Gröbner geometry of Schubert polynomials

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Abstract

Schubert polynomials, which a priori represent cohomology classes of Schubert varieties in the flag manifold, also represent torus-equivariant cohomology classes of certain determinantal loci in the vector space of $n \times n$ complex matrices. Our central result is that the minors defining these “matrix Schubert varieties” are Gröbner bases for any antidiagonal term order. The Schubert polynomials are therefore positive sums of monomials, each monomial representing the torus-equivariant cohomology class of a component (a scheme-theoretically reduced coordinate subspace) in the limit of the resulting Gröbner degeneration. Interpreting the Hilbert series of the flat limit in equivariant $K$-theory, another corollary of the proof is that Grothendieck polynomials represent the classes of Schubert varieties in $K$-theory of the flag manifold.

An inductive procedure for listing the limit coordinate subspaces is provided by the proof of the Gröbner basis property, bypassing what has come to be known as Kohnert’s conjecture [Mac91]. The coordinate subspaces, which are the facets of a simplicial complex, are in an obvious bijection with the rographs of Fomin and Kirillov [FK96b]. Thus our positive formula for Schubert polynomials agrees with (and provides a geometric proof of) the combinatorial formula of Billey-Jockusch-Stanley [BJS93]. Moreover, we shell this complex (as one of a new class of vertex-decomposable complexes we introduce), which shows that the initial ideal of the minors is a Cohen–Macaulay Stanley–Reisner ideal. This provides a new proof that Schubert varieties are Cohen–Macaulay.

The multidegree of any finitely generated multigraded module, defined here based on torus-equivariant cohomology classes, generalizes the usual $\mathbb{Z}$-graded degree to finer gradings. Part of the Gröbner basis theorem includes formulae for the multidegrees and Hilbert series of determinantal ideals in terms of Schubert and Grothendieck polynomials. In the special case of vexillary determinantal loci, which include all one-sided ladder determinantal varieties, the multidegree formulae are themselves determinantal, and our new antidiagonal Gröbner basis statement contrasts with known diagonal Gröbner basis statements.

Interpreting the Schubert polynomials as equivariant cohomology classes on matrices gives a topological reason (see also [FR01]) why Schubert polynomials are the characteristic classes for degeneracy loci [Ful92]: the mixing space

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construction of Borel that computes this equivariant cohomology is identified as the classifying space for maps between flagged vector bundles.

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1 Introduction

1.1 Summary of main results

1.1.1 Motivation

Combinatorialists have recognized for some time the intrinsic interest of the Schubert polynomials $\mathcal{S}_w$ of Lascoux and Schützenberger indexed by permutations $w \in S_n$, and are therefore producing a wealth of interpretations for their coefficients; see [Ber92], [Mac91, Appendix to Chapter IV, by N. Bergeron], [BJS93], [FK96b], [FS94], [Koh91], and [Win99]. Geometers, on the other hand, who take Schubert classes for granted, generally remain less convinced of the naturality of Schubert polynomials, even though they arise in certain universal geometric contexts [Ful92], and there are geometric proofs of positivity for their coefficients [BS00, Kog00].

Our primary motivation for undertaking this project was to provide a geometric context in which both (i) polynomial representatives for the Schubert classes $[X_w]$ in $H^*(\mathcal{F}_\ell_n)$ are uniquely singled out, with no choices other than a Borel subgroup of $GL_n$; and (ii) it is geometrically obvious that these representatives have nonnegative coefficients. The fact that these polynomials turn out to be the Schubert polynomials is a testament to their naturality. Here is the kernel of our idea.

We first undertake to replace topology on the flag manifold with multigraded commutative algebra, as follows. The ordinary cohomology $H^*(\mathcal{F}_\ell_n)$ of the manifold of flags in $\mathbb{C}^n$ is naturally isomorphic to the $B$-equivariant cohomology $H^*_B(GL_n)$ for the Borel subgroup $B$ inside $GL_n \mathbb{C}$ whose coeast space is $\mathcal{F}_\ell_n$. Furthermore, $H^*(\mathcal{F}_\ell_n)$ is naturally surjected onto the $B$-equivariant cohomology $H^*_B(M_n) \cong \mathbb{Z}[x_1, \ldots, x_n]$ for the $n \times n$ matrices $M_n$ over $\mathbb{C}$. We therefore have a geometric explanation for Borel’s presentation of $H^*(\mathcal{F}_\ell_n)$ as a quotient of a polynomial ring (Lemma A.5.2). In this presentation, the closure $\overline{X}_w \subset M_n$ of the preimage of the Schubert variety $X_w$ in $GL_n \mathbb{C}$ yields a $B$-equivariant cohomology class $[\overline{X}_w]_B$ on $M_n$, well-defined as a polynomial and mapping to $[X_w]$. Making use of the standard equality $H^*_B(M_n) = H^*_T(M_n)$ for the maximal torus $T \subset B$ allows us to work instead with the corresponding class $[\overline{X}_w]_T$ in the $T$-equivariant cohomology of $M_n$.

Let $W = \mathbb{Z}^n$ denote the weight lattice (character group) of our torus $T$. Working now with a $T$-action on a vector space $V = \mathbb{C}^m$ (thought of as $M_n$, with $m = n^2$), we can at this point replace the technology of equivariant cohomology by the more algebraically appealing notion of multidegree for subschemes of $V$ stable under the action of $T$. More algebraically, the coordinate ring of $V$, the polynomial ring $\mathbb{C}[z]$ in $m$ variables $z_1, \ldots, z_m$, becomes $\mathbb{Z}^n$-graded by virtue of the torus action, with the weight of $z_i$ being the weight of $T$ on the $i$th coordinate subspace of $V$. The multidegree is then an invariant associated to each graded ideal in $\mathbb{C}[z]$, taking values in the polynomial ring $\mathbb{Z}[x_1, \ldots, x_n] = \text{Sym}_2(W)$. We define the multidegree for a $T$-stable subscheme of $V$ as follows:

- If $V' \subseteq V$ is a (reduced) linear subspace, then its multidegree $|V'|$ is the product of the weights of $T$’s action on $V/V'$. In particular, when $V'$ is the zero set of $\langle z_i, \ldots, z_i \rangle$, its multidegree equals the product weight(z_i) \cdots weight(z_i).
• If $X$ is a (nonreduced) scheme supported on a subspace $V'$, and generically of length $l$, then $[X] = l [V']$.

• If $X = \bigcup X_i$ is a union of irreducible components $X_i$, then $[X] = \sum_i [X_i]$ where the sum runs over the components of top dimension.

• If there is a $T$-equivariant flat family connecting two subschemes $X$ and $Y$, then they have the same multidegree. In particular, if $X$ is the zero scheme of $I$ and $Y$ is the zero scheme of the initial ideal $\text{in}(I)$ for some term order, then $X$ and $Y$ have equal multidegrees.

Part of what we show is that this gives a well-defined multidegree to any $T$-invariant subscheme. In the case of affine cones, where $T = \mathbb{C}^*$ and $\mathbb{C}[z]$ carries its usual $\mathbb{Z}$-grading, this reduces to the ordinary notion of degree of the corresponding projective variety. As far as we know, this material is new, though nonetheless quite basic, being just an algebraic reformulation of the $T$-equivariant cohomology of affine space, or equivalently the Chow ring [Tot99, EG98]. Note that it works just as well over any field $k$, not necessarily algebraically closed or of characteristic zero.

Returning to the case of $M_n$, once we relax the condition of stability under the action of $B$ to just $T$, we can degenerate the matrix Schubert variety $X_w$ to a union $\mathcal{L}_w$ of linear subspaces of $M_n$ by finding a Gröbner basis for the ideal of $X_w$. The resulting flat deformation is $T$-equivariant and the limit is $T$-stable, so the multidegrees $[X_w]_T$ and $[\mathcal{L}_w]_T$ are equal. The class of a single $T$-invariant subspace is a monomial, so the positivity of coefficients in $[X_w]_T$ comes from the identity $[\mathcal{L}_w]_T = \sum_{L \in \mathcal{L}_w} [L]_T$ reflecting the additivity of multidegrees on irreducible components.

**Example 1.1.1** Let $w = 2143 \in S_4$ (all our permutations will be written in one-line, not cycle, notation). The matrix Schubert variety $X_w$ is then the set of $4 \times 4$ matrices $Z = (z_{ij})$ whose upper left entry is zero, and upper left $3 \times 3$ block is rank at most one. The equations defining $X_{2143}$ are the determinants

$$\langle z_{11}, \det \begin{vmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \\ z_{31} & z_{32} & z_{33} \end{vmatrix} = -z_{13} z_{22} z_{31} + \ldots \rangle.$$

Note that this is *not* a Gröbner basis with respect to term orders that pick out the diagonal term $z_{11} z_{22} z_{33}$ of the second generator, since $z_{11}$ divides that. The term orders that interest us pick out the antidiagonal term $-z_{13} z_{22} z_{31}$.

When we Gröbner-degenerate the matrix Schubert variety to the scheme defined by the initial ideal $\langle z_{11}, -z_{13} z_{22} z_{31} \rangle$, we get a union of three coordinate subspaces

$L_{11,13}, L_{11,22}$, and $L_{11,31}$, with ideals $\langle z_{11}, z_{13} \rangle, \langle z_{11}, z_{22} \rangle$, and $\langle z_{11}, z_{31} \rangle$.

The resulting equation in $T$-equivariant cohomology, or on multidegrees, reads:

$$[\overline{X}_w] = [L_{11,13}] + [L_{11,22}] + [L_{11,31}]$$

$$= \text{wt}(z_{11} z_{13}) + \text{wt}(z_{11} z_{22}) + \text{wt}(z_{11} z_{31})$$

$$= x_1^2 + x_1 x_2 + x_1 x_3.$$
in $\mathbb{Z}[x_1, x_2, x_3, x_4] \cong H^2_\Sigma(M_4)$. Pictorially, we can represent the subspaces $L_{11,13}$, $L_{11,22}$, and $L_{11,31}$ by drawing the $4 \times 4$ grid and placing a ‘+’ in each box containing a generator for its ideal—that is, place a ‘+’ at $(i,j)$ if every matrix in the subspace has $(i,j)$ entry zero:

$$\langle z_{11}, z_{13} \rangle = \begin{bmatrix} \hline + & + & & \\ \hline & + & + & \\ \hline & & + & \\ \hline & & & + \hline \end{bmatrix}, \quad \langle z_{11}, z_{22} \rangle = \begin{bmatrix} \hline + & & & \\ \hline + & + & & \\ \hline & + & + & \\ \hline & & & + \hline \end{bmatrix}, \quad \langle z_{11}, z_{31} \rangle = \begin{bmatrix} \hline + & & & \\ \hline & + & + & \\ \hline & & + & \\ \hline & & & + \hline \end{bmatrix}$$

In the figures below, we draw the zero entries ‘+’ by crossing pipes, and the nonzero entries by elbow joints (imagine that the lower-right triangle is filled with elbows).

These are the three “re-graphs”, or “planar histories”, for the permutation 2143, and we recover the combinatorial formula from [FK96b, BB93] for the Schubert polynomial $[\overline{X}_w]$ in the case $w = 2143$. 

