ALTERNATING FORMULAS FOR K-THEORETIC QUIVER POLYNOMIALS

EZRA MILLER

Abstract. The main theorem here is the K-theoretic analogue of the cohomological 'stable double component formula' for quiver polynomials in [KMS03]. This K-theoretic version is still in terms of lacing diagrams, but nonminimal diagrams contribute terms of higher degree. The motivating consequence is a conjecture of Buch on the sign-alternation of the coefficients appearing in his expansion of quiver K-polynomials in terms of stable Grothendieck polynomials for partitions [Buc02a].

Introduction

The study of combinatorial formulas for the degeneracy loci of quivers of vector bundles with arbitrary ranks was initiated by Buch and Fulton [BF99]. In their paper they studied the case of a sequence $E_0 \to E_1 \to \cdots \to E_n$ of vector bundles over a fixed base. Their Main Theorem implies that given an integer array $\mathbf{r} = (r_{ij})_{i \le j}$, the cohomology class of the locus $\Omega_{\mathbf{r}}$ in the base where $E_i \to E_j$ has rank at most r_{ij} can be expressed, under suitably general conditions, as an integer sum of products of Schur polynomials evaluated on the Chern classes of the bundles E_i. After giving an explicit algorithmic—but nonpositive—expression for the *quiver coefficients* appearing in the sum, they also conjectured a positive combinatorial formula for them. This conjecture was proved in [KMS03] (for a natural choice of an array of rectangular tableaux) by way of three other positive combinatorial formulas for the *quiver polynomials*.

The cohomological ideas of [BF99] were extended to K-theory in [Buc02a], where the class $K\mathcal{Q}_{\mathbf{r}}$ of the structure sheaf of the degeneracy locus $\Omega_{\mathbf{r}}$ is expressed as an integer sum of products of stable double Grothendieck polynomials for Grassmannian permutations. Buch proved an algorithmic combinatorial formula for the coefficients in this expansion of $K\mathcal{Q}_{\mathbf{r}}$ [Buc02a, Theorem 4.1], and conjectured that the signs of these coefficients alternate in a simple manner [Buc02a, Conjecture 4.2].

The purpose of this paper is to prove Buch's conjecture (Theorem 18) by way of combinatorial formulas for $K\mathcal{Q}_{\mathbf{r}}$. The starting point is a formula from [KMS03] that expresses a 'doubled' version of the Laurent polynomial $K\mathcal{Q}_{\mathbf{r}}$ as a ratio of two Grothendieck polynomials (Definition 1). Consequently, the arguments and results in this paper do not require any geometry of degeneracy loci or K-theory. For an introduction to those geometric perspectives, see [Buc02a].

The main result here is Theorem 13, which gives a K-theoretic extension of the *stable double component formula* that appeared in [KMS03, Theorem 6.20]. That formula from [KMS03] was cohomological, and was stated as a sum over *lacing diagrams*

Date: 10 June 2004.

The author was partly supported by the National Science Foundation, grant DMS-0304789.

(graphical representations of sequences of partial permutations) that are *minimal*. In Theorem 13, nonminimal diagrams contribute terms of higher degree. The word 'stable' refers to the limit in Theorem 13 obtained by adding a large constant m to all of the ranks r_{ij}. When specialized to the ordinary (non-doubled) Laurent polynomial $K\mathcal{Q}_{\mathbf{r}}$, Theorem 13 holds without taking limits, and Buch's conjecture follows using a sign-alternation theorem of Lascoux [Las01, Theorem 4] (see Section 6).

The proof of Theorem 13 generalizes a procedure suggested by [KMS03] (see Remark 6.21 there), and carried out in [Yon03], for constructing pipe dreams associated to given lacing diagrams. This technique is combined with those developed in [KM03b] for dealing with nonreduced subwords of reduced expressions for permutations. The K-theory analogue of a formula [KMS03, Theorem 5.5] for quiver polynomials in terms of the *pipe dreams* of Fomin and Kirillov [FK96] enters along the way (Theorem 3).

Buch [Buc03] independently arrived at the main results and definitions here (and more) by applying general techniques of Fomin and Kirillov [FK96, FK94]. A special case of the sign conjecture and K-component formula already appeared in [BKTY03].

Organization. A notion of double quiver K-polynomial is identified via a ratio formula in Section 1, in analogy with the way (cohomological) double quiver polynomials arise in [KMS03]. The 'pipe formula' for quiver K-polynomials is proved in Section 2, after background on nonreduced pipe dreams and Demazure products. The condition on nonminimal lacing diagrams that turns out to make them occur with sign ± 1 in the K-component formula is defined in Section 3. Rank stability of these nonminimal lacing diagrams, proved in Section 4, leads to the the stable K-component formula in Section 5, after reviewing basics regarding Grothendieck polynomials and their stable limits. Finally, Buch's sign alternation conjecture is derived in Section 6.

1. Double quiver K-polynomials

A $k\times\ell$ *partial permutation* is a $k\times\ell$ matrix w whose entries are either 0 or 1, with at most one nonzero entry in each row or column. Each such matrix w can be completed to a permutation matrix—that is, with exactly one 1 in each row and column—having w as its upper-left $k\times\ell$ corner. Viewing permutations as lying in the union $S_\infty = \bigcup_m S_m$ of all symmetric groups S_m, there is a unique completion $\widetilde{w}$ of w that has minimal length $l(\widetilde{w})$. For any partial permutation w, we write $q = w(p)$ if the entry w_{pq} at (p,q)—that is, in row p and column q—equals 1. If v is a permutation matrix, then the assignment $p \mapsto v(p)$ defines a permutation in S_∞.

Let $\mathbf{z} = z_1, z_2, \ldots$ and $\dot{\mathbf{z}} = \dot{z}_1, \dot{z}_2, \ldots$ be alphabets. Writing a given polynomial f in these two alphabets over the integers $\mathbb{Z}$ as a polynomial in z_i and z_{i+1} with coefficients that are polynomials in the other variables, the i^{th} *Demazure operator* $\overline{\partial}_i$ sends f to

$$\overline{\partial}_i f \quad = \quad \frac{z_{i+1} f(z_i, z_{i+1}) - z_i f(z_{i+1}, z_i)}{z_{i+1} - z_i}.$$

Let w_0^m be the permutation of maximal length in S_m, and write $s_i \in S_\infty$ for the transposition switching i and $i+1$. Following [LS82], the *double Grothendieck polynomial* for a permutation $v \in S_m$ is defined from the "top" double Grothendieck polynomial

$\mathcal{G}_{w_0^m}(\mathbf{z}/\dot{\mathbf{z}}) = \prod_{i+j\leq m}(1 - z_i/\dot{z}_j)$ by the recursion

$$\mathcal{G}_{vs_i}(\mathbf{z}/\dot{\mathbf{z}}) \;=\; \overline{\partial}_i \mathcal{G}_v(\mathbf{z}/\dot{\mathbf{z}})$$

whenever vs_i is lower in Bruhat order than v. This definition is independent of the choice of m [LS82]. If w is a partial permutation, then set $\mathcal{G}_w(\mathbf{z}/\dot{\mathbf{z}}) = \mathcal{G}_{\widetilde{w}}(\mathbf{z}/\dot{\mathbf{z}})$. The notation here differs from [FK94], where their polynomial $\mathfrak{L}_w^{(-1)}(y,x)$ is obtained from $\mathcal{G}_w(\mathbf{z}/\dot{\mathbf{z}})$ by replacing z_i with $1 - x_i$ and $\dot{z}_i^{-1}$ with $1 - y_i$.

