Z meet transversally has many additional consequences; it

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implies, for example, that $Y \cap gZ$ is smooth, which is the better known conclusion of the Kleiman–Bertini theorem. In our situation, where $\mathcal{E}$ and $\mathcal{F}$ need not be structure sheaves, there is no analogous smoothness conclusion to draw. Our result shows that we may at least continue to disregard higher $\mathcal{T}or$ sheaves in this general setting.

There are many results which generalize the Kleiman–Bertini theorem by saying that for X as in our theorem, if Y and Z are subvarieties of X obeying various conditions on their singularities, then for generic $g \in G$ the intersection $Y \cap gZ$ obeys the same conditions (sometimes we must assume that the ground field has characteristic zero). The cases where the condition on singularities is reducedness (in characteristic zero), equidimensionality, local connectivity in codimension c, Serre's conditions R_d (in characteristic zero) or S_d, or normality can be obtained by methods similar to those of Spreafico [Spr98]. (Spreafico considers Kleiman–Bertini-like results in Section 5.2 of her paper. In order to avoid imposing Spreafico's condition (b), which is not true for many interesting examples such as grassmannians, one should remove her notion of generic geometric regularity and instead impose on P the axioms that (1) if A and B have property P then $A \times B$ has property P, and (2) if A has property P, and B is a regular domain with a morphism $A \to B$, then $A \times_B \overline{\eta}$ has property P, where $\overline{\eta}$ is a geometric generic point of B. It is checking this second property that introduces characteristic dependencies.) The condition of rationality of singularities is essentially Lemmas 1 and 2 of [Bri02]. These lemmas also provide an alternative proof when the condition on singularities is Cohen–Macaulayness, and it is straightforward to modify this argument to apply to Gorenstein singularities. In the same paper, Brion also proves the special case of our result where $\mathcal{E}$ and $\mathcal{F}$ are the structure sheaves of Cohen–Macaulay subschemes of X. We could not find in the literature a result dealing with sheaves that are not structure sheaves.

The main use we see for our result is in K-theory computations for homogeneous spaces. This is an active field of research, with many results both in pure geometry and in combinatorics; see [Bri05] for a good introduction to the geometric side of the theory and [Buc02, Las90] for a sample of the combinatorial side. The K-ring $K^\circ(X)$ of a smooth variety X can be defined additively to be the abelian group generated by the collection of coherent sheaves on X subject to the relations $[\mathcal{A}] + [\mathcal{C}] = [\mathcal{B}]$ whenever there is a short exact sequence $0 \to \mathcal{A} \to \mathcal{B} \to C \to 0$. The multiplication in $K^\circ(X)$ is induced by the tensor product; more precisely, it is given by

$$[\mathcal{E}][\mathcal{F}] = \sum_i (-1)^i [\mathcal{T}or_i^X(\mathcal{E}, \mathcal{F})]. \tag{1}$$

(The smoothness of X guarantees that this sum terminates.) Whenever G is a connected linear algebraic group acting transitively on X, the G-action on $K^\circ(X)$ is trivial, so we may compute $[\mathcal{E}][g\mathcal{F}]$ instead of $[\mathcal{E}][\mathcal{F}]$. Our result says that when g is chosen generically, only the $i = 0$ term in (1) can be nonzero. This is useful, as the tensor product is a far more familiar object than the higher $\mathcal{T}or$ sheaves. In particular, when $\mathcal{E} = \mathcal{O}_Y$ and $\mathcal{F} = \mathcal{O}_Z$ are structure sheaves for arbitrary closed subschemes of X, the tensor product $\mathcal{E} \otimes \mathcal{F}$ is simply the structure sheaf $\mathcal{O}_{Y \cap Z}$ of the scheme-theoretic intersection $Y \cap Z$.

Proof of Theorem

From now on, G, X, $\mathcal{E}$ and $\mathcal{F}$ have the meanings in the statement of our Theorem. Let Γ denote the subvariety of $X \times X \times G$ consisting of those points (x_1, x_2, g) such that $x_1 = gx_2$. Thus Γ is an isomorphic copy of $X \times G$ embedded in $X \times (X \times G)$ by the graph of the multiplication map $X \times G \to X$. We will write p_1, p_2, and q for the projections of Γ onto the two X factors and G, respectively.