1.1.2 Algebra

The Gröbner bases that arise during these computations have many applications. In general, the importance of Gröbner bases in recent work on determinantal ideals and their relatives, such as powers and symbolic powers, cannot be overstated. They are used in treatments of questions about Cohen–Macaulayness, rational singularities, multiplicity, dimension, $a$-invariants, and divisor class groups; see [CGG90, Stu90, HT92, Con95a, Con95b, MS96a, MS96b, CH97, BC98, GM00] for a sample. Since determinantal ideals as well as their Gröbner bases also arise in the study of (partial) flag varieties and their Schubert varieties (see Section 3.5 and [BL00, GL00, GL97, GM00], for instance), it is surprising to us that Gröbner bases for the determinantal ideals defining matrix Schubert varieties $\overline{X}_w$ do not seem to be in the literature, even though the ideals themselves appeared in [Ful92].

Our main results along these commutative algebraic lines are as follows. Consider the $\mathbb{Z}^n$-grading on the polynomial ring $k[z]$ in $n^2$ variables $(z_{ij})_{i,j=1}^n$ over a field $k$ in which $z_{ij}$ has weight $x_i$. The ideal $I_w$ of $\overline{X}_w$ is generated by determinants and therefore homogeneous for this grading. We prove:

(i) The multidegree of the ideal $I_w$ is $\mathfrak{S}_w(x_1, \ldots, x_n)$, the Schubert polynomial of Lascoux and Schützenberger.

(ii) The minors in $I_w$ form a Gröbner basis for any antidiagonal term order. This result is new even for the determinantal ideals we found in the literature, as they generally use diagonal term orders.
(iii) The $\mathbb{Z}^n$-graded Hilbert series of $k[z]/I_w$ is $G_w(x_1, \ldots, x_n)/\prod_{i=1}^n (1 - x_i)^n$, where
the numerator is the Grothendieck polynomial of Lascoux and Schützenberger.

(iv) The initial complex $L_w$ is a shellable ball or sphere, and hence Cohen–Macaulay.

The first statement is Theorem 1.4.3, and while quite close to results that are
well-known, our proof introduces some new techniques. The next two statements
appear in Theorem 2.1.2, and the last is Corollary 4.5.7 along with Corollary 4.6.8.
Formulae for the $\mathbb{Z}$-graded Hilbert series and degree follow respectively from (i) by
setting $(x_1, \ldots, x_n) = (1, \ldots, 1)$, and from (iii) by setting $(x_1,\ldots,x_n) = (t, \ldots, t)$.

It follows from (iv) that $\overline{X}_w$ is Cohen–Macaulay, and an easy consequence is a new
proof of the theorem of Ramanathan [Ram85] to the effect that Schubert varieties
$X_w \subseteq \mathcal{F}_n$ are Cohen–Macaulay (Corollary 4.5.8). Conversely, we show that the
product of $\overline{X}_w$ with a certain affine space embeds as an open subvariety of a Schubert
variety in $\mathcal{F}_N$ for some $N \geq n$ (Corollary 3.5.3), so the Cohen–Macaulayness of $\overline{X}_w$
follows from that of Schubert varieties. More generally, any local statement about
the class of all Schubert varieties in flag manifolds that remains valid after taking
products with affine spaces is equivalent to the corresponding statement for matrix
Schubert varieties (Theorem 3.5.4). Since rationality of singularities and normality
are among such local statements, $\overline{X}_w$ possesses these properties because Schubert
varieties do [Ram85, RR85].

Fulton used a similar (but different) argument in [Ful92] to derive the Cohen–
Macaulayness of $\overline{X}_w$, and could just as easily have concluded the normality and
rationality of singularities. We compare his methods to ours and others in the latter
parts of Section 3. In particular, we observe in Section 3.3 that Fulton’s notion of
essential set for a permutation and the resulting characterization of vexillary permutations
identifies the popular class of one-sided ladder determinantal varieties (see
the references above) as the class of vexillary matrix Schubert varieties (Proposition 3.4.1). Thus our results contain as special cases many known and some unknown
statements about one-sided ladder determinantal ideals. For instance, the multidegree
formula (i) becomes completely explicit, taking the form of a determinantal expression
(Theorem 3.4.2). As we mentioned in (ii), even the Gröbner basis theorem is new
for ladder determinantal ideals, because the term order differs from those appearing
in the literature (Section 3.5).

To prove (i)–(iv), we introduce a collection of geometric and combinatorial tools to
the study of determinantal ideals that we hope will be useful to algebrists in future
investigations. At present, “almost all of the approaches one can choose for the
investigation of determinantal rings use standard bitableaux and the straightening law” [BC00, p. 3], and are thus intimately tied to the Knuth–Robinson–Schensted
correspondence. However, our Gröbner basis statements above do not seem to yield
to KRS techniques (Example 3.3.5 and the discussion in Section 3.5 explain one
reason why). We find it necessary instead to rely on our algebraic reformulation of
equivariant cohomology and $K$-theory of flag manifolds and related spaces, as well as
on combinatorial methods involving rc-graphs, antidiagonals, Stanley–Reisner theory,
and a new class of shellable simplicial complexes defined using subwords of a fixed
word in a Coxeter group.

For example, we calculate the multidegree of \( \overline{X}_w \) using an inductive technique that, while based on calculations using localization of equivariant cohomology to torus-fixed points, is nonetheless entirely commutative algebraic, requiring no localization theorems. (An alternative calculation for the degree of \( \overline{X}_w \), in Appendix A.5, proceeds by comparing the ordinary cohomology of \( \mathcal{F} \ell_n \) to the equivariant cohomology of \( M_n \) for the torus action of left multiplication by diagonal invertible matrices.) The Hilbert series calculation for \( \mathcal{L}_w \) in Sections 2.3 and 2.4, which is fundamental to our proof of (ii), is really a computation in equivariant \( K \)-theory, as demonstrated by Corollary 3.2.1: \( G_w \) represents the \( K \)-theory class of \( X_w \) on \( \mathcal{F} \ell_n \) [LS82b, Las90]. In fact, though, the Hilbert series calculation proceeds via the combinatorics of the antidiagonals that generate the initial ideal (Sections 2.3, 2.4, and 2.5). The Stanley–Reisner theory in Section 1.2 then enables the multidegree calculation for \( \mathcal{L}_w \).

1.1.3 Combinatorics

The arguments of Section 4 involved in the proof of Cohen–Macaulayness of the initial complex (iv) actually benefit the combinatorics as much as they do the algebra and geometry. Our main results here include the following.

(v) The facets \( \mathcal{L}_w \) are in natural bijection with the rc-graphs for \( w \) of Fomin and Kirillov [FK96b].

(vi) Our positive formula for Schubert polynomials agrees with—and provides a new geometric proof/explanation of—the combinatorial formula of Billey, Jockusch, and Stanley [BJS93].

(vii) There is a simple combinatorial rule, called mitosis, to list all of the rc-graphs for a given permutation positively (i.e. without cancellation) by induction on the weak Bruhat order.

(viii) The simplicial complex \( \mathcal{L}_w \) is a subword complex in the Coxeter group \( S_n \), and is therefore a vertex-decomposable ball or sphere.

The rc-graphs for a permutation in \( S_n \) are subsets of \( [n] \times [n] \), and can be thought of as a generalization to flag manifolds of semistandard Young tableaux for Grassmannians, there being a natural bijection between tableaux and rc-graphs for Grassmannian permutations (as can be found in [Kog00], for instance).

Our argument for Theorem 4.2.11, which contains the statement of (v), uses the proof of (ii) along with combinatorial results of Bergeron and Billey [BB93]. We then prove (vi) as a consequence of (v) and (i). (Another proof of (vi), based on symplectic geometry, appears in [Kog00].)

The combinatorics that goes into our proof of the Gröbner basis theorem (ii) in Section 2 translates via (v) into (vii), which appears as Theorem 4.3.1. Mitosis serves as a geometrically motivated substitute for Kohnert’s conjecture [Koh91, Mac91, Win99]. To demonstrate its simplicity, we outline in Section 4.4 a self-contained combinatorial proof using rc-graphs and the formula of [BJS93], in addition to our earlier proof via antidiagonals and Gröbner bases.
The origin of (viii) was our desire, at the outset, to find some existing family of Cohen–Macaulay complexes containing $L_w$. The seeming absence of such a family in the literature motivated us instead to define here the new class of subword simplicial complexes, containing $L_w$ as a special case. Given a Coxeter group $\Pi$, there is a subword complex associated to each pair consisting of a word in $\Pi$ and an element of $\Pi$. The structures governing subword complexes relax the axioms for greedoids in a manner that still preserves the fundamental properties of the dual greedoid complex. In particular, Theorem 4.5.6 says that subword complexes are vertex-decomposable, and hence shellably Cohen–Macaulay. In fact, more is true: Theorem 4.6.7 says that subword complexes are homeomorphic to balls or spheres, and identify their boundary faces (if there are any).

Sections 3.1, 4.2, and 4.7 contain our main applications of the results in Section 2 to facts concerning Schubert and Grothendieck polynomials that are mostly known, but find here either generalizations or new proofs (or both). In addition to the motivational considerations already mentioned at the beginning of this Introduction, we find particularly intriguing the interaction with the Eagon-Reiner theorem [ER98] to $G_w$ in Section 4.7: in concert with the Gröbner basis theorem, the Alexander inversion formula (Proposition 4.7.2), and the Cohen–Macaulayness of $L_w$, it implies that the coefficients on each homogeneous piece of $G_w(1 - x)$ all have the same sign (Remark 4.7.5). We derive a combinatorial formula for these coefficients (Corollary 4.7.4) in terms of nonreduced expressions and compatible sequences, which we call pipe dreams (Definition 4.1.1). This formula, which agrees with that of Fomin and Kirillov [FK94], is a special case of a computation for subword complexes (Theorem 4.7.3) achieved by applying Theorems 4.5.6 and 4.6.7 along with Hochster’s theorem from Stanley-Reisner theory concerning Betti numbers for squarefree monomial ideals.

1.1.4 Geometry

Sections 3.1 and 3.2 relate our algebraic and combinatorial methods in Section 2 to the geometry of Schubert polynomials and Grothendieck polynomials. In particular, this is where we

(ix) explain the positivity of Schubert coefficients by identifying the intrinsic geometric objects counted by their coefficients; and

(x) recover the role of Grothendieck polynomials as the $K$-theory classes of structure sheaves of Schubert varieties on the flag manifold [LS82b].

We have already said a good deal about (ix). One interesting point about our derivation of (x) is that it requires no assumptions about the rational singularities of Schubert varieties: the multidegree proof of the Hilbert series calculation (iii), which is based on cohomological considerations that ignore phenomena at complex codimension 1 or more, automatically produces the $K$-classes as numerators of Hilbert series. Section 3.6 contains an explanation of how to

(xi) derive the connection between matrix Schubert varieties and Fulton’s theory of degeneracy loci for maps between flagged vector bundles.
This is due to the double appearance of $BB \times BB_+$ as the base space for $(B \times B_+)$-equivariant cohomology, and as the classifying space for pairs of flagged vector bundles. Borel’s mixing construction from the equivariant cohomology context, when applied to $X_w$, gives the universal degeneracy locus inside the universal Hom-bundle from the classifying space context. Since completing this work, we learned of the nice paper [FR01] and refer the reader to it for greater detail about the topological considerations and applications (as Section 3.6 is rather a departure from the commutative algebra framework underlying the rest of our exposition).