The permutation matrices v of central importance here are those defined as follows. Fix a positive integer d and an expression $d = \sum_{j=0}^n r_j$ of d as a sum of $n+1$ "ranks" r_j. Endow each $d \times d$ permutation matrix v with a block decomposition in which the j^{th} block row from the top has height r_j, and the i^{th} block column *from the right* has width r_i. Thus each $d \times d$ permutation matrix v is composed of $(n+1)^2$ blocks B_{ij}, each of size $r_j \times r_i$ and lying at the intersection of block column i and block row j. The matrix v is a *Zelevinsky permutation* if B_{ij} has all zero entries whenever $i \geq j+2$, and the nonzero entries of v proceed from northwest to southeast within every block row and within every block column (so v has no 1 entry that is northeast of another within the same block row or block column).

For each Zelevinsky permutation v, there is a *rank array* $\mathbf{r} = (r_{ij})_{i\leq j}$ such that v equals the Zelevinsky permutation $v(\mathbf{r})$ associated to $\mathbf{r}$ as in the original construction from [KMS03, Proposition 1.6]. Indeed, it can be checked that the following suffices: define r_{ij} to be the number of nonzero entries of v in the union of all blocks B_{pq} for which $i \geq p$ and $j \leq q$ (that is, blocks B_{pq} weakly southeast of B_{ij}). In particular, $r_{ii} = r_i$, and the number of nonzero entries in the block $B_{j+1,j}$ of v is $r_{j,j+1}$. Pictures and examples can be found in [KMS03, Section 1.2], but see also Example 4, below.

Double Grothendieck polynomials for Zelevinsky permutations $v(\mathbf{r})$ are naturally written as $\mathcal{G}_{v(\mathbf{r})}(\mathbf{x}/\mathring{\mathbf{y}})$, using two alphabets $\mathbf{z} = \mathbf{x}$ and $\dot{\mathbf{z}} = \mathring{\mathbf{y}}$ each of which is an ordered sequence of $n+1$ alphabets of sizes $r_0, \ldots, r_n$ and $r_n, \ldots, r_0$, respectively:

$$\begin{aligned} \mathbf{x} = \mathbf{x}^0, \ldots, \mathbf{x}^n \quad &\text{and} \quad \mathring{\mathbf{y}} = \mathbf{y}^n, \ldots, \mathbf{y}^0, \\ \text{where} \quad \mathbf{x}^j = x_1^j, \ldots, x_{r_j}^j \quad &\text{and} \quad \mathbf{y}^j = y_1^j, \ldots, y_{r_j}^j. \end{aligned}$$

It is convenient to think of the $\mathbf{x}$ variables as labeling the rows of the $d \times d$ grid (from top to bottom, in the above ordering on the $\mathbf{x}$ variables), while the $\mathring{\mathbf{y}}$ variables label its columns (from left to right, in the above ordering on the $\mathring{\mathbf{y}}$ variables). See [KMS03, Section 2.2] for pictures and examples. Most partial permutations w that occur in the sequel will have size $r_{j-1} \times r_j$ for some $j \in \{1, \ldots, n\}$; in that case we usually consider $\mathcal{G}_w(\mathbf{x}^{j-1}/\mathbf{y}^j)$, so $\mathbf{z} = \mathbf{x}^{j-1}$ and $\dot{\mathbf{z}} = \mathbf{y}^j$.

Among all $d \times d$ Zelevinsky permutations with block decompositions determined by $d = \sum_{j=0}^n r_j$, there is a unique one $v(\mathit{Hom})$ whose rank array $\mathbf{r}(\mathit{Hom})$ is maximal, in the sense that $r_{ij}(\mathit{Hom}) \geq r_{ij}$ for all other $d \times d$ Zelevinsky permutations $v(\mathbf{r})$.

Definition 1. The **double quiver K-polynomial** is the ratio

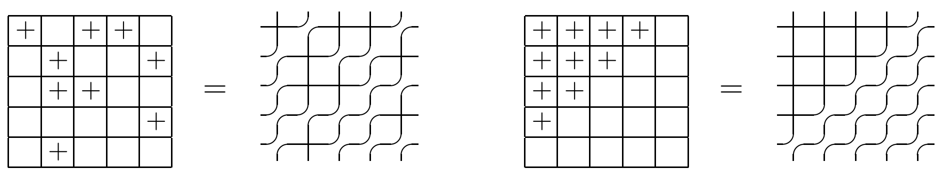
$$K\mathcal{Q}_{\mathbf{r}}(\mathbf{x}/\mathring{\mathbf{y}}) \;=\; \frac{\mathcal{G}_{v(\mathbf{r})}(\mathbf{x}/\mathring{\mathbf{y}})}{\mathcal{G}_{v(\mathit{Hom})}(\mathbf{x}/\mathring{\mathbf{y}})}$$

of double Grothendieck polynomials for $v(\mathbf{r})$ and $v(\mathit{Hom})$.

The "ordinary" specialization of the polynomial $K\mathcal{Q}_{\mathbf{r}}(\mathbf{x}/\mathring{\mathbf{y}})$ appears in the K-theoretic ratio formula [KMS03, Theorem 2.7]. It will follow from Theorem 3, below, that $\mathcal{G}_{v(Hom)}(\mathbf{x}/\mathring{\mathbf{y}})$ divides $\mathcal{G}_{v(\mathbf{r})}(\mathbf{x}/\mathring{\mathbf{y}})$, so the right hand side of Definition 1 is actually a (Laurent) polynomial rather than simply a rational function.

2. Nonreduced pipe dreams

A $k\times\ell$ *pipe dream* is a subset of the $k\times\ell$ grid, identified as the set of crosses in a tiling of the $k\times\ell$ grid by *crosses* ┼ and *elbow joints* ╯╭, as in the following diagrams:



The square tile boundaries are omitted from the tilings forming the newtworks of *pipes* on right sides of these equalities. Pipe dreams are special cases of diagrams introduced by Fomin and Kirillov [FK96]; for more background, see [KM03a, Section 1.4].

A pipe dream P yields a word in the Coxeter generators $s_1, s_2, s_3, \ldots$ of S_∞ by reading the antidiagonal indices of the crosses in P along rows, right to left, starting from the top row and proceeding downward [BB93, FK96]. The *Demazure product* $\delta(P)$ is obtained (as in [KM03b, Definition 3.1]) by omitting adjacent transpositions that decrease length. More precisely, $\delta(P)$ is obtained by multiplying the word of P using the idempotence relation $s_i^2 = s_i$ along with the usual braid relations $s_is_{i+1}s_i = s_{i+1}s_is_{i+1}$ and $s_is_j = s_js_i$ for $|i-j| \geq 2$. Up to signs, this amounts to taking the product of the word of P in the degenerate Hecke algebra [FK96]. Let

$$\mathcal{P}(w) \quad = \quad \{\text{pipe dreams } P \mid \delta(P) = \widetilde{w}\}$$

for a $k\times\ell$ partial permutation w be the set of pipe dreams whose Demazure product is the minimal completion of w to a permutation $\widetilde{w} \in S_\infty$. Every pipe dream in $\mathcal{P}(w)$ fits inside the $k\times\ell$ rectangle, and is to be considered as a pipe dream of size $k\times\ell$. The subset of $\mathcal{P}(w)$ consisting of *reduced* pipe dreams (or *rc-graphs* [BB93]), where no pair of pipes crosses more than once, is denoted by $\mathcal{RP}(w)$.