Lemma 1. *The map $p_1 \times p_2 : \Gamma \to X \times X$ is flat.*

Proof. Extending the base field k, we assume X has a k-rational point, with stabilizer H. (Note that H need not be reduced; this causes no difficulty here or elsewhere.) By virtue of the transitivity of the group action, X is smooth, the map $\mu : G \to X$ is surjective, and all fibers of μ have the same dimension. By our standing assumption, G is smooth, so the map μ is faithfully flat by [Gro66, Proposition 15.4.2]. Base extending $p_1 \times p_2$ by $G \times G \to X \times X$, we need only prove that $G \times G \times_{X \times X} \Gamma \to G \times G$ is flat. Thus, if $\Delta = \{(g_1, g_2, g) : gg_2 \in g_1 H\} \subseteq G \times G \times G$, we need the projection p_{12} of Δ to the first two factors to be flat. But the automorphism of $G \times G \times G$ taking $(g_1, g_2, g) \mapsto (g_1, g_2, g_1^{-1} g g_2)$ commutes with p_{12} and takes Δ to $G \times G \times H$. □

Lemma 1 would be false without the reduced (no nilpotents) hypothesis on X, as can be seen by taking $X = \mathrm{Spec}(k[\varepsilon]/\varepsilon^2)$ and setting $G = \mathrm{Spec}(k)$, the trivial group.

Let $\mathcal{G}$ be the coherent sheaf $p_1^* \mathcal{E} \otimes p_2^* \mathcal{F}$ on Γ. Thus $\mathcal{G} = (p_1 \times p_2)^* (\mathcal{E} \boxtimes \mathcal{F})$, where $\boxtimes$ denotes the tensor product of the pullbacks to $X \times X$ along its two projections to X. For $g \in G(k)$, let $\iota^g : X \hookrightarrow \Gamma$ be the map $x \mapsto (x, g^{-1}x, g)$. Let k_g be the skyscraper sheaf on G concentrated at g. The heart of our proof is the following computation.

Proposition 2. *With the above notation, we have*

$$\iota^g_* \mathcal{T}or_i^X(\mathcal{E}, g\mathcal{F}) = \mathcal{T}or_i^\Gamma(q^* k_g, \mathcal{G}).$$

Proof. X is smooth, by virtue of the transitive group action, so X has enough locally frees (see Exercises 6.4, 6.8, and 6.9 in [Har77, Chapter III]). Let $0 \leftarrow \mathcal{K}_0 \leftarrow \mathcal{K}_1 \leftarrow \cdots$ and $0 \leftarrow \mathcal{L}_0 \leftarrow \mathcal{L}_1 \leftarrow \cdots$ be resolutions of $\mathcal{E}$ and $\mathcal{F}$ by locally free $\mathcal{O}_X$-modules. Then $\mathcal{C}_{i,j} = \mathcal{K}_i \boxtimes \mathcal{L}_j$, is a double complex on $X \times X$. The rows $\mathcal{C}_{\bullet,j}$ resolve $\mathcal{E} \boxtimes \mathcal{L}_j$, the columns $\mathcal{C}_{i,\bullet}$ resolve $\mathcal{K}_i \boxtimes \mathcal{F}$, and the total complex resolves $\mathcal{E} \boxtimes \mathcal{F}$. Let

$$\mathcal{D}_{i,j} = (p_1 \times p_2)^* (\mathcal{K}_i \boxtimes \mathcal{L}_j) = p_1^* \mathcal{K}_i \otimes p_2^* \mathcal{L}_j.$$

Because $p_1 \times p_2$ is a flat morphism, the rows, columns, and total complex of $\mathcal{D}_{\bullet,\bullet}$ resolve $p_1^* \mathcal{E} \otimes p_2^* \mathcal{L}_j$, $p_1^* \mathcal{K}_i \otimes p_2^* \mathcal{F}$, and $\mathcal{G} = p_1^* \mathcal{E} \otimes p_2^* \mathcal{F}$, respectively.

We claim that the rows of $\mathcal{D}_{\bullet,\bullet} \otimes q^* k_g$ can only have nonzero homology in homological degree 0, and that in row j this homology is $p_1^* \mathcal{E} \otimes p_2^* \mathcal{L}_j \otimes q^* k_g$. To see this, note that $p_1^* \mathcal{M} \otimes q^* k_g = \iota^g_* \mathcal{M}$ for all coherent sheaves $\mathcal{M}$ on X. Since ι^g is a closed embedding, ι^g_* is exact, so the complex $0 \leftarrow p_1^* \mathcal{K}_0 \otimes q^* k_g \leftarrow p_1^* \mathcal{K}_1 \otimes q^* k_g \leftarrow \cdots$ is a resolution of $\iota^g_* \mathcal{E} = p_1^* \mathcal{E} \otimes q^* k_g$. Tensoring this resolution with the locally free sheaf $p_2^* \mathcal{L}_j$ preserves exactness, resulting in a resolution of $p_1^* \mathcal{E} \otimes p_2^* \mathcal{L}_j \otimes q^* k_g$, as claimed.