Everything we have to say about Lie groups concerns type $A$. While the definition of Schubert class is clear for arbitrary groups $G$, the definition of Schubert polynomial is less clear for groups other than $GL_n$ [FK96a], and we do not have anything to say here about their geometry. In particular, some flavors do not have positive coefficients, and so cannot arise from Gröbner degenerations to (automatically) effective classes.

**Prerequisites.** We have tried to make the material below as accessible as possible to combinatorialists, geometers, and commutative algebraists alike. Therefore, when there was a choice of whether to include or omit the reason for a basic fact, we usually included it. In particular, we have assumed no specific knowledge of the geometry or combinatorics of flag manifolds, or of Schubert varieties, Schubert polynomials, and Grothendieck polynomials. The remaining parts of this Introduction aim to fill in this background, as seen from our point of view using multidegrees.

Other background material can be found in the appendices on: equivariant $K$-theory, multigradings, and Hilbert series; the new algebraic machinery of multidegrees; the equivariant cohomology on which the intuition for the latter is based, in the situations to which we apply it; some background on the topology and algebraic geometry of flag manifolds, including a geometric argument for (i) and (ii) above; and finally, term orders, Gröbner bases, and initial ideals.

Many of our examples interpret the same underlying data in varying contexts, to highlight common themes. In particular this is true of Examples 2.1.4, 2.2.6, 2.2.7, 2.3.2, 2.3.7, 2.3.8, 3.3.4, 4.1.2, 4.1.7, 4.1.9, 4.2.3, 4.2.7, and 4.3.4.

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### 1.2 Multidegrees and squarefree monomial ideals

We review here some basic facts of Stanley–Reisner theory concerning squarefree monomial ideals. These facts then help us define and justify the notion of multidegree for the ideals and subschemes appearing in the main body of this monograph. The general introduction to multidegrees can be found in Appendix A.3.
Since Lemma 1.2.1 will be applied in Section 4.7 to squarefree monomial ideals in any number of variables, we begin by working with a $\mathbb{Z}^m$-graded polynomial ring $k[z] = k[z_1, \ldots, z_m]$, although the reader may assume $m = n^2$ for the applications to the antidiagonal ideals $J_w$ (introduced in Section 2.1). The field $k$ can have arbitrary characteristic, and can be assumed algebraically closed for convenience, although this hypothesis is unnecessary if one considers things like vector subspaces and orbits of group actions in their natural scheme-theoretic sense.

For notation, we write $\mathbb{Z}^m$-graded Hilbert series in the variables $z = z_1, \ldots, z_m$. The Hilbert series of any $\mathbb{Z}^m$-graded $k[z]$-module $\Gamma$ satisfies

$$H(\Gamma; z) = \frac{K(\Gamma; z)}{\prod_{i=1}^{m}(1 - z_i)}$$

for the Hilbert numerator Laurent polynomial $K(\Gamma; z)$, as in Proposition A.1.5 of the first appendix. For a coordinate subspace $L \subseteq k^m$, let $D_L \subseteq \{1, \ldots, m\}$ be the subset such that the ideal $\langle z_i \mid i \in D_L \rangle$ consists of those functions vanishing on $L$. Finally, set $z^D = \prod_{i \in D} z_i$ for $D \subseteq \{1, \ldots, m\}$.

**Lemma 1.2.1** Suppose that the squarefree monomial ideal $J$ has zero set $\mathcal{L}$, and that $k[z]/J$ has $\mathbb{Z}^m$-graded Hilbert numerator $K(z)$. If $\mathcal{C}(z)$ is the sum of lowest weight terms in the polynomial $K(1 - z)$ obtained from $K(z)$ by substituting $1 - z_i$ for $z_i$, then

$$\mathcal{C}(z) = \sum z^{D_L},$$

where the sum is over all subspaces $L \subseteq \mathcal{L}$ of maximal dimension.

By “the sum of lowest weight terms” here, we mean the homogeneous component (in the usual sense) of minimal total degree.

**Proof.** By definition, $J = \bigcap_{L \subseteq \mathcal{L}} \langle z_i \mid i \in D_L \rangle$. The Hilbert series of $k[z]/J$ is therefore the sum of all monomials outside every one of the ideals $\langle z_i \mid i \in D_L \rangle$ for coordinate subspaces $L \subseteq \mathcal{L}$. This sum is over the monomials $z^b$ for $b \in \mathbb{Z}^m$ having support exactly $\{1, \ldots, m\} \setminus D_L$ for some $L \subseteq \mathcal{L}$:

$$H(k[z]/J; z) = \sum_{L \subseteq \mathcal{L}} \prod_{i \not\in D_L} \frac{z_i}{1 - z_i} = \sum_{L \subseteq \mathcal{L}} \prod_{i \not\in D_L} (z_i) \prod_{i \in D_L} (1 - z_i).$$

(1)

After substituting $1 - z_i$ for $z_i$ in the last expression above, the lowest weight terms in the numerator correspond to the subspaces $L \subseteq \mathcal{L}$ of highest dimension. □

Many of our applications concern gradings that aren’t as fine as the $\mathbb{Z}^m$-grading. Specifically, let $m = n^2$, so that $z = (z_{ij})_{i,j=1}^{n}$ and $k[z]$ is the coordinate ring of the $n \times n$ matrices $M_n$ over $k$. Consider the following two gradings on $k[z]$, in addition to the $\mathbb{Z}^{n^2}$-grading:

- the $\mathbb{Z}^n$-grading in which the weight of the variable $z_{ij}$ is defined by $\text{wt}(z_{ij}) = x_i$, where $x = x_1, \ldots, x_n$ is the standard basis of $\mathbb{Z}^n$; and
the $\mathbb{Z}^{2n}$-grading in which $\text{wt}(z_{ij}) = x_i - y_j$, where $y = y_1, \ldots, y_n$ is the standard basis for another copy of $\mathbb{Z}^n$.

In the next subsection, where we justify geometrically why we consider these gradings, we shall also have occasion to consider the $\mathbb{Z}^{2n}$-graded polynomial ring $k[z][u]$ in $n^2 + 1$ variables, where the extra variable $u$ has weight $x_i - x_{i+1}$.

Consider one of the polynomial rings $R = k[z]$ or $k[z][u]$, with one of the multigradings above. These are all positively graded (as in Appendix A.1), so the graded pieces of $R$ have finite dimension. We write multigraded Hilbert series of multigraded quotients of $R$ using the $x$ and $y$ variables, so that (for instance), the Hilbert series of a $\mathbb{Z}^{2n}$-graded quotient $R/I$ of $R = k[z][u]$ always has the form

$$H(R/I; x, y) = \frac{\mathcal{K}(R/I; x, y)}{(1 - x_i/x_{i+1}) \prod_{i,j=1}^{n^2}(1 - x_i/y_j)}.$$

The Hilbert numerator $\mathcal{K}(R/I; x, y)$ here is a Laurent polynomial, while the factors in the denominator correspond to the variables in $R$, which is $k[z][u]$ in this case. Note that the weights of the variables show up in the denominator in exponential form: the exponential weight $x_i/y_j$ corresponds to the variable $z_{ij}$ of weight $x_i - y_j$.

Transition between the exponential and ordinary forms of the weights can be accomplished by analogy with the usual exponential function: expanding $1 - e^{-x}$ as a power series in $x$ and taking lowest degree terms yields back $x$ again. (This analogy is explained in Remark A.5.1 as an algebraic reflection of the Poincaré homomorphism and the Riemann–Roch isomorphism of cohomology with the associated graded ring of $K$-theory.) In the case of an arbitrary Laurent polynomial $\mathcal{K}(x, y)$, the rational function $\mathcal{K}(1 - x, 1 - y)$, in which each $x_i$ has been replaced by $1 - x_i$ and each $y_j$ by $1 - y_j$, can be expanded as a well-defined power series in $x$ and $y$.

**Definition 1.2.2** Let $R = k[z]$ or $k[z][u]$ as above. The $\mathbb{Z}^n$-graded or $\mathbb{Z}^{2n}$-graded multidegree of a respectively $\mathbb{Z}^n$-graded or $\mathbb{Z}^{2n}$-graded quotient $R/I$ is the sum $\langle \text{Spec}(k[z]/I) \rangle_{\mathbb{Z}^n}$ or $\langle \text{Spec}(R/I) \rangle_{\mathbb{Z}^{2n}}$ of all lowest weight terms in the substituted Laurent polynomial $\mathcal{K}(k[z]/I; 1 - x)$ or $\mathcal{K}(R/I; 1 - x, 1 - y)$.

The brackets are supposed to suggest some sort of cohomology class; see Remark 1.2.8. The subscript $\mathbb{Z}^n$ might be dropped, if the grading is either clear from context or irrelevant. It will sometimes be useful to have the variables $x$ and possibly $y$ explicitly in the notation. When required, we will instead write

$$\langle \text{Spec}(k[z]/I) \rangle_{\mathbb{Z}^n} = \mathcal{C}(k[z]/I; x) \quad \text{or} \quad \langle \text{Spec}(R/I) \rangle_{\mathbb{Z}^{2n}} = \mathcal{C}(R/I; x, y).$$

**Example 1.2.3** The Hilbert numerator of $k = k[z]/\langle z_{ij} \mid i, j = 1, \ldots, n \rangle$ is the universal denominator: $\mathcal{K}(k; x, y) = \prod_{i,j=1}^{n^2}(1 - x_i/y_j)$. The multidegree of $k$ is the product $\mathcal{C}(k; x, y) = \prod_{i,j=1}^{n^2}(x_i - y_j)$. The $\mathbb{Z}^n$-graded versions of these calculations, $\mathcal{K}(k; x) = \prod_{i=1}^{n^2}(1 - x_i)^n$ and $\mathcal{C}(k; x) = \prod_{i=1}^{n^2} x_i^n$, are obtained from the $\mathbb{Z}^{2n}$ versions by setting all of the $y$ variables to 1 and 0, respectively.
If a subspace \( L \subseteq M_n \) is the zero set of \( \langle z_{ij} \mid (i, j) \in D_L \rangle \), then its multidegree is

\[
[L] = \prod_{(i,j) \in D_L} \text{wt}(z_{ij}).
\]

The absence of the \( \mathbb{Z}^n \) subscript is useful here, as (2) holds for any multigrading. □

The next two results generalize the observations in the previous example to the point where they become useful in later sections; the second is in fact applied directly in the proof of Theorem 2.1.2.

**Lemma 1.2.4** If \( I \subseteq k[z] \) is a \( \mathbb{Z}^{2n} \)-graded ideal, then \( I \) is also \( \mathbb{Z}^n \)-graded. The \( \mathbb{Z}^n \)-graded and \( \mathbb{Z}^{2n} \)-graded Hilbert series and multidegrees of \( k[z]/I \) satisfy

\[
H(k[z]/I; x) = H(k[z]/I; x, 1) \quad \text{and} \quad \mathcal{C}(k[z]/I; x) = \mathcal{C}(k[z]/I; x, 0).
\]

**Proof.** The statement about Hilbert series follows by thinking of them as generating functions for the number of monomials having given weight. The multidegree statement is an easy consequence, as it can be checked on a single Laurent monomial. □

**Proposition 1.2.5** If some squarefree monomial ideal has zero set \( \mathcal{L} \), then

\[
[L] = \sum_{\substack{L \subseteq \mathcal{L} \\ \dim L = \dim \mathcal{L}}} [L]
\]

in any grading; i.e. sum expressions (2) over subspaces \( L \subseteq \mathcal{L} \) of maximal dimension.