Here is the observation that will make the limiting arguments in [KMS03, Section 6] for reduced pipe dreams work on nonreduced pipe dreams (see Proposition 5, below).

Lemma 2. *Suppose that $P \in \mathcal{P}(w)$. Then the crossing tiles in P lie in the union of all reduced pipe dreams for w.*

Proof. The statement is obvious if P is reduced, so suppose otherwise. Then some pipe dream $P' \in \mathcal{P}(w)$ be can be obtained by deleting a single crossing tile from P. By induction, each crossing tile in P' lies in some reduced pipe dream for w. On the other hand, a second pipe dream $P'' \in \mathcal{P}(w)$ can be obtained from P by deleting a different crossing tile (using [KM03b, Theorem 3.7], for example). Induction shows that each crossing tile in P'', including the tile $P \smallsetminus P'$, lies in a reduced pipe dream for w. $\square$

The *exponential reverse monomial* associated to a $d \times d$ pipe dream P is

$$(\mathbf{1} - \tilde{\mathbf{x}}/\tilde{\mathbf{y}})^P \;=\; \prod_{+\,\in P} (1 - \tilde{x}_+/\tilde{y}_+),$$

where the variable $\tilde{x}_+$ sits at the left end of the row containing $+$ after reversing each of the $\mathbf{x}$ alphabets before Definition 1, and the variable $\tilde{y}_+$ sits atop the column containing $+$ after reversing each of the $\mathbf{y}$ alphabets there (see Example 4).

Fix $d = r_0 + \cdots + r_n$ as in Section 1, and let D_{Hom} be the Ferrers shape of all locations strictly above the block superantidiagonal (as in [KMS03, Definition 1.10]). The region corresponding to D_{Hom} in the $d \times d$ grid is filled with crossing tiles in every reduced pipe dream for every $d \times d$ Zelevinksy permutation v (to deduce this one needs only that v is devoid of nonzero entries in that region). By Lemma 2, every nonreduced pipe dream $P \in \mathcal{P}(v)$ contains D_{Hom}, as well. (In [KMS03], the crossing tiles in $D_{Hom} \subseteq P$ are the $*$ entries of P, but it is necessary here to consider all of the crosses in P, including those in D_{Hom}, to make the meaning of $\delta(P)$ clear.)

All of the crossing tiles in every pipe dream $P \in \mathcal{P}(v)$ lie above the main antidiagonal (this holds for any permutation $v \in S_d$, since the other crosses correspond to Coxeter generators that lie outside of S_d). Hence the "interesting" crossing tiles in each pipe dream $P \in \mathcal{P}(v)$ all lie in the block antidiagonal and the block superantidiagonal. In particular, any pipe dream $P \in \mathcal{P}(v)$ with no crossing tiles in its antidiagonal blocks has its "interesting" crosses confined to the block superantidiagonal. These kinds of pipe dreams $P \in \mathcal{P}(v(\mathbf{r}))$ will be central to the next section.

Here is the K-theoretic analogue of the reversed version [KMS03, Proposition 6.9] of the *pipe formula* for cohomological quiver polynomials [KMS03, Theorem 5.5].

Theorem 3 (Pipe formula)**.** *The double quiver K-polynomial is the alternating sum*

$$K\mathcal{Q}_{\mathbf{r}}(\mathbf{x}/\mathring{\mathbf{y}}) \;=\; \sum_{\delta(P)=v(\mathbf{r})} (-1)^{|P|-l(v(\mathbf{r}))} (\mathbf{1} - \tilde{\mathbf{x}}/\tilde{\mathbf{y}})^{P \smallsetminus D_{Hom}}$$

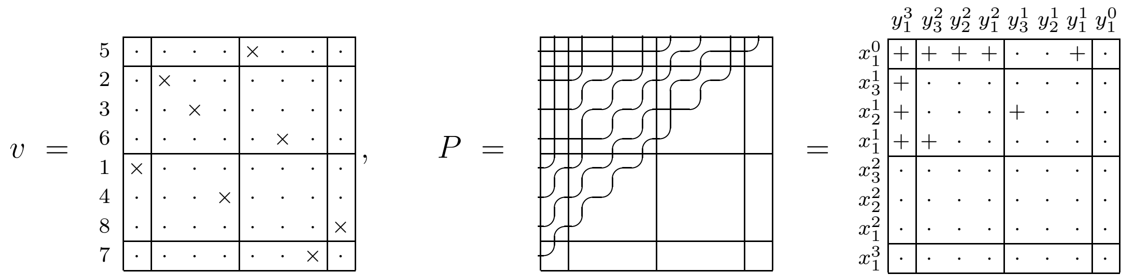
of exponential reverse monomials associated to pipe dreams $P \smallsetminus D_{Hom}$ for $P \in \mathcal{P}(v(\mathbf{r}))$. The exponent on -1 is the number crosses in P minus the length $l(v(\mathbf{r}))$ of $v(\mathbf{r})$.

Proof. The formula of Fomin and Kirillov [FK94, Theorem 2.3 and p. 190] (or see [KM03b, Theorem 4.1 and Corollary 5.4]) implies that for any permutation w, the Grothendieck polynomial for w can be expressed as the alternating sum

$$\mathcal{G}_w(\mathbf{z}/\dot{\mathbf{z}}) \;=\; \sum_{\delta(P)=w} (-1)^{|P|-l(w)} (\mathbf{1} - \mathbf{z}/\dot{\mathbf{z}})^P,$$

where $(\mathbf{1} - \mathbf{z}/\dot{\mathbf{z}})^P = \prod_{+\in P}(1 - z_+/\dot{z}_+)$ is the product of the factors $(1 - z_p/\dot{z}_q)$ such that P has a crossing tile at (p, q). On the other hand, it follows immediately from the isobaric divided difference recursion that $\mathcal{G}_w(\mathbf{z}/\dot{\mathbf{z}})$ is symmetric in z_p and z_{p+1} if $w(p) < w(p+1)$—that is, if w has no descent at p, or equivalently, if the nonzero entry of w at $(p, w(p))$ is northwest of $(p + 1, w(p + 1))$. Applying the same logic to w^{-1}, we find that $\mathcal{G}_w(\mathbf{z}/\dot{\mathbf{z}})$ is symmetric in $\dot{z}_q$ and $\dot{z}_{q+1}$ whenever $(w^{-1}(q), q)$ is northwest of $(w^{-1}(q + 1), q + 1)$. Consequently, the northwest-to-southeast progression of nonzero entries in block rows and columns implies that the double Grothendieck polynomial for any Zelevinsky permutation is symmetric in each of its $2n+2$ alphabets. The ratio in Definition 1 now gives the desired result via the Fomin–Kirillov formula, after using the symmetry in the $2n+2$ alphabets to reverse each one. $\square$

Example 4. Let $d = 8 = 1+3+3+1$ and consider the Zelevinsky permutation $v = 52361487$ at left below. The pipe dream P below has Demazure product $\delta(P) = v$.