Now, consider the horizontal homology of $\mathcal{D}_{\bullet,\bullet} \otimes q^* k_g$, which is concentrated in the zeroth column, as a complex under the vertical differential, namely

$$0 \leftarrow p_1^* \mathcal{E} \otimes p_2^* \mathcal{L}_0 \otimes q^* k_g \leftarrow p_1^* \mathcal{E} \otimes p_2^* \mathcal{L}_1 \otimes q^* k_g \leftarrow \cdots . \tag{2}$$

By an argument similar to that of the previous paragraph, we may rewrite (2) as the pushforward of a complex on X; namely, (2) is the image under ι^g_* of

$$0 \leftarrow \mathcal{E} \otimes g\mathcal{L}_0 \leftarrow \mathcal{E} \otimes g\mathcal{L}_1 \leftarrow \cdots, \tag{3}$$

where we have used that $p_2^*\mathcal{M} \otimes q^*k_g = \iota^g_*(g\mathcal{M})$ for all coherent sheaves $\mathcal{M}$ on X. The i^{th} homology of (3) is $\mathcal{T}or_i^X(\mathcal{E}, g\mathcal{F})$. As ι^g is a closed immersion, ι^g_* is exact; hence the i^{th} homology of (2) is $\iota^g_*\,\mathcal{T}or_i^X(\mathcal{E}, g\mathcal{F})$.

To summarize, $\mathcal{D}_{\bullet,\bullet} \otimes q^*k_g$ is a double complex whose rows are acyclic and whose horizontal homology is a vertical complex computing $\iota^g_*\,\mathcal{T}or_i^X(\mathcal{E}, g\mathcal{F})$. On the other hand, the total complex of $\mathcal{D}_{\bullet,\bullet}$ is a locally free resolution of $\mathcal{G}$ over $\mathcal{O}_\Gamma$, so the i^{th} homology of $\mathcal{D}_{\bullet,\bullet} \otimes q^*k_g$ is $\mathcal{T}or_i^\Gamma(q^*k_g, \mathcal{G})$. Standard homological techniques (see [Wei94, Section 5.6], for example) produce a natural isomorphism from the homology of the total complex to that of the vertical complex, given the horizontal acyclicity. □

Due to the isomorphism in Proposition 2, we are interested in $\mathcal{T}or_i^\Gamma(q^*k_g, \mathcal{G})$.

Proposition 3. *For $i > 0$ and generic $g \in G$, we have $\mathcal{T}or_i^\Gamma(q^*k_g, \mathcal{G}) = 0$.*

The proof is based on the following general result due to Grothendieck.

Generic Freeness. *Let A be a generically reduced noetherian scheme, $q : B \to A$ a finite type A-scheme, and $\mathcal{M}$ a coherent sheaf on B. Then there is a dense open subset U of A such that (the pushforward to U of) $\mathcal{M}|_{q^{-1}(U)}$ is a locally free $\mathcal{O}_U$-module.*

Proof. See [Gro63, IV$_3$, 11.2.6.1], or see [Eis95, Theorem 14.4] for a very readable proof in the affine case. □

Proof of Proposition 3. Apply generic freeness twice to find a dense open $U \subset G$ such that $\mathcal{G}|_{q^{-1}(U)}$ and $\mathcal{O}_\Gamma|_{q^{-1}(U)}$ are both locally free $\mathcal{O}_U$ modules. We claim that $\mathcal{T}or_i^\Gamma(q^*k_g, \mathcal{G}) = 0$ for all $i > 0$ when $g \in U$. This computation may be checked locally: if R is the local ring of $\mathcal{O}_G$ at $g \in U$, with maximal ideal $\mathfrak{m}_g$, and S is the local ring of any point $\gamma \in \Gamma$ mapping to g, then we only need the S-module $\mathrm{Tor}_i^S(S \otimes_R R/\mathfrak{m}_g, M)$ to vanish for $i > 0$, where $M = \mathcal{G}_\gamma$ is the stalk of $\mathcal{G}$ at γ.

Considering $\mathrm{Tor}_i^S(S \otimes_R R/\mathfrak{m}_g, M)$ as an R-module, it agrees with $\mathrm{Tor}_i^R(R/\mathfrak{m}_g, M)$, since any resolution $F_\bullet$ of M by free S-modules is already a resolution of M by free R-modules (by our choice of U), and $F_\bullet \otimes_S (S \otimes_R R/\mathfrak{m}_g) = F_\bullet \otimes_R (R/\mathfrak{m}_g)$. The desired vanishing follows because M is a free (hence flat) R-module (by our choice of U). □

The proof of our Theorem is now easy to complete. Propositions 2 and 3 imply that for $g \in G$ generic, $\iota^g_*\,\mathcal{T}or_i^X(\mathcal{E}, g\mathcal{F}) = \mathcal{T}or_i^\Gamma(q^*k_g, \mathcal{G}) = 0$ whenever $i > 0$. But ι^g is a closed immersion, so ι^g_* is exact and faithful; in particular, $\iota^g_*\,\mathcal{T}or_i^X(\mathcal{E}, g\mathcal{F}) = 0$ implies that $\mathcal{T}or_i^X(\mathcal{E}, g\mathcal{F}) = 0$, as desired. □

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