**Proof.** Translate the Hilbert series (1) into the \( m = n^2 \) notation, and use the \( \mathbb{Z}^n \)- or \( \mathbb{Z}^{2n} \)-grading. Now substitute \( x \mapsto 1 - x \) and \( y \mapsto 1 - y \) and calculate. □

Proposition 1.2.5 is easier than the corresponding general result, Proposition A.3.3, precisely because we have an explicit formula for the Hilbert series in (1). It justifies the geometric content of multidegree for monomial ideals by taking initial ideals:

**Corollary 1.2.6** If the multigraded ideal \( I \subseteq k[z] \) of a subscheme \( X \subseteq M_n \) has initial ideal \( J \) for some term order, then \( \mathcal{C}(k[z]/I; x, y) = \mathcal{C}(k[z]/J; x, y) \). In particular, if \( J \) is a squarefree monomial ideal with zero set \( \mathcal{L} \), then

\[
[X] = \sum_{\substack{L \subseteq \mathcal{L} \\ \dim L = \dim \mathcal{L}}} [L].
\]

**Proof.** Hilbert series, and thus multidegrees, are preserved by taking intial ideals. □

There is one final result we need concerning multidegrees. Unfortunately, its proof requires the full machinery of Appendix A.3, and thus fails to fit the flow of this introductory section. Therefore, we reluctantly banish its proof to Corollary A.3.9. The result is used twice in the main text, in the proofs of Theorem 1.4.3 and Lemma 2.1.5. For all of our other applications, Corollary 1.2.6 suits us better.
Proposition 1.2.7 Let $R = k[z][u]$ or $k[z]$, and $I \subseteq R$ is a $\mathbb{Z}^{2n}$-graded ideal. If $I_1, \ldots, I_r$ are the top-dimensional primary components of $I$, then

$$C(R/I; x, y) = \sum_{\ell=1}^r C(R/I_\ell; x, y).$$

Remark 1.2.8 The letters $C$ and $K$ throughout this text stand for ‘cohomology’ and ‘$K$-theory’. When $k = \mathbb{C}$ is the complex numbers, the (Laurent) polynomials they denote are honest torus-equivariant cohomology and $K$-classes on affine space. Since affine space is equivariantly contractible, the equivariant cohomology and $K$-theory rings are naturally polynomial and Laurent polynomial rings over $\mathbb{Z}$, respectively. Had we defined the $\mathbb{Z}$-graded multidegree (which we don’t need here, surprisingly, but appears in the Appendix), we would see that the $\mathbb{Z}$-graded multidegree of a $\mathbb{Z}$-graded ideal equals the usual degree of its corresponding projective scheme times $t^{\text{odd}}$. See Appendix A.3 for details. In particular, see Remark A.5.1 for the history behind the odd practice of using $x_i$ as both an exponential and ordinary variable. 

1.3 Matrix Schubert varieties

Let $M_n$ be the $n \times n$ matrices over $k$, with coordinate ring $k[z]$ in indeterminates $\{z_{ij}\}_{i,j=1}^n$. Throughout the paper, $q$ and $p$ will be integers with $1 \leq q, p \leq n$, and $Z$ will stand for an $n \times n$ matrix. Most often, $Z$ will be the generic matrix of variables $(z_{ij})$, although occasionally $Z$ will be an element of $M_n$. Denote by $Z_{[q,p]}$ the northwest $q \times p$ submatrix of $Z$. Given a permutation $w \in S_n$, for instance, we find that

$$\text{rank}(w_{[q,p]}^T) = \#\{(i, j) \leq (q, p) \mid w(i) = j\}$$

is the number of 1’s in the submatrix $w_{[q,p]}^T$ (recall that the 1’s lie at $w_{i,w(i)}^T$).

The following definition was made by Fulton in [Ful92].

Definition 1.3.1 Let $w \in S_n$. The matrix Schubert variety $X_w \subseteq M_n$ consists of the matrices $Z \in M_n$ such that $\text{rank}(Z_{[q,p]}) \leq \text{rank}(w_{[q,p]}^T)$ for all $q, p$.

Example 1.3.2 The smallest matrix Schubert variety is $X_w$, where $w_0$ is the long permutation $n \cdots 2 1$ reversing the order of $1, \ldots, n$. The variety $X_{w_0}$ is just the linear subspace of lower-right-triangular matrices whose ideal is $\langle z_{ij} \mid i + j \leq n \rangle$. 

Example 1.3.3 Five of the six $3 \times 3$ matrix Schubert varieties are linear subspaces:

- $X_{123} = M_3$
- $X_{213} = \{Z \in M_3 \mid z_{11} = 0\}$
- $X_{231} = \{Z \in M_3 \mid z_{11} = z_{12} = 0\}$
- $X_{312} = \{Z \in M_3 \mid z_{11} = z_{21} = 0\}$
- $X_{321} = \{Z \in M_3 \mid z_{11} = z_{21} = z_{12} = 0\}$
The last one, $\overline{X}_{132}$, is the set of matrices

$$\overline{X}_{132} = \{ Z \in M_3 \mid z_{11}z_{22} - z_{12}z_{21} = 0 \}$$

whose upper left 2 × 2 block is singular. \hfill \Box

Matrix Schubert varieties are clearly stable under rescaling any row or column. Moreover, since we only impose rank conditions on submatrices that are as far north and west as possible, any operation that adds a multiple of some row to a row below it (“sweeping downward”), or that adds a multiple of some column to another column to its right (“sweeping to the right”) preserves every matrix Schubert variety.

These observations have useful group-theoretic restatements. Let $B$ denote the group of invertible lower triangular matrices and $B_+$ the invertible upper triangular matrices, which intersect in the invertible diagonal matrices $T = B \cap B_+$. The previous paragraph says exactly that each matrix Schubert variety $\overline{X}_w$ is preserved by the action\(^1\) of $B \times B_+$ on $M_n$ in which $(b, b_+) \cdot Z = bZb_+^{-1}$. Proposition 1.3.8 will actually say much more.

The next three lemmas, which we require for Proposition 1.3.8, are basically standard results on Bruhat order for $S_n$, enhanced slightly for $M_n$ instead of $GL_n$; we provide proofs to remain completely self-contained. Recall that a partial permutation matrix $Z \in M_n$ has at most one 1 in each row and column, and all remaining entries equal to zero.

**Lemma 1.3.4** In each $B \times B_+$ orbit on $M_n$ lies a unique partial permutation matrix.

*Proof.* By doing row and column operations that sweep down and to the right, we can get from an arbitrary matrix $Z'$ to a partial permutation matrix $Z$. Such sweeping preserves the ranks of northwest $q \times p$ submatrices, and $Z$ can be reconstructed uniquely by knowing only rank($Z'_{[q,p]}$) for $1 \leq q, p \leq n$. \hfill \Box

By definition, the length of a partial permutation matrix $Z$ is the number of zeros in $Z$ that lie neither due south nor due east of a 1. In other words, for every 1 in $Z$, cross out all the boxes beneath it in the same column as well as to its right in the same row, and count the number of uncrossed-out boxes to get the length of $Z$. When $Z = w^T$ is a permutation matrix, define length($w$) to be the length of $w^T \in M_n$. Write $Z \subseteq Z'$ for partial permutation matrices $Z$ and $Z'$ if the 1’s in $Z$ are a subset of the 1’s in $Z'$. Also, to simplify notation, write $|Z_{[q,p]}|$ instead of rank($Z_{[q,p]}$). Finally, let $t_{i,i'} \in S_n$ be the transposition switching $i$ and $i'$.

**Lemma 1.3.5** Fix a permutation $w \in S_n$ and a partial permutation matrix $Z$. If $Z \in \overline{X}_w$ and length($Z$) ≤ length($w$), then either $Z \subseteq w^T$, or there is a transposition $t_{i,i'}$ such that $v = wt_{i,i'}$ satisfies: $Z \in \overline{X}_v$ and length($v$) > length($w$).

---

\(^1\)This is a left group action, in the sense that $(b, b_+) \cdot ((b', b_+) \cdot Z)$ equals $(b, b_+) \cdot (b', b_+) \cdot Z$ instead of $((b, b_+) \cdot (b', b_+) \cdot Z$, even though—in fact because—the $b_+$ acts via its inverse on the right.
Proof. The proof is by downward induction on the number of 1’s shared by $w^T$ and $Z$, the case where this number is $n$ (so $Z = w^T$) being trivial. Let $i$ be the first row in which $w^T$ has a 1 where $Z$ does not, and let $j = w(i)$ be the column index of this 1. Comparing rank conditions along row $i$, this means that either (a) row $i$ of $Z$ is blank, or (b) the 1 in row $i$ of $Z$ lies in some column $j'$ to the right of column $j$.

In case (b), choose $i'$ satisfying $w(i') = j'$, so that $w^T$ has a 1 at $(i', j')$. Noting that $i < i'$ by minimality of $i$, switch rows $i$ and $i'$ of $w^T$ to get $v^T$, so that $v = wt_{i,i'}$. We leave to the reader the standard check of the following statement.

Claim 1.3.6 Suppose $Z$ is a partial permutation matrix with 1’s at $(i, j)$ and $(i', j')$, where $(i, j) \leq (i', j')$. Switching rows $i$ and $i'$ of $Z$ creates a partial permutation matrix $Z'$ satisfying length($Z'$) = length($Z$) + 1 + twice the number of 1’s strictly inside the rectangle enclosed by $(i, j)$ and $(i', j')$. \hfill \square

In case (a), one of two things can happen: (a') there is some spot $(i', j')$ strictly southeast of $(i, j)$ such that $w(i') = j'$; or (a'') not. In case (a'), choose $i'$ minimal and switch rows $i$ and $i'$ of $w^T$—that is, take $v = wt_{i,i'}$. Compare ranks to show $Z \in \overline{X}_v$:

- In any spot $(q, p)$ not satisfying $(i, j) \leq (q, p) \leq (i'-1, j'-1)$, we have $|Z_{(q,p)}| \leq |w_{(q,p)}^T| \leq |v_{(q,p)}^T|$, the first inequality by definition and the second by construction.

- Setting $q = i$ and assuming $j \leq p \leq j'-1$ yields $|Z_{[i,p]}| = |Z_{[i-1,p]}| \leq |w_{[i-1,p]}^T| = |w_{[i,p]}^T| - 1 = |v_{[i,p]}^T|$. 

- If $i + \ell = q \leq i'-1$ for some $\ell \geq 0$, and $j \leq p \leq j'-1$, then $|Z_{[q,p]}| \leq |Z_{[i,p]}| + \ell \leq |w_{[i,p]}^T| + \ell = |w_{[q,p]}^T|$. The second inequality is the previous item, and the last equality is because all the 1’s in rows strictly between $i$ and $i'$ of $w^T$ lie to the left of column $j$, by minimality of $i'$.