Indeed, the transposition s_7 from the cross at $(3,5)$ is canceled by the idempotence relation from the word associated to P, so $\delta(P) = s_7 s_4 s_3 s_2 s_1 s_2 s_3 s_5 s_4 = v$. The exponential reverse monomial associated to $P \smallsetminus D_{Hom}$, which uses the given labeling of the rows and columns by variables, is

$$(\mathbf{1} - \tilde{\mathbf{x}}/\tilde{\mathbf{y}})^{P \smallsetminus D_{Hom}} \quad = \quad \Big(1 - \frac{x_1^0}{y_1^1}\Big)\Big(1 - \frac{x_2^1}{y_3^1}\Big)\Big(1 - \frac{x_1^1}{y_3^2}\Big).$$

The summand in Theorem 3 corresponding to P contributes the negative of this product, since $l(v) = 9$ but P has 10 crosses.

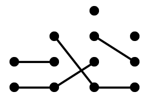
Let $m+\mathbf{r}$ be the rank array obtained from $\mathbf{r}$ by adding the nonnegative integer m to each entry of $\mathbf{r}$. Let $\mathbf{x}_{m+\mathbf{r}}$ be a list of finite alphabets of sizes $m+r_0, \ldots, m+r_n$, and let the alphabets in $\mathring{\mathbf{y}}_{m+\mathbf{r}}$ have sizes $m+r_n, \ldots, m+r_0$. This notation agrees with that in [KMS03, Section 4.4]. Denote by $D_{Hom}(m)$ the unique reduced pipe dream for the Zelevinsky permutation $v(m+\mathbf{r}(Hom))$ in $S_{d+m(n+1)}$ associated to the maximal irreducible rank array.

Proposition 5. *There is a fixed integer ℓ, independent of m, such that for every pipe dream $P \in \mathcal{P}(v(m+\mathbf{r}))$ with at least one cross $+$ in an antidiagonal block, setting the last ℓ variables to 1 in every finite alphabet from the lists $\mathbf{x}_{m+\mathbf{r}}$ and $\mathring{\mathbf{y}}_{m+\mathbf{r}}$ kills the exponential reverse monomial $(\mathbf{1} - \tilde{\mathbf{x}}/\tilde{\mathbf{y}})^{P \smallsetminus D_{Hom}(m)}$.*

Proof. The result is immediate from Lemma 2 and [KMS03, Proposition 6.10], the latter being the analogue for reduced pipe dreams and 'ordinary' (as opposed to exponential) monomials. $\square$

3. Nonminimal lacing diagrams

Suppose that $\mathbf{w} = (w_1, \ldots, w_n)$ is a list of partial permutations in which w_j has size $r_{j-1} \times r_j$. The list $\mathbf{w}$ can be identified with the (nonembedded) graph in the plane called its *lacing diagram* in [KMS03, Section 3.1], based on diagrams of Abeasis and Del Fra [AD80]. The vertex set of the graph consists of r_j bottom-justified dots in column j for $j = 0, \ldots, n$, with an edge connecting the dot at height α (from the bottom) in column $j-1$ with the dot at height β in column j if and only if the entry of w_j at (α, β) is 1. A *lace* is a connected component of a lacing diagram. For example, here is the lacing diagram associated to a partial permutation list:



$$\longleftrightarrow \quad \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right)$$

The goal of this section is to define what it means for a rank array to equal the Demazure product $\delta(\mathbf{w})$ of a lacing diagram $\mathbf{w}$. That $\delta(\mathbf{w})$ is a rank array rather than a minimal lacing diagram is in analogy with Demazure products of lists of simple reflections, which are permutations rather than reduced decompositions. Usually $\delta(\mathbf{w})$ will not equal the rank array of $\mathbf{w}$ itself. In analogy with Demazure products of *reduced* words, however, the Demazure product of a *minimal* lacing diagram will equal its own rank array.

Definition 6. Suppose $P_1, \ldots, P_n$ are pipe dreams of sizes $r_0 \times r_1, \ldots, r_{n-1} \times r_n$, and set $d = r_0 + \cdots + r_n$. Denote by $P(P_1, \ldots, P_n)$ the $d \times d$ pipe dream in which every block strictly above the block superantidiagonal is filled with crossing tiles, and the superantidiagonal $r_{j-1} \times r_j$ block in block row $j-1$ is the pipe dream P_j.

Given a $k \times \ell$ pipe dream P, let $\check{P}$ be the $k \times \ell$ pipe dream that results after rotating P through 180°. Also, if $m \in \mathbb{N}$, write $m + P_j$ for the $k \times (\ell + m)$ pipe dream that results by shifting all the crosses in P_j to the right by m.

Proposition 7. *Fix a lacing diagram* $\mathbf{w} = (w_1, \ldots, w_n)$. *The Demazure product of* $P(P_1, \ldots, P_n)$ *is independent of* $P_1, \ldots, P_n$, *as long as* $\check{P}_j \in \mathcal{P}(w_j)$ *for all* $j = 1, \ldots, n$.

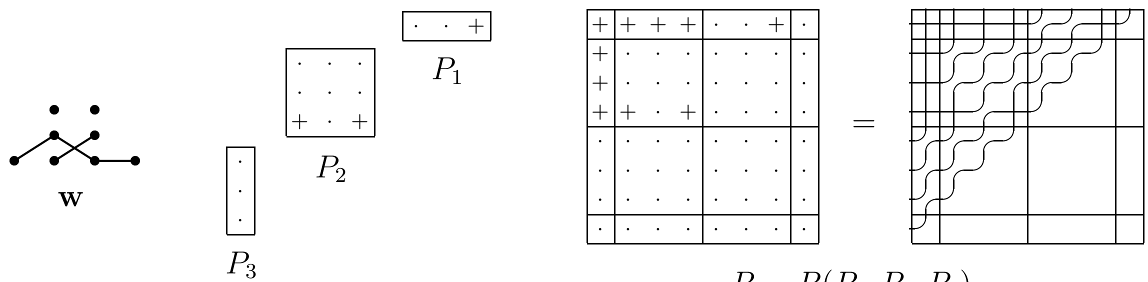
Proof. First consider an arbitrary pipe dream P. Instead of the usual word for P, consider the word in $s_1, s_2, s_3, \ldots$ gotten by reading the antidiagonal indices of the crosses in P from top to bottom in each column, starting in the right column and proceeding leftward. The results of [FK96] imply that using the idempotence and Coxeter relations to multiply this word again yields the Demazure product $\delta(P)$.

We need $P(P_1, \ldots, P_n)$ and $P(P'_1, \ldots, P'_n)$ to have equal Demazure products whenever $\delta(\check{P}_j) = \delta(\check{P}'_j)$ for all j. It follows from the definitions that $\delta(\check{P}_j) = \delta(\check{P}'_j)$ if and only if $\delta(P_j) = \delta(P_j)$, and this latter equality is equivalent to $\delta(m + P_j) = \delta(m + P'_j)$ for all $m \in \mathbb{N}$.