In case (a''), we may replace $Z$ by the matrix $Z'$ obtained by adding a 1 to the $(i, j)$ spot of $Z$. The new matrix $Z'$ still lies in $\overline{X}_v$, by checking the number of 1’s in $Z_{[q,p]}$:

- If $p < j$ or $q < i'$ then $Z_{[q,p]} = Z'_{[q,p]}$; that is, $Z$ and $Z'$ agree strictly north of row $i$ and strictly west of column $j$.

- Assuming $q \geq i'$ and setting $p = j$, we get $|Z_{[q,j]}| = |Z_{[q,j-1]}| \leq |w_{[q,j-1]}^T| = |w_{[q,j]}^T| - 1$, and it follows that $|Z'_{[q,j]}| \leq |w_{[q,j]}^T|$. 

- If $q \geq i'$ and $p = j + \ell$ for some $\ell \geq 0$, then $|Z_{[q,p]}'| \leq |Z_{[q,j]}'| + \ell \leq |w_{[q,j]}^T| + \ell = |w_{[q,p]}^T|$, where the last equality is because all the 1’s in columns east of $j$ in $w^T$ lie north of row $i$.

As $Z'$ shares more 1’s with $w^T$ than $Z$, induction completes the proof. \hfill \square

Lemma 1.3.7 Let $Z$ be a partial permutation matrix with orbit closure $\overline{O}_Z$ in $M_n$. If length($s_iZ$) < length($Z$), then

$$n^2 - \text{length}(s_iZ) = \dim(\overline{O}_{s_iZ}) = \dim(\overline{O}_Z) + 1 = n^2 - \text{length}(Z) + 1,$$

and $s_i(\overline{O}_{s_iZ}) = \overline{O}_{s_iZ}$. 

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Proof. Let \( P_i \subseteq GL_n \) be the \( i \)th parabolic subgroup containing \( B \), in which the only non-zero entry outside the lower triangle may lie at \( (i, i + 1) \). Consider the image \( Y \) of the multiplication map \( P_i \times \overline{O}_Z \to M_n \) sending \( (p, x) \mapsto p \cdot x \). This map factors through the quotient \( B\backslash(P_i \times \overline{O}_Z) \) by the diagonal action of \( B \), which is an \( \overline{O}_Z \)-bundle over \( P_i/B \cong \mathbb{P}^1 \) and hence has dimension \( \dim(\overline{O}_Z) + 1 \). Thus \( \dim(Y) \leq \dim(\overline{O}_Z) + 1 \).

The variety \( Y \) is \( B \times B_+ \)-stable by construction. Since \( B \times B_+ \) has only finitely many orbits on \( M_n \), the irreducibility of \( Y \) implies that \( Y \) is an orbit closure \( \overline{O}_Z \) for some partial permutation matrix \( Z' \), by Lemma 1.3.4. Clearly \( \overline{O}_Z \subseteq Y \), and \( s_iZ \in Y \), so the dimension bound will imply \( Z' = s_iZ \) as soon as we show that \( Z \in \overline{O}_{s_iZ} \).

By checking all possible placements of 1’s in rows \( i \) and \( i + 1 \) of \( s_iZ \), the hypothesis length \( s_iZ < \) length \( Z \) means either that \( s_iZ \) has all zeros in row \( i + 1 \) and a 1 somewhere in row \( i \), or that \( s_iZ \) has 1’s at \( (i, j) \) and \( (i + 1, j') \) with column indices \( j < j' \). In the \( 2 \times 2 \) case, we have

\[
\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
t & 0 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
t & 0 \\
0 & 1
\end{pmatrix} =
\begin{pmatrix}
t & 0 \\
0 & 1
\end{pmatrix}
\]

and

\[
\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
t & 1 \\
0 & -t^{-1}
\end{pmatrix}
\begin{pmatrix}
t & 1 \\
0 & -t^{-1}
\end{pmatrix} =
\begin{pmatrix}
t & 1 \\
0 & -t^{-1}
\end{pmatrix}
\]

View these equations as occurring in the two rows \( i, i + 1 \) and the two columns \( j, j' \) in equations \( b(t) \cdot s_iZ \cdot b_+(t)^{-1} = Z(t) \). Inserting the appropriate extra identity rows and columns into \( b(t) \) and \( b_+(t)^{-1} \) while completing the middle matrix to \( s_iZ \), yields, in each equation, a 1-parameter family of matrices in \( \mathcal{O}_Z = Bs_iZB_+ \). The limit at \( t = 0 \) is \( Z \) in both cases, as seen from the right hand sides.

That \( \dim(\overline{O}_Z) = n^2 - \text{length}(Z) \) follows by direct calculation whenever \( Z \subseteq id_n \) has 1’s only along the main diagonal. Every other partial permutation matrix can be obtained from a diagonal one by repeatedly switching adjacent rows. After fixing the number of 1’s, induction on length \( (Z) \) proves the formula.

The argument above shows that \( \overline{O}_{s_iZ} = Y \) is stable under multiplication by \( P_i \) on the left. In particular, \( s_i \in P_i \) takes \( \overline{O}_{s_iZ} \) to itself. \( \square \)

The next result appears in [Ful92], where it is derived from the corresponding result on flag manifolds. Again, we prove it here to keep the exposition self-contained.

Proposition 1.3.8 The matrix Schubert variety \( \overline{X}_w \) is the closure \( \overline{Bw^T}B_+ \) of the \( B \times B_+ \) orbit on \( M_n \) through the permutation matrix \( w^T \). Thus \( \overline{X}_w \) is irreducible of dimension \( n^2 - \text{length}(w) \), and \( w^T \) is a smooth point of it.

Proof. The stability of \( \overline{X}_w \) under \( B \times B_+ \), which we explained earlier, means that \( \overline{X}_w \) is a union of orbits. By Lemma 1.3.4 and the obvious containment \( \overline{O}_{w^T} \subseteq \overline{X}_w \) (sweeping down and right preserves northwest ranks), it therefore suffices to show that any partial permutation matrix \( Z \) lying in \( \overline{X}_w \) lies also in \( \overline{Bw^T}B_+ \).

Scaling the rows independently, we see that the \( B \times B_+ \) orbit through any partial permutation matrix contains all of the so-called “monomial matrices” supported on it,
in which the ‘1’ entries can be replaced by arbitrary elements of \( \mathbf{k}^* \). In particular, 
\( Z \) lies in \( \overline{Bw^TB} \) if \( Z \subseteq w^T \). On the other hand, if \( Z \not\subseteq w^T \) but still \( Z \in \overline{X_w} \), then Lemma 1.3.5 produces a permutation \( v = wt_{i'j} \) for some \( i < i' \) such that \( Z \in \overline{X_v} \) and length(\( v \)) > length(\( w \)). It is enough to show that \( v^T \in \overline{Bw^TB} \).

View (3) as the two rows \( i, i' \) and the two columns \( w(i), w(i') \) in an equation
\[ b(t) \cdot w^T \cdot b_+(t)^{-1} = v(t). \]
Inserting the appropriate extra identity rows and columns into \( b(t) \) and \( b_+(t)^{-1} \) while completing the middle matrix to \( w^T \) yields a 1-parameter family of matrices in \( \overline{Bw^TB} \), whose limit at \( t = 0 \) is \( v^T \).

The last sentence of the Proposition is standard for orbit closures, except for the
dimension count, which comes from Lemma 1.3.7.

Throughout the rest of Sections 1 and 2, we shall repeatedly invoke the hypothesis
\[ \text{length}(ws_i) < \text{length}(w). \]
In terms of permutation matrices, this means that \( w^T \) differs from \((ws_i)^T\) only in rows \( i \) and \( i + 1 \), where they look heuristically like
\begin{align*}
\begin{array}{cccccccc}
& & & & & w(i) & & \\
& & & & & \downarrow & & \\
& \cdots & & 1 & & \cdots & & \\
i & & & & & & & \\
& & & & & \uparrow & & \\
& \cdots & & 1 & & \cdots & & \\
i + 1 & & & & & & & \\
\end{array}
\end{align*}
\begin{align*}
\begin{array}{cccccccc}
& & & & & w(i+1) & & \\
& & & & & \downarrow & & \\
& \cdots & & 1 & & \cdots & & \\
i & & & & & & & \\
& & & & & \uparrow & & \\
& \cdots & & 1 & & \cdots & & \\
i + 1 & & & & & & & \\
\end{array}
\end{align*}

between columns \( w(i + 1) \) and \( w(i) \). Since reversing the inequality in \( \text{length}(ws_i) < \text{length}(w) \) makes so much difference, we always write the hypothesis this way, for consistency, even though we may actually use one of the following equivalent formulations in any given lemma or proposition. We hope that collecting this list of standard—and at this point almost trivially equivalent—statements (“shorter permutation \( \Leftrightarrow \) bigger variety”) will prevent the reader from stumbling on this subtlety as many times as we did. In particular, the string of characters ‘\( \text{length}(ws_i) < \text{length}(w) \)’ can serve as a visual cue to this frequent assumption; we shall never assume the opposite inequality.

**Corollary 1.3.9** The following are equivalent for a permutation \( w \in S_n \).

1. \( \text{length}(ws_i) < \text{length}(w) \).
2. \( \text{length}(ws_i) = \text{length}(w) - 1 \).
3. \( w(i) > w(i + 1) \).
4. \( ws_i(i) < ws_i(i + 1) \).
5. \( I(\overline{X_{ws_i}}) \subseteq I(\overline{X_w}) \).
6. \( \dim(\overline{X_{ws_i}}) > \dim(\overline{X_w}) \).
7. \( \dim(\overline{X_{ws_i}}) = \dim(\overline{X_w}) + 1 \).
8. \( s_i\overline{X_w} \neq \overline{X_w} \).
9. \( s_i\overline{X_{ws_i}} = \overline{X_{ws_i}} \).
10. \( \overline{X_{ws_i}} \supset \overline{X_w} \).

Here, the transposition \( s_i \) acts on the left of \( M_n \), switching rows \( i \) and \( i + 1 \).

**Proof.** The equivalence of 1–4 comes from Claim 1.3.6. The equivalence of these with
5–7 and 10 uses Proposition 1.3.8 and Lemma 1.3.5. Finally, 8–10 are equivalent by
Lemma 1.3.7 and its proof. \( \Box \)
1.4 Multidegrees of matrix Schubert varieties

This subsection provides a proof of the recursion satisfied by the multidegrees of matrix Schubert varieties, stated in Theorem 1.4.3.

Let \( T \times T^{-1} \) be the 2n-dimensional torus inside \( B \times B_+ \). We write \( T^{-1} \) for the right factor to indicate that it acts by inverse multiplication, and to distinguish it from the left factor. Note, however, that \( T^{-1} \) is the same torus \( B \cap B_+ \) as the left factor, but with a different action. We use the left factor much more often and call it simply \( T \).

Using the action of \( T \times T^{-1} \), we can speak of \( \mathbb{Z}^{2n} \)-graded multidegrees for subschemes of \( M_n \). Let \( x_i : T \to k^* \) be the character that picks out the \((i, i)\) entry of every diagonal matrix, so \( \{x_i\}_{i=1}^n \) constitutes the standard basis for the weight lattice \( \mathbb{Z}^n \) of \( T \). Likewise let \( \{y_i\}_{i=1}^n \) be the standard basis for the weight lattice of \( T^{-1} \). Then \( \mathbb{Z}^{2n} \)-graded multidegrees live in \( \mathbb{Z}[x, y] := \mathbb{Z}[x_1, \ldots, x_n, y_1, \ldots, y_n] \), with the weight of \( z_{ij} \) being \( x_i - y_j \). When we speak of \( \mathbb{Z}^n \)-graded multidegrees for subschemes of \( M_n \), we always mean the grading by virtue of the \( T \) action, in which \( z_{ij} \) has weight \( x_i \).

**Example 1.4.1** The matrix Schubert variety \( \overline{X}_{w_0} \) from Example 1.3.2 has \( \mathbb{Z}^{2n} \)-graded multidegree \( \prod_{i+j \leq n} (x_i - y_j) \). If one only wants to consider the left half of the torus action, the \( \mathbb{Z}^n \)-multidegree is \( \prod_{i=1}^n x_i^{n-i} \), obtained from the previous formula by setting \( y_j \) to 0 for all \( j \), as per Lemma 1.2.4. \( \square \)

**Example 1.4.2** Five of the six \( 3 \times 3 \) matrix Schubert varieties in Example 1.3.3 have \( \mathbb{Z}^{2n} \)-multidegrees that are products of expressions having the form \( x_i - y_j \) as in (2):

\[
\begin{align*}
[\overline{X}_{123}] &= 1 \\
[\overline{X}_{213}] &= x_1 - y_1 \\
[\overline{X}_{231}] &= (x_1 - y_1)(x_1 - y_2) \\
[\overline{X}_{312}] &= (x_1 - y_1)(x_2 - y_1) \\
[\overline{X}_{321}] &= (x_1 - y_1)(x_1 - y_2)(x_2 - y_1)
\end{align*}
\]

The last one, \( \overline{X}_{132} \), has a multidegree

\[
\overline{X}_{132} = x_1 + x_2 - y_1 - y_2
\]

that can be written as a sum of expressions \((x_i - y_j)\) in two different ways. To see why, pick term orders that choose different leading monomials for \( z_{11}z_{22} - z_{12}z_{21} \). Geometrically, these degenerate \( \overline{X}_{132} \) either to the scheme defined by \( z_{11}z_{22} \) or to the scheme defined by \( z_{12}z_{21} \), while preserving the multidegree in both cases. The degenerate limits break up as unions

\[
\begin{align*}
\overline{X}_{132} &= \{Z \mid z_{11} = 0\} \cup \{Z \mid z_{22} = 0\} = \{Z \in M_3 \mid z_{11}z_{22} = 0\} \\
\overline{X}_{132}' &= \{Z \mid z_{12} = 0\} \cup \{Z \mid z_{21} = 0\} = \{Z \in M_3 \mid z_{12}z_{21} = 0\}
\end{align*}
\]

and therefore have multidegrees

\[
\begin{align*}
[\overline{X}_{132}] &= (x_1 - y_1) + (x_2 - y_2) \\
[\overline{X}_{132}'] &= (x_1 - y_2) + (x_2 - y_1).
\end{align*}
\]
Either way lets us calculate \( [X_{132}] = x_1 + x_2 - y_1 - y_2 \) by Corollary 1.2.6. For most
permutations \( w \), only the antidiagonal degeneration \( X''_{132} \) will generalize.

Our goal in this section is to show that the multidegrees \( \{[X_w]\}_{w \in S_n} \) of matrix
Schubert varieties satisfy a recurrence relation, with which they can be determined
from the top one \( [X_w] \). Define the **divided difference operator** \( \partial_i \) that takes a
polynomial \( f \) in \( \{x_1, \ldots, x_n\} \) and possibly some other variables to

\[
\partial_i f(x_1, x_2, \ldots) = \frac{f(x_1, x_2, \ldots) - f(x_1, \ldots, x_{i-1}, x_{i+1}, x_i, x_{i+2}, \ldots)}{x_i - x_{i+1}}.
\]

Since the numerator vanishes when \( x_i = x_{i+1} \), and polynomials enjoy unique factorization, this is again a polynomial, of degree one lower than that of \( f \).

We will want to apply these operators to \( \mathbb{Z}^n \)-graded as well as \( \mathbb{Z}^{2n} \)-graded multidegrees. Note that in this case, when we switch the variables \( x_i \) and \( x_{i+1} \), we do **not** switch the corresponding \( y \) variables.

**Theorem 1.4.3** If the permutation \( w \) satisfies \( \text{length}(ws_i) < \text{length}(w) \), then

\[
[X_{ws_i}] = \partial_i[X_w]
\]

hold for both the \( \mathbb{Z}^{2n} \)-graded and \( \mathbb{Z}^n \)-graded multidegrees.

Two lemmas are required before getting into the multidegree part of the proof.

**Lemma 1.4.4** Let \( Z \) be a partial permutation matrix and \( w \in S_n \). If the orbit closure \( \overline{O}_{Z} \) has codimension 1 inside \( X_{ws_i} \), then \( \overline{O}_{Z} \) is mapped to itself by \( s_i \) unless \( Z = w \).

**Proof.** First note that \( \text{length}(Z) = \text{length}(ws_i) + 1 \) by Lemma 1.3.7 and Proposition 1.3.8. Using Lemma 1.3.5 and Claim 1.3.6, we find that \( Z \) is obtained from \( (ws_i)^T \) by switching some pairs of rows to make partial permutations of strictly larger length and then deleting some 1’s. Since the length of \( Z \) is precisely one less than that of \( ws_i \), we can switch exactly one pair of rows of \( (ws_i)^T \), or we can delete a single 1 from \( (ws_i)^T \).

Any 1 that we delete from \( (ws_i)^T \) must have no 1’s southeast of it, or else the length increases by more than one, as direct calculation shows. Therefore the 1 in row \( i \) of \( (ws_i)^T \) cannot be deleted by part 4 of Corollary 1.3.9, leaving us in the situation of Lemma 1.3.7 with \( Z = w^T \) (see (3) and the sentence before it).

The 1’s in any pair of switched rows must enclose no additional 1’s in their rectangle, by Claim 1.3.6. Suppose now that \( v \neq w \), but that switching rows \( q \) and \( q' \) of \( (ws_i)^T \) results in a permutation matrix \( v^T \) for which \( v(i) > v(i+1) \). Then exactly one of \( q \) and \( q' \) must lie in \( \{i, i+1\} \), because switching rows \( i \) and \( i + 1 \) yields \( v = w \), but moving neither leaves \( v(i) < v(i+1) \). Either the 1 at \( (i, w(i+1)) \) or the 1 at \( (i+1, w(i)) \) lies inside the rectangle formed by the switched 1’s.

**Lemma 1.4.5** If \( \text{length}(ws_i) < \text{length}(w) \), and \( m_{ws_i} \) is the maximal ideal in the local ring of \( (ws_i)^T \in X_{ws_i} \), then the variable \( z_{i+1, w(i+1)} \) maps to a regular parameter in \( m_{ws_i} \). In other words \( z_{i+1, w(i+1)} \) lies in \( m_{ws_i} \setminus m_{ws_i}^2 \).

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Proof. Let \( v = w_{s_i} \), and consider the map \( B \times B_+ \to M_n \) sending \((b, b^+) \mapsto b \cdot v^T \cdot b^+\). The image of this map is contained in \( \overline{X}_v \) by Proposition 1.3.8, and the identity \( \text{id} := (\text{id}_B, \text{id}_{B_+}) \) maps to \( v^T \). The induced map of local rings the other way therefore takes \( \mathfrak{m}_v \) to the maximal ideal

\[
\mathfrak{m}_{\text{id}} := \langle b_{ii} - 1, b_{ii}^+ - 1 \mid 1 \leq i \leq n \rangle + \langle b_{ij}, b_{ji}^+ \mid i > j \rangle
\]

in the local ring at the identity \( \text{id} \in B \times B_+ \). It is enough to demonstrate that the image of \( z_{i+1, w(i+1)} \) lies in \( \mathfrak{m}_{\text{id}} \setminus \mathfrak{m}_{\text{id}}^2 \).

Direct calculation shows that \( z_{i+1, w(i+1)} \) maps to

\[
b_{i+1, i} b_{w(i+1), w(i+1)} + \sum_{q \in Q} b_{i+1, q} b_{w(i+1), w(i+1)}^+ \quad \text{where} \quad Q = \{ q < i \mid w(q) < w(i + 1) \}.
\]

In particular, all of the summands \( b_{i+1, q} b_{w(i+1), w(i+1)}^+ \) lie in \( \mathfrak{m}_{\text{id}}^2 \). On the other hand, \( b_{w(i+1), w(i+1)} \) is a unit near the identity, so \( b_{i+1, i} b_{w(i+1), w(i+1)}^+ \) lies in \( \mathfrak{m}_{\text{id}} \setminus \mathfrak{m}_{\text{id}}^2 \). \( \square \)

The argument forming the proof of the previous lemma is an alternative way to calculate the dimension as in Lemma 1.3.7.

Proof of Theorem 1.4.3. The proof here works for the \( \mathbb{Z}^n \)-grading as well as the \( \mathbb{Z}^{2n} \)-grading, simply by ignoring all occurrences of \( y \), or setting them to zero.

Let \( j = w(i + 1) - 1 \), and suppose \( \text{rank}(w_{[i, j]}^T) = r - 1 \). Then the permutation matrix \((w_{s_i})^T\) has \( r \) entries equal to 1 in the submatrix \((w_{s_i})^T_{[i, j]}\). Consider the \( r \times r \) minor \( \Delta \) using the rows and columns in which \((w_{s_i})^T_{[i, j]}\) has 1’s. Thus \( \Delta \) is not the zero function on \( \overline{X}_{w_{s_i}} \); in fact, \( \Delta \) is nonzero everywhere on its interior \( B(w_{s_i})^T B_+ \).

Therefore the subscheme \( X_\Delta \) defined by \( \Delta \) inside \( \overline{X}_{w_{s_i}} \) is supported on a union of orbit closures \( \overline{O}_Z \) contained in \( \overline{X}_{w_{s_i}} \) with codimension 1.

Compare the subscheme \( X_\Delta \) to its image \( s_i X_\Delta = X_{s_i \Delta} \) under switching rows \( i \) and \( i + 1 \). Lemma 1.4.4 says that \( s_i \) induces an automorphism of the local ring at the generic point (i.e. the prime ideal) of \( \overline{O}_Z \) inside \( \overline{X}_{w_{s_i}} \), for every irreducible component \( \overline{O}_Z \) of \( X_\Delta \) other than \( \overline{X}_w \). This automorphism takes \( \Delta \) to \( s_i \Delta \), so these two functions have the same multiplicity along \( \overline{O}_Z \). The only remaining codimension 1 irreducible component of \( X_\Delta \) is \( \overline{X}_w \), and we shall now verify that the multiplicity equals 1 there.

As a consequence, the multiplicity of \( s_i X_\Delta \) along \( s_i \overline{X}_w \) also equals 1.