Reading right to left in each row as usual, by associativity of Demazure products we only need the corresponding block rows of $P = P(P_1, \ldots, P_n)$ and $P' = P(P'_1, \ldots, P'_n)$ to have equal Demazure products. Suppose that block row $j-1$ in the the Ferrers shape D_{Hom} has m columns, and let v_j be the Demazure product of this block row. Applying the first paragraph to corresponding block rows of P and P', we find that the Demazure products of block row $j-1$ in P and P' are obtained by using the idempotence and Coxeter relations to multiply $\delta(m + P_j)v_j$ and $\delta(m + P'_j)v_j$. □

Definition 8. Fix a lacing diagram $\mathbf{w}$. If, for some (and hence, by Proposition 7, every) sequence $P_1, \ldots, P_n$ of pipe dreams satisfying $\check{P}_j \in \mathcal{P}(w_j)$ for all j, the Demazure product of $P(P_1, \ldots, P_n)$ is a Zelevinsky permutation $v(\mathbf{r})$, then we write $\delta(\mathbf{w}) = \mathbf{r}$ and call the rank array $\mathbf{r}$ the **Demazure product** of the lacing diagram $\mathbf{w}$.

Example 9. For the lacing diagram $\mathbf{w}$ below, the pipe dreams P_j satisfy $\check{P}_j \in \mathcal{P}(w_j)$.



$P = P(P_1, P_2, P_3)$

The Demazure product $\delta(P)$ of the pipe dream $P = P(P_1, P_2, P_3)$ at right above equals the Zelevinsky permutation v from Example 4. If $\mathbf{r}$ is the rank array satisfying $v = v(\mathbf{r})$, then we conclude that $\delta(\mathbf{w}) = \mathbf{r}$. Observe that although P_1, P_2, and P_3 are all reduced pipe dreams, the pipe dream P is not reduced, so $\mathbf{w}$ is not minimal. Thus the lacing diagram $\mathbf{w}$ will give rise to a higher degree summand in Theorem 13, below.

4. Rank stability of lacing diagrams

Next we show that lacing diagrams with Demazure product $\mathbf{r}$ are stable, in the appropriate sense, under uniformly increasing ranks obtained by replacing $\mathbf{r}$ with $m + \mathbf{r}$. The results and methods in this section rely on certain parts of [KMS03], namely Proposition 5.7 (which is actually [BB93, Theorem 3.7]) and Section 5.3. That material describes the elementary combinatorics behind the transition from $v(\mathbf{r})$ to $v(1 + \mathbf{r})$ [KMS03, Lemma 5.11], and the resulting effect on reduced pipe dreams of the form $P(P_1, \ldots, P_n)$ [KMS03, Propositions 5.7 and 5.15]. To avoid excessive use of the word 'block' in what follows, we use 'horizontal strip j' as a synonym for 'block row j'.

Lemma 10. *If $P(P_1, \ldots, P_n) \in \mathcal{P}(v(1 + \mathbf{r}))$ and each $\check{P}_j$ is the top pipe dream for a $(1 + r_{j-1}) \times (1 + r_j)$ partial permutation w_j, then all crossing tiles of P_j lie in the southwest $r_{j-1} \times r_j$ rectangle of the superantidiagonal block in horizontal strip $j - 1$.*

Thus the superantidiagonal block in the Lemma is supposed to have a blank row above and a blank column to the right of the southwest $r_{j-1} \times r_j$ rectangle in question.

Proof. No reduced pipe dream for $v(1+\mathbf{r})$ has a crossing tile on the main superantidiagonal, by [KMS03, Proposition 5.15]. Lemma 2 implies that the same is true of P. It follows that $w_j = 1 + w'_j$ for some $r_{j-1} \times r_j$ partial permutation w'_j. Consequently, the left column of $\check{P}_j$ has no crossing tiles, and shifting all crossing tiles in $\check{P}_j$ one unit to the left results in the top pipe dream for w'_j. This top pipe dream fits inside the rectangle of size $r_{j-1} \times r_j$. □

Suppose $P = P(P_1, \ldots, P_n)$ is a pipe dream in which

(SW) P_j has size $(1 + r_{j-1}) \times (1 + r_j)$, but every $+$ in P_j lies in the southwest $r_{j-1} \times r_j$ rectangle.

Write P'_j for the $r_{j-1} \times r_j$ pipe dream consisting of the southwest rectangle of P_j, and then write $P' = P(P'_1, \ldots, P'_n)$. Thus P has block sizes consistent with ranks $1 + \mathbf{r}$ (so the i^{th} antidiagonal block is square of size $1 + r_{ii}$), while P' has block sizes consistent with ranks $\mathbf{r}$ (so the i^{th} antidiagonal block is square of size r_{ii}). The construction can also be reversed to create P having been given the pipe dream called P'.

For a pipe dream P, as in [KM03b, Theorem 4.4] say that P *simplifies* to $D \subseteq P$ if D is the lexicographically first subword of P with Demazure product $\delta(P)$. Equivalently, if $P_{\leq m}$ is the length m initial string of simple reflections in P, the simplification D is gotten by omitting the m^{th} reflection from P for all m such that $\delta(P_{\leq m-1}) = \delta(P_{\leq m})$.

Given a reduced pipe dream D, an elbow tile is *absorbable* [KM03b, Section 4] if the two pipes passing through it intersect in a crossing tile to its northeast. It follows from the definitions that a pipe dream P simplifies to D if and only if P is obtained from D by changing (at will) some of its absorbable elbow tiles into crossing tiles.

Lemma 11. *Suppose $D = (D_1, \ldots, D_n)$ satisfies the (SW) condition. Then D is a reduced pipe dream for $v(1+\mathbf{r})$ if and only if $D' = (D'_1, \ldots, D'_n)$ is a reduced pipe dream for $v(\mathbf{r})$. In this case, the absorbable elbow tiles in horizontal strip $j-1$ of D' are in bijection with the absorbable elbow tiles in the southwest $r_{j-1} \times r_j$ rectangle of the superantidiagonal block in horizontal strip $j-1$ of D.*

Proof. The first claim is a straightforward consequence of [KMS03, Proposition 5.15]. The second claim follows because the corresponding pairs of pipes in D and D' pass through corresponding elbow tiles. The rest of the proof makes this statement precise.

Given a nonzero entry of the Zelevinsky permutation $v(1+\mathbf{r})$, exactly one of the following three conditions must hold: (i) the entry lies in the northwest corner of some superantidiagonal block; (ii) the entry lies in the southeast corner of the whole matrix; or (iii) there is a corresponding nonzero entry in $v(\mathbf{r})$. This means that the pipes in D' are in bijection with those pipes in D corresponding to nonzero entries of $v(1+\mathbf{r})$ that do not satisfy (i) or (ii). Furthermore, it is easily checked that the pipes in D of type (i) or (ii) can only intersect a superantidiagonal block in its top row or rightmost column. Hence to say

> the two pipes passing through an elbow tile in the southwest $r_{j-1} \times r_j$ rectangle of the superantidiagonal block in horizontal strip $j-1$ of D correspond to the pipes passing through the corresponding elbow in D'

actually makes sense. That this claim is true follows from [KMS03, Proposition 5.15], and it immediately proves the lemma. □

In [KMS03, Corollary 5.16] it was proved that the set of lacing diagrams obtained from reduced pipe dreams for $v(\mathbf{r})$ is in canonical bijection with the set of (automatically minimal) lacing diagrams obtained from reduced pipe dreams for $v(1+\mathbf{r})$. Here is the extension to nonreduced pipe dreams, via nonminimal lacing diagrams. For notation, given $m \in \mathbb{N}$ and a $k \times \ell$ partial permutation w, define the $(m+k) \times (m+\ell)$ partial permutation $m+w$ to fix $1, \ldots, m$ and act on $\{m+1, \ldots, m+k\}$ just as w acts on $\{1, \ldots, k\}$. For a list $\mathbf{w} = (w_1, \ldots, w_n)$ of partial permutations, set $m+\mathbf{w} = (m+w_1, \ldots, m+w_n)$. This notation agrees with that in [KMS03, Section 4.4].