By Proposition 1.3.8, the local ring of \((w_{s_i})^T\) in \( \overline{X}_{w_{s_i}} \) is regular. Since \( s_i \) is an automorphism of \( \overline{X}_{w_{s_i}} \), we find that the local ring of \( w^T \in \overline{X}_{w_{s_i}} \) is also regular. In a neighborhood of \( w^T \), the variables \( z_{qp} \), corresponding to the locations of the 1’s in \( w^T_{[i, j]} \) are units. This implies that the coefficient of \( z_{i, w(i+1)} \) in \( \Delta \) is a unit in the local ring of \( w^T \in \overline{X}_{w_{s_i}} \). On the other hand, the set of variables in spots where \( w^T \) has zeros generate the maximal ideal in the local ring at \( w^T \in \overline{X}_{w_{s_i}} \). Therefore, all of the terms of \( \Delta \) lie in the square of this maximal ideal, except for the unit times \( z_{i, w(i+1)} \) term produced above. Hence, to prove multiplicity one, it is enough to prove that \( z_{i, w(i+1)} \) itself is a regular parameter at \( w^T \in \overline{X}_{w_{s_i}} \), or equivalently (after applying \( s_i \)) that \( z_{i+1, w(i+1)} \) is a regular parameter at \((w_{s_i})^T \in \overline{X}_{w_{s_i}} \). This is Lemma 1.4.5.
Now we come to the multidegree trick. Consider \( \Delta \) and \( s_i \Delta \) not as elements in \( k[z] \), but as elements in the ring \( k[z][u] \) from Section 1.2, whose spectrum we denote by \( \mathbb{P} \). Then \( \Delta \) and the product \( \Delta \Delta u \) in \( k[z][u] \) have the same (ordinary) weight \( f := f(x, y) \). Since the affine coordinate ring of \( \mathbb{X}_{w_1} \) is a domain, and neither \( \Delta \) nor \( s_i \Delta \) vanishes on \( \mathbb{X}_{w_1} \), we get two short exact sequences

\[
0 \to k[z][u]/I(\mathbb{X}_{w_1})k[z][u](-f) \xrightarrow{\Theta} k[z][u]/I(\mathbb{X}_{w_1})k[z][u] \to Q(\Theta) \to 0, \tag{5}
\]

in which \( \Theta \) equals either \( \Delta \) or \( us_i \Delta \). The quotients \( Q(\Delta) \) and \( Q(us_i \Delta) \) therefore have equal \( \mathbb{Z}^{2n} \)-graded Hilbert series, and hence equal multidegrees.

Note that \( Q(\Delta) \) is the coordinate ring of \( X_{\Delta} \times A^1 \), while \( Q(us_i \Delta) \) is the coordinate ring of \( (X_{\Delta} \times A^1) \cup (\mathbb{X}_{w_1} \times \{0\}) \), the latter component being the zero scheme of \( \Delta \) in \( k[z][u]/I(\mathbb{X}_{w_1})k[z][u] \). Breaking up the multidegrees of \( Q(\Delta) \) and \( Q(us_i \Delta) \) into sums over irreducible components as in Proposition 1.2.7, the analysis of multiplicity above says that almost all the terms in the equation

\[
[X_{\Delta} \times A^1] = [s_i X_{\Delta} \times A^1] + [\mathbb{X}_{w_1} \times \{0\}]
\]
cancel, leaving us only with

\[
[\mathbb{X}_{w} \times A^1] = [s_i \mathbb{X}_{w} \times A^1] + [\mathbb{X}_{w_1} \times \{0\}]. \tag{6}
\]

The brackets in these equations denote multidegrees over \( k[z][u] \). However, the ideals in \( k[z][u] \) of \( \mathbb{X}_{w} \times A^1 \) and \( s_i \mathbb{X}_{w} \times A^1 \) are extended from the ideals in \( k[z] \) of \( \mathbb{X}_{w} \) and \( s_i \mathbb{X}_{w} \). Therefore they have the same Hilbert numerators, whence \( [\mathbb{X}_{w} \times A^1] = [\mathbb{X}_{w}] \) and \( [s_i \mathbb{X}_{w} \times A^1] = [s_i \mathbb{X}_{w}] \) as polynomials in \( x \) and \( y \). The same argument shows that \( [\mathbb{X}_{w_1} \times \{0\}] = [\mathbb{X}_{w_1}] \). The coordinate ring of \( [\mathbb{X}_{w_1} \times \{0\}] \), on the other hand, is the right hand term of the exact sequence that results after replacing \( f \) by \( x_i - x_{i+1} \) and \( \Theta \) by \( u \) in (5). We therefore find that

\[
[\mathbb{X}_{w_1} \times \{0\}] = (x_i - x_{i+1})[\mathbb{X}_{w_1} \times A^1] = (x_i - x_{i+1})[\mathbb{X}_{w_1}]
\]
as polynomials in \( x \) and \( y \). Substituting back into (6) yields the equation \( [\mathbb{X}_{w}] = [s_i \mathbb{X}_{w}] + (x_i - x_{i+1})[\mathbb{X}_{w_1}] \), which produces the desired result after moving the \( [s_i \mathbb{X}_{w}] \) to the left and dividing by \( x_i - x_{i+1} \).

**Remark 1.4.6** This proof, although translated into the language of multigraded commutative algebra, is actually derived from a standard proof of divided difference formulae by localization in equivariant cohomology, when \( k = \mathbb{C} \). To see how, our two functions \( \Delta \) and \( s_i \Delta \) yield a map \( \mathbb{X}_{w_1} \to \mathbb{C}^2 \), where the preimage of one axis is \( \mathbb{X}_{w} \) union some stuff, and the preimage of the other axis is \( s_i \mathbb{X}_{w} \) union the same stuff. Therefore all of the unwanted (canceling) contributions map to the point \((0, 0) \in \mathbb{C}^2 \). Essentially, the standard equivariant localization proof makes the map to \( \mathbb{C}^2 \) into a map to \( \mathbb{CP}^1 \), thus avoiding the extra components, and pulls back the localization formula on \( \mathbb{CP}^1 \) to a formula on whatever \( \mathbb{X}_{w_1} \) has become (a Schubert variety).

If \( w \in S_n \) does not have a descent at \( i \), so \( w(i) > w(i + 1) \), then the result of applying the divided difference operator is less interesting:

\[
\partial_i [\mathbb{X}_{w}] = \partial_i \partial_i [\mathbb{X}_{w}] = 0
\]
because \( \partial_i f \) is symmetric in \( x_i \) and \( x_{i+1} \) for any polynomial \( f \), and thus \( \partial_i^2 f \) is zero.
1.5 Schubert and Grothendieck polynomials

The polynomials appearing in Theorem 1.4.3 are our title characters:

**Definition 1.5.1** The Schubert polynomial and double Schubert polynomial for a permutation \( w \in S_n \) are the \( \mathbb{Z}^n \)-graded and \( \mathbb{Z}^{2n} \)-graded multidegrees

\[
\mathcal{G}_w(x) := [X_w]_{\mathbb{Z}^n} \quad \text{and} \quad \mathcal{G}_w(x, y) := [X_w]_{\mathbb{Z}^{2n}}.
\]

The definition says that \( w \in S_n \), but in fact \( n \) plays no role, because the multidegrees \( [X_w] \) are stable under the inclusion \( S_n \hookrightarrow S_{n+1} \), in the following sense. Given a permutation \( w_n \) in \( S_n \), let \( w_{n+1} \) be its extension to a permutation in \( S_{n+1} \) defined by fixing the last element, \( n+1 \). Then \( X_{w_{n+1}} \) is the set of \( (n+1) \times (n+1) \) matrices where the upper left \( n \times n \) block is in \( X_{w_n} \), while the last row and column are arbitrary (and so drop out of the multidegree calculation). Thus \( [X_{w_n}] = [X_{w_{n+1}}] \).

Lascoux and Schützenberger [LS82a] used the recursion in Theorem 1.4.3 as the definition of Schubert polynomials and double Schubert polynomials, and they proved the stability property, which singles \( \mathcal{G}_w \) out uniquely among polynomials representing the cohomology classes of the Schubert variety corresponding to \( w \) in the flag manifold (see Appendix A.5). To have it on record, we state their definition as a corollary.

**Corollary 1.5.2** Set \( \mathcal{G}_{w_0}(x) = \prod_{i=1}^n x_i^{a_i} \) and \( \mathcal{G}_{w_0}(x, y) = \prod_{i+j \leq n} (x_i - y_j) \). The Schubert and double Schubert polynomials for \( w \) are \( \mathcal{G}_w(x) = \partial_{i_k} \cdots \partial_{i_1} \mathcal{G}_{w_0}(x) \) and \( \mathcal{G}_w(x, y) = \partial_{i_k} \cdots \partial_{i_1} \mathcal{G}_{w_0}(x, y) \), where \( w_0w = s_{i_1} \cdots s_{i_k} \) and length\( (w_0w) = k \). \( \square \)

The condition length\( (w_0w) = k \) means by definition that \( w_0w = s_{i_1} \cdots s_{i_k} \) is a reduced expression for \( w_0w \). The formulation in Corollary 1.5.2 seems to make it less clear that \( \mathcal{G}_w \) is well-defined: it is necessary that the divided differences satisfy the Coxeter relations, \( \partial_i \partial_{i+1} \partial_i = \partial_{i+1} \partial_i \partial_{i+1} \) and \( \partial_i \partial_{i'} = \partial_{i'} \partial_i \) when \( |i - i'| \geq 2 \). They do, of course (by Theorem 1.4.3, for the cases we care about), and this is not hard to check directly.

As the ordinary Schubert polynomials are much more common in the literature than double Schubert polynomials, we have phrased much of our coming exposition in terms of Schubert polynomials (left torus actions, or \( \mathbb{Z}^n \)-graded multidegrees), although the double Schubert version (with the \( T \times T^{-1} \) action, or \( \mathbb{Z}^{2n} \)-graded multidegrees) holds with the same proof, mutatis mutandis. This choice has the advantage of simplifying the notation.