Proposition 12. *For each array $\mathbf{r}$, let $L(\mathbf{r}) = \{\mathbf{w} \mid \delta(\mathbf{w}) = \mathbf{r}\}$ be the set of lacing diagrams $\mathbf{w}$ with Demazure product $\mathbf{r}$. Then $L(\mathbf{r})$ and $L(m+\mathbf{r})$ are in canonical bijection:*

$$L(m+\mathbf{r}) \quad = \quad \{m+\mathbf{w} \mid \mathbf{w} \in L(\mathbf{r})\}.$$

Proof. It suffices to prove the case $m = 1$, so suppose $\mathbf{w} \in L(1+\mathbf{r})$. Let $P = P(P_1, \ldots, P_n)$ be the pipe dream in $\mathcal{P}(v(1+\mathbf{r}))$ for which each $\check{P}_j$ is the top pipe dream in $\mathcal{RP}(w_j)$. Then P simplifies to a reduced pipe dream $D \in \mathcal{RP}(v(1+\mathbf{r}))$. By Lemma 10 there is a corresponding pipe dream $D' \in \mathcal{RP}(v(\mathbf{r}))$, constructed via the procedure after Lemma 10. On the other hand, the pipe dream P' constructed from P results by changing back into crossing tiles those elbow tiles in D' that correspond to the $+$ tiles deleted from P to get D. Lemma 11 says that P' has Demazure product $v(\mathbf{r})$. Defining $\mathbf{w}'$ by the equality $1 + \mathbf{w}' = \mathbf{w}$, which can be done by Lemma 10, it follows that $\mathbf{w}' \in L(\mathbf{r})$.

In summary, we have constructed P' from P via the intermediate steps

$$P \in \mathcal{P}(v(1+\mathbf{r})) \ \rightsquigarrow\ D \in \mathcal{RP}(v(1+\mathbf{r})) \ \rightsquigarrow\ D' \in \mathcal{RP}(v(\mathbf{r})) \ \rightsquigarrow\ P' \in \mathcal{P}(v(\mathbf{r})),$$

where the first and third steps are simplification and "unsimplification". Consequently, $L(1+\mathbf{r}) \subseteq \{1 + \mathbf{w}' \mid \mathbf{w}' \in L(\mathbf{r})\}$. But the arguments justifying these steps are all reversible, so the reverse containment holds, as well. □

5. Stable double component formula

The main result in this paper, namely Theorem 13, involves *stable double Grothendieck polynomials* $\hat{\mathcal{G}}_w(\mathbf{z}/\dot{\mathbf{z}})$ for $k\times\ell$ partial permutations w [FK94], which we recall presently. Suppose that the argument of a Laurent polynomial $\mathcal{G}$ is naturally a pair of alphabets $\mathbf{z}$ and $\dot{\mathbf{z}}$ of sizes k and ℓ, respectively. In this section and the next, the convention is that if $\mathcal{G}(\mathbf{z}/\dot{\mathbf{z}})$ is written, but $\mathbf{z}$ or $\dot{\mathbf{z}}$ has *fewer* than the required number of letters, then the rest of the letters are assumed to equal 1. For example, the notation $K\mathcal{Q}_{m+\mathbf{r}}(\mathbf{x_r}/\mathring{\mathbf{y}}_\mathbf{r})$ indicates that all variables in $\mathbf{x}_{m+\mathbf{r}} \smallsetminus \mathbf{x_r}$ and $\mathring{\mathbf{y}}_{m+\mathbf{r}} \smallsetminus \mathring{\mathbf{y}}_\mathbf{r}$ (see the paragraph preceding Proposition 5) are to be set equal to 1.

Under this convention, let w be a $k\times\ell$ partial permutation, and write $\mathcal{G}_{m+w}(\mathbf{z}_k/\dot{\mathbf{z}}_\ell)$ for each $m \geq 0$ to mean the Laurent polynomial $\mathcal{G}_{m+w}$ applied to alphabets $\mathbf{z}_k$ and $\dot{\mathbf{z}}_\ell$ of fixed sizes k and ℓ. As m gets large, these Laurent polynomials eventually stabilize, allowing the notation $\hat{\mathcal{G}}_w(\mathbf{z}/\dot{\mathbf{z}}) = \lim_{m\to\infty} \mathcal{G}_{m+w}(\mathbf{z}_k/\dot{\mathbf{z}}_\ell)$ for the stable double Grothendieck polynomial.

Given a lacing diagram $\mathbf{w}$ with r_j dots in column j, for $j = 0, \ldots, n$ denote by

$$\mathcal{G}_\mathbf{w}(\mathbf{x}/\mathring{\mathbf{y}}) \ = \ \mathcal{G}_{w_1}(\mathbf{x}^0/\mathbf{y}^1)\cdots\mathcal{G}_{w_n}(\mathbf{x}^{n-1}/\mathbf{y}^n)$$

the product of double Grothendieck polynomials taken over partial permutations in the list $\mathbf{w} = (w_1, \ldots, w_n)$. Add hats over every $\mathcal{G}$ for the stable Grothendieck case.

Here now is the main result, the K-theoretic analogue of the (cohomological) component formula for stable double quiver polynomials [KMS03, Theorem 6.20]. For notation, recall the set $L(\mathbf{r}) = \{\mathbf{w} \mid \delta(\mathbf{w}) = \mathbf{r}\}$ from Proposition 12, define the *length* of the lacing diagram $\mathbf{w}$ to be the sum $l(\mathbf{w}) = \sum_{i=1}^n l(\widetilde{w}_i)$ of the lengths of the minimal extensions of $w_1, \ldots, w_n$ to permutations, and set $d(\mathbf{r}) = l(v(\mathbf{r})) - l(v(\mathit{Hom}))$. (Thus $d(\mathbf{r})$ is the "expected codimension" of the locus $\Omega_\mathbf{r}$ from the introduction, and $d(\mathbf{r})$ can be described equivalently as $\sum_{i<j}(r_{i,j-1} - r_{ij})(r_{i+1,j} - r_{ij})$, or as $l(v(\mathbf{r})) - |D_{\mathit{Hom}}|$.)

Theorem 13. *The limit of the double quiver K-polynomials $K\mathcal{Q}_{m+\mathbf{r}}(\mathbf{x_r}/\mathring{\mathbf{y}}_\mathbf{r})$ for m approaching ∞ exists and equals the alternating sum*

$$G_\mathbf{r}(\mathbf{x}/\mathring{\mathbf{y}}) \;:=\; \lim_{m\to\infty} K\mathcal{Q}_{m+\mathbf{r}}(\mathbf{x_r}/\mathring{\mathbf{y}}_\mathbf{r}) \;=\; \sum_{\mathbf{w}\in L(\mathbf{r})} (-1)^{l(\mathbf{w})-d(\mathbf{r})}\hat{\mathcal{G}}_\mathbf{w}(\mathbf{x}/\mathring{\mathbf{y}})$$

of products of stable double Grothendieck polynomials. The limit polynomial $G_\mathbf{r}(\mathbf{x}/\mathring{\mathbf{y}})$ is symmetric separately in each of the $2n+2$ finite alphabets $\mathbf{x}^0,\ldots,\mathbf{x}^n,\mathbf{y}^n,\ldots,\mathbf{y}^0$.

Definition 14. $G_\mathbf{r}(\mathbf{x}/\mathring{\mathbf{y}})$ is called the **stable double quiver K-polynomial**.

As we shall see in Corollary 17 and the comments after it, the Laurent polynomial $G_\mathbf{r}(\mathbf{x}/\mathring{\mathbf{y}})$ is not a new object: it is obtained from Buch's power series P_r [Buc02a, Section 4] by substituting $1-x_i$ for x_i and $1-y_j^{-1}$ for y_j in each polynomial G_{μ_k} there.

Proof. Define $K\mathcal{Q}_{m+\mathbf{r}}(\mathbf{x}/\mathring{\mathbf{y}})_\ell$ by setting the last ℓ variables to 1 in every finite alphabet from the lists $\mathbf{x}_{m+\mathbf{r}}$ and $\mathring{\mathbf{y}}_{m+\mathbf{r}}$. Similarly, for each lacing diagram $\mathbf{w}$, define $\mathcal{G}_{m+\mathbf{w}}(\mathbf{x}/\mathring{\mathbf{y}})_\ell$ by setting the same variables to 1 in $\mathcal{G}_{m+\mathbf{w}}(\mathbf{x}/\mathring{\mathbf{y}})$. Because of the nature of the limit in question, and the defining properties of stable Grothendieck polynomials, it suffices to prove that for all $m\geq 0$ and some fixed ℓ independent of m,

$$K\mathcal{Q}_{m+\mathbf{r}}(\mathbf{x}/\mathring{\mathbf{y}})_\ell \;=\; \sum_{\mathbf{w}\in L(\mathbf{r})} (-1)^{l(\mathbf{w})-d(\mathbf{r})}\mathcal{G}_{m+\mathbf{w}}(\mathbf{x}/\mathring{\mathbf{y}})_\ell.$$

Fix ℓ as in Proposition 5, and apply Theorem 3 to $m+\mathbf{r}$ instead of $\mathbf{r}$. Setting the last ℓ variables in each alphabet to 1 on the right hand side there kills all summands corresponding to pipe dreams P that are not expressible as $P(P_1,\ldots,P_n)$ for some list of pipe dreams P_j of sizes $(m+r_{j-1})\times(m+r_j)$; this is the content of Proposition 5. What remains on the right side of Theorem 3 is a sum of terms having the form $(-1)^{|P|-l(v(m+\mathbf{r}))}(1-\tilde{\mathbf{x}}/\tilde{\mathbf{y}})_\ell^{P\smallsetminus D_{Hom}(m)}$ for pipe dreams $P=P(P_1,\ldots,P_n)$ in $\mathcal{P}(v(m+\mathbf{r}))$. If $P_j\in\mathcal{P}(m+w_j)$ for each j, then this term equals the product

$$(*)\qquad (-1)^{l(\mathbf{w})-d(\mathbf{r})}\prod_{i=1}^{n}(-1)^{|P_j|-l(\tilde{w}_j)}(1-\tilde{\mathbf{x}}^{j-1}/\tilde{\mathbf{y}}^j)_\ell^{P_j}$$

for $\mathbf{w}=(w_1,\ldots,w_n)$. The signs in $(*)$ are correct because $|P|-l(v(m+\mathbf{r}))=\sum_j|P_j|-d(m+\mathbf{r})$, and $d(m+\mathbf{r})=d(\mathbf{r})$. To make sense of $(1-\tilde{\mathbf{x}}^{j-1}/\tilde{\mathbf{y}}^j)^{P_j}$, identify P_j with the $d\times d$ pipe dream consisting of just P_j on the j^{th} superantidiagonal block.

For each lacing diagram $\mathbf{w}\in L(\mathbf{r})$, let $\mathcal{P}_\mathbf{w}(m+\mathbf{r})$ be the set of pipe dreams $P(P_1,\ldots,P_n)\in\mathcal{P}(v(m+\mathbf{r}))$ such that $\check{P}_j\in\mathcal{P}(m+w_j)$ for all j. Summing the products in $(*)$ over pipe dreams $P\in\mathcal{P}_\mathbf{w}(m+\mathbf{r})$ yields $(-1)^{l(\mathbf{w})-d(\mathbf{r})}\mathcal{G}_{m+\mathbf{w}}(\mathbf{x}/\mathring{\mathbf{y}})_\ell$ by [FK94, Theorem 2.3 and p. 190] (see also [KM03b, Section 5]). Summing over $\mathbf{w}\in L(\mathbf{r})$ completes the proof, by Proposition 12. □

Remark 15. Theorem 13 implies that $G_{m+\mathbf{r}}(\mathbf{x_r}/\mathring{\mathbf{y}}_\mathbf{r})=G_\mathbf{r}(\mathbf{x}/\mathring{\mathbf{y}})$, in analogy with the (defining) stability properties of stable double Grothendieck polynomials.

Remark 16. Theorem 13 gives an explicit combinatorial formula, but the characterization of the Demazure product $\delta(\mathbf{w})$ of a lacing diagram via Zelevinsky permutations would be more satisfying if it were intrinsic. That is, it would be better to identify those partial permutation lists that fit stripwise into a pipe dream with Demazure product $v(\mathbf{r})$ using the language of lacing diagrams, without referring to Zelevinsky permutations or pipe dreams. Such an intrinsic method appears in [BFR03].

6. Sign alternation

A permutation $\mu \in S_\infty$ is *Grassmannian* if it has at most one descent—that is, at most one index p such that $\mu(p) > \mu(p+1)$. A crucial property of arbitrary stable double Grothendieck polynomials, proved in [Buc02b, Theorem 6.13], is that every such polynomial $\hat{\mathcal{G}}_w(\mathbf{z}/\dot{\mathbf{z}})$ has a unique expression

$$\hat{\mathcal{G}}_w(\mathbf{z}/\dot{\mathbf{z}}) \;=\; \sum_{\text{Grassmannian } \mu} \alpha_w^\mu \hat{\mathcal{G}}_\mu(\mathbf{z}/\dot{\mathbf{z}})$$

as a sum of stable Grothendieck polynomials $\hat{\mathcal{G}}_\mu$ for Grassmannian permutations. If $\underline{\mu} = (\mu_1, \ldots, \mu_n)$ is a sequence of partial permutations such that the minimal completions $\widetilde{\mu}_1, \ldots, \widetilde{\mu}_n$ are Grassmannian, then let us call $\underline{\mu}$ a *Grassmannian* lacing diagram.

Corollary 17. *If $\alpha_{\mathbf{w}}^{\underline{\mu}} = \prod_{i=1}^n \alpha_{w_i}^{\mu_i}$ for lacing diagrams $\mathbf{w}$ and Grassmannian $\underline{\mu}$, then*

$$G_{\mathbf{r}}(\mathbf{x}/\mathring{\mathbf{y}}) \;=\; \sum_{\underline{\mu}} c_{\underline{\mu}}(\mathbf{r}) \hat{\mathcal{G}}_{\underline{\mu}}(\mathbf{x}/\mathring{\mathbf{y}})$$

for the constants
$$c_{\underline{\mu}}(\mathbf{r}) \;=\; \sum_{\mathbf{w} \in L(\mathbf{r})} (-1)^{l(\mathbf{w})-d(\mathbf{r})} \alpha_{\mathbf{w}}^{\underline{\mu}},$$

where the first sum above is over all Grassmannian lacing diagrams $\underline{\mu}$.

Proof. Expand the right hand side of Theorem 13 using $\hat{\mathcal{G}}_w = \sum_\mu \alpha_w^\mu \hat{\mathcal{G}}_\mu$. □

Let $G_{\mathbf{r}}(\mathbf{x}/\mathring{\mathbf{x}})$ be the specialization of the stable double quiver K-polynomial obtained by setting $\mathbf{y}^j = \mathbf{x}^j$ for $j = 0, \ldots, n$. Independently from Corollary 17, it follows from [Buc02a, Theorem 4.1] that the (ordinary) stable quiver K-polynomial

$$G_{\mathbf{r}}(\mathbf{x}/\mathring{\mathbf{x}}) \;=\; \sum_{\underline{\mu}} c_{\underline{\mu}}(\mathbf{r}) \hat{\mathcal{G}}_{\underline{\mu}}(\mathbf{x}/\mathring{\mathbf{x}})$$

is a sum of products of stable double Grothendieck polynomials $\hat{\mathcal{G}}_{\mu_j}(\mathbf{x}^{j-1}/\mathbf{x}^j)$ for Grassmannian permutations $\widetilde{\mu}_j$, with uniquely determined integer coefficients $c_{\underline{\mu}}(\mathbf{r})$. That these coefficients are the same as in Corollary 17 follows from the fact that the right side above determines the same element in the n^{th} tensor power of Buch's bialgebra Γ [Buc02b, Buc02a] as does the right side of the top formula in Corollary 17.

In addition to proving the expansion of $\hat{\mathcal{G}}_w$ as a sum of terms $\alpha_w^\mu \hat{\mathcal{G}}_\mu$, Buch showed in [Buc02b, Theorem 6.13] that the coefficients α_w^μ can only be nonzero if $l(\mu) \geq l(w)$, and he conjectured that the sign of α_w^μ equals $(-1)^{l(\mu)-l(w)}$. This was proved by Lascoux [Las01, Theorem 4] as part of his extension of "transition" from Schubert polynomials to Grothendieck polynomials. Since, as shown in [Buc02a, Section 5], the coefficients α_w^μ are special cases of the coefficients $c_{\underline{\mu}}(\mathbf{r})$, Lascoux's result is evidence for the following more general statement that was surmised by Buch (prior to [Las01]).

Theorem 18 ([Buc02a, Conjecture 4.2])**.** *The coefficients* $c_{\underline{\mu}}(\mathbf{r})$ *alternate in sign; that is,* $(-1)^{l(\underline{\mu})-d(\mathbf{r})}c_{\underline{\mu}}(\mathbf{r}) \geq 0$ *is a nonnegative integer.*

Proof. By [Las01, Theorem 4] the sign of $\alpha^{\underline{\mu}}_{\mathbf{w}}$ is $(-1)^{l(\underline{\mu})-l(\mathbf{w})}$. Thus the sign of $c_{\underline{\mu}}(\mathbf{r})$ is $(-1)^{l(\mathbf{w})-d(\mathbf{r})}(-1)^{l(\underline{\mu})-l(\mathbf{w})} = (-1)^{l(\underline{\mu})-d(\mathbf{r})}$, by the second formula in Corollary 17. □

References

[AD80] S. Abeasis and A. Del Fra, *Degenerations for the representations of a quiver of type* A_m, Boll. Un. Mat. Ital. Suppl. (1980) no. **2**, 157–171.

[BB93] Nantel Bergeron and Sara Billey, *RC-graphs and Schubert polynomials*, Experimental Math. **2** (1993), no. 4, 257–269.

[BF99] Anders Skovsted Buch and William Fulton, *Chern class formulas for quiver varieties*, Invent. Math. **135** (1999), no. 3, 665–687.

[BFR03] Anders S. Buch, László M. Fehér, and Richárd Rimányi, *Positivity of quiver coefficients through Thom polynomials*, preprint, 2003. arXiv:math.AG/0311203

[BKTY03] Anders S. Buch, Andrew Kresch, Harry Tamvakis, and Alexander Yong, *Grothendieck polynomials and quiver formulas*, 2003.

[Buc02a] Anders S. Buch, *Grothendieck classes of quiver varieties*, Duke Math. J. **115** (2002), no. 1, 75–103.

[Buc02b] Anders S. Buch, *A Littlewood–Richardson rule for the K-theory of Grassmannians*, Acta Math. **189** (2002), 37–78.

[Buc03] Anders S. Buch, *Alternating signs of quiver coefficients*, preprint, 2003.

[FK94] Sergey Fomin and Anatol N. Kirillov, *Grothendieck polynomials and the Yang–Baxter equation*, 1994, Proceedings of the Sixth Conference in Formal Power Series and Algebraic Combinatorics, DIMACS, pp. 183–190.

[FK96] Sergey Fomin and Anatol N. Kirillov, *The Yang-Baxter equation, symmetric functions, and Schubert polynomials*, Discrete Math. **153** (1996), no. 1–3, 123–143, Proceedings of the 5th Conference on Formal Power Series and Algebraic Combinatorics (Florence, 1993).

[KM03a] Allen Knutson and Ezra Miller, *Gröbner geometry of Schubert polynomials*, to appear in Ann. of Math (2), 2003.

[KM03b] Allen Knutson and Ezra Miller, *Subword complexes in Coxeter groups*, to appear in Adv. in Math., 2003. arXiv:math.AG/0309259

[KMS03] Allen Knutson, Ezra Miller, and Mark Shimozono, *Four positive formulae for type A quiver polynomials*, preprint, 2003. arXiv:math.AG/0308142

[Las01] A. Lascoux, *Transition on Grothendieck polynomials*, Physics and combinatorics, 2000 (Nagoya), 164–179, World Sci. Publishing, River Edge, NJ, 2001.

[LS82] Alain Lascoux and Marcel-Paul Schützenberger, *Structure de Hopf de l'anneau de cohomologie et de l'anneau de Grothendieck d'une variété de drapeaux*, C. R. Acad. Sci. Paris Sér. I Math. **295** (1982), no. 11, 629–633.

[Yon03] Alexander Yong, *On combinatorics of quiver component formulas*, preprint, 2003.

University of Minnesota, Minneapolis, Minnesota
E-mail address: ezra@math.umn.edu