**Example 1.5.3** Here are all of the Schubert polynomials for permutations in \( S_3 \), along with the rules for applying divided differences.

\[
\begin{array}{ccc}
\partial_2 & \partial_3 & [X_{321}] \\
\downarrow & \downarrow & \downarrow \\
\partial_2 & \partial_3 & [X_{312}] \\
\downarrow & \downarrow & \downarrow \\
0 & \partial_1 & [X_{132}]
\end{array}
\quad
\begin{array}{ccc}
\partial_3 & \partial_1 & x_1^2x_2 \\
\downarrow & \downarrow & \downarrow \\
\partial_3 & \partial_1 & x_2 \\
\downarrow & \downarrow & \downarrow \\
0 & \partial_1 & [X_{123}]
\end{array}
\quad
\begin{array}{ccc}
\partial_2 & \partial_1 & x_1 \\
\downarrow & \downarrow & \downarrow \\
\partial_2 & \partial_1 & x_1x_2 \\
\downarrow & \downarrow & \downarrow \\
0 & \partial_2 & [X_{123}]
\end{array}
\]

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For a taste of what’s to come, compare these diagrams to the one in Example 4.3.5. □

Note that the use of multidegrees instead of Hilbert series greatly simplified life in the proof of Theorem 1.4.3, by virtue of their additivity on irreducible components (Proposition 1.2.7). Of course, Schubert polynomials are therefore only the leading terms of a richer structure, coming from the Hilbert numerators themselves. In view of Theorem 1.4.3, it is natural to ask whether the Hilbert numerators of matrix Schubert varieties satisfy a similar recurrence. They do, as we shall see in Theorem 2.1.2. The recurrence uses a “homogenized” operator (sometimes called an isobaric divided difference operator):

**Definition 1.5.4** Let $R$ be a unique factorization domain. The $i$th Demazure operator $\overline{\partial}_i : R[[x]] \to R[[x]]$ sends a power series $f(x)$ to

$$x_{i+1} f(x_1, \ldots, x_n) - x_i f(x_1, \ldots, x_{i-1}, x_{i+1}, x_i, x_{i+2}, \ldots, x_n)$$

$$x_{i+1} - x_i$$

**Definition 1.5.5** The Grothendieck polynomial $G_w(x)$ is defined recursively from the top one $G_{w_0}(x) := \prod_{i=1}^n (1 - x_i)^{n-i}$, and the recurrence

$$G_{w_{\delta_i}}(x) = \overline{\partial}_i G_w(x)$$

whenever $\text{length}(w_{\delta_i}) < \text{length}(w)$. The double Grothendieck polynomials are defined by the same recurrence, but start from $G_{w_0}(x, y) := \prod_{i+j \leq n} (1 - x_i y^{-1} y_j^{-1})$.

It will follow from our inductive method of proof in Section 2 that Grothendieck polynomials turn out to be well-defined; however, as with Schubert polynomials, one can check directly that the $\overline{\partial}_i$ satisfy the Coxeter relations. Grothendieck polynomials enjoy the same stability property as do Schubert polynomials, but again, this will follow immediately from Theorem 2.1.2.

Lascoux and Schützenberger [LS82b], building on ideas of Demazure [Dem74] and Bernstein-Gelfand-Gelfand [BGG73], showed that Grothendieck polynomials represent the $K$-classes of the structure sheaves of Schubert varieties in the flag manifold. Their methods require some vanishing of sheaf cohomology, namely the rationality of singularities for Schubert varieties. Working in reverse, we shall also prove that the $K$-classes of matrix Schubert varieties satisfy this recurrence (Corollary 3.2.1), but indirectly via multidegrees, which are easier to deal with geometrically (as seen in the proof of Theorem 1.4.3). We then exploit a key property of squarefree monomial ideals (Lemma 2.1.5) to draw conclusions about $K$-theory upon Gröbner deformation.

It is automatic that our multidegree-defined Schubert polynomials $\mathcal{S}_w$ relate to the recursively-defined Grothendieck polynomials as in the next lemma, which gives hope for a Hilbert numerator statement.

**Lemma 1.5.6** The Schubert polynomial $\mathcal{S}_w(x)$ is the sum of all lowest-degree terms in $G_w(1 - x)$. Similarly, the double Schubert polynomial $\mathcal{G}_w(x, y)$ is the sum of all lowest-degree terms in $G_w(1 - x, 1 - y)$.
Proof. Assuming $f(1 - x)$ is homogeneous, plugging $1 - x$ for $x$ into the displayed equation in Definition 1.5.4 and taking the lowest degree terms yields $\partial_i f(1 - x)$. Since $\mathfrak{S}_{w_0}$ is homogeneous, the result follows by induction on length($w_0 w$).  

Although the Demazure operators are usually applied only to polynomials in $x$, it will be crucial in our applications to use them on power series in $x$. Note that $\overline{\partial}_i$ takes a power series to a power series, again because $R[[x]]$ has unique factorization. Since the standard denominator $f(x) = \prod_{i=1}^n (1 - x_i)^n$ for $\mathbb{Z}^n$-graded Hilbert series over $k[z]$ is symmetric in $x_1, \ldots, x_n$, an easy check shows that applying $\overline{\partial}_i$ to a Hilbert series $g/f$ simplifies: $\overline{\partial}_i (g/f) = (\overline{\partial}_i g)/f$. The same comment applies when $f(x) = \prod_{i,j=1}^n (1 - x_i/y_j)$ is the standard denominator for $\mathbb{Z}^{2n}$-graded Hilbert series.

Remark 1.5.7 We consciously chose our notational conventions (with considerable effort) to mesh with those of [Ful92], [LS82a], [FK94], [HT92], and [BB93] concerning permutations ($w^T$ versus $w$), the indexing on (matrix) Schubert varieties and polynomials (open orbit corresponds to identity permutation and smallest orbit corresponds to long word), the placement of one-sided ladders (in the northwest corner as opposed to the southwest), and re-graphs. These conventions dictated our seemingly idiosyncratic choices of Borel subgroups as well as the identification $F\ell_n \cong B/\text{GL}_n$ as the set of right cosets, and resulted in our use of row vectors in $k^n$ instead of the usual column vectors. That there even existed consistent conventions came as a relief to us; that they remained consistent when combined with the signs and weights that enter into computations in equivariant cohomology and $K$-theory, which have their own natural bases, was more than we could have requested.  

## 2 Gröbner bases

### 2.1 The Gröbner basis theorem

Recall our notation: $M_n$ is the $n \times n$ matrices over $k$, with coordinate ring $k[z]$ in indeterminates $\{z_{ij}\}_{i,j=1}^n$, the northwest $q \times p$ submatrix of a matrix $Z$ is $Z_{[q,p]}$, and the matrix Schubert variety $\overline{X}_w \subseteq M_n$ consists of the matrices $Z \in M_n$ such that $\text{rank}(Z_{[q,p]}) \leq \text{rank}(w^T_{[q,p]})$ for all $q, p$. We now define two associated ideals.

**Definition 2.1.1** Let $w \in S_n$ be a permutation.

1. The **Schubert determinantal ideal** $I_w \subseteq k[z]$ is generated by all minors in $Z_{[q,p]}$ of size $1 + \text{rank}(w^T_{[q,p]})$ for all $q, p$, where $Z = (z_{ij})$ is the matrix of variables.
2. The **antidiagonal ideal** $J_w$ is generated by the antidiagonals of the minors of $Z = (z_{ij})$ generating $I_w$.

Here, the **antidiagonal** of a square matrix or a minor is the product of the entries on the main antidiagonal.

It is clear from the definition that $\overline{X}_w$ is the reduced subscheme of $M_n$ underlying the subscheme defined by $I_w$. It is equally clear that given any antidiagonal term
order $>$, which by definition picks off from each minor its antidiagonal term, $J_w$ is contained in the initial ideal $\text{in}_>(I_w)$ of $I_w$. There are lots of antidiagonal term orders, including: the reverse lexicographic term order that snakes its way from the northwest corner to the southeast corner, $z_{11} > z_{12} > \cdots > z_{1n} > z_{21} > \cdots > z_{nn}$; or the lexicographic term order making its way from northeast to southwest. That these term orders exist (Example A.2.2) ensures that our main result isn’t vacuous.

The ring $k[z]$ is both $\mathbb{Z}^n$-graded and $\mathbb{Z}^{2n}$-graded, with the exponential weight of $z_{ij}$ being respectively $x_i$ or $x_i/y_j$, where $x = x_1, \ldots, x_n$ and $y = y_1, \ldots, y_n$ are standard bases for copies of $\mathbb{Z}^n$. (Other gradings will occur later on; background on gradings for the coordinate ring of $M_n$ can be found in Section 1.2 or Appendix A.1, especially Example A.1.1 and Example A.1.9.) Observe that $I_w$, $J_w$, and the radical ideal $I(\overline{X}_w)$ of the matrix Schubert variety $\overline{X}_w$ are all $\mathbb{Z}^{2n}$-graded (and hence also $\mathbb{Z}^n$-graded), the first two by checking the algebra and the third because it is the radical of the $\mathbb{Z}^{2n}$-graded ideal $I_w$ (see Lemma A.1.6 for a geometric explanation). We denote the $\mathbb{Z}^n$-graded Hilbert series of a $\mathbb{Z}^n$-graded module $\Gamma$ over $k[z]$ by $H(\Gamma; x, y)$, and write $H(\Gamma; x, y)$ for the $\mathbb{Z}^{2n}$-graded Hilbert series if $\Gamma$ is $\mathbb{Z}^{2n}$-graded.

**Theorem 2.1.2** If $>$ is an antidiagonal term order, then $\text{in}_>(I_w) = J_w$; in other words, the $(\text{rank}(w^T_{q,p}) + 1)$-minors of $Z_{q,p}$ for all $q, p$ constitute a Gröbner basis. Furthermore, $I_w = I(\overline{X}_w)$, and the variously graded Hilbert series of $k[z]/I_w$ are

$$H(k[z]/I_w; x) = \frac{G_w(x)}{\prod_{i=1}^n (1 - x_i)^n} \quad \text{and} \quad H(k[z]/I_w; x, y) = \frac{G_w(x, y)}{\prod_{i,j=1}^n (1 - x_i/y_j)}, \quad (7)$$

the numerators being Grothendieck and double Grothendieck polynomials for $w$.

**Remark 2.1.3** Fulton proved that $I_w$ is the radical ideal of $\overline{X}_w$ in [Ful92], where he also related the ideals $I_w$ to Schubert varieties. In Sections 3.3–3.5, we compare his point of view to ours, as well as to statements about ladder determinantal ideals (which we define later) by other authors. In the course of our comparisons, Proposition 3.3.3 will pinpoint those minors forming a minimal Gröbner basis. \hfill $\Box$

The proof of Theorem 2.1.2, which involves a great deal of tailor-made combinatorics, spans the entirety of Section 2, although we encounter some other results of independent interest along the way. In the rest of this subsection we provide a brief argument that constitutes a complete proof of Theorem 2.1.2, if one assumes the results it quotes from later subsections. Besides providing the logical framework, the argument serves as an overview of the rest of Section 2.

Before getting to the proof, we present an example (that will recur a few times).

**Example 2.1.4** Let $w = 13865742$, so that $w^T$ is given by the left matrix below.

$$
\begin{bmatrix}
* & * & & & & * & & \\
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
\end{bmatrix}
\Rightarrow
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
1 & 1 & & & & & & & & & & & & & & \\
1 & 1 & & & & & & & & & & & & & & \\
1 & 1 & & & & & & & & & & & & & & \\
1 & 1 & & & & & & & & & & & & & & \\
1 & 1 & & & & & & & & & & & & & & \\
1 & 1 & & & & & & & & & & & & & & \\
1 & 1 & & & & & & & & & & & & & & \\
\end{bmatrix}
$$

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Then each matrix in $\overline{X}_w \subseteq M_n$ has the property that every rectangular submatrix contained in the region filled with 1's has rank $\leq 1$, and every rectangular submatrix contained in the region filled with 2's has rank $\leq 2$. The ideal $I_w$ contains the 21 minors of size $2 \times 2$ in the first region and the 144 minors of size $3 \times 3$ in the second region. Moreover, it is easy to show directly that all of the larger minors in $I_w$ stipulated by the definition have antidiagonals divisible by the antidiagonals of some 2- or 3-minor as above (this will also follow from Proposition 3.3.3, where we describe explicitly the minimal Gröbner basis). Therefore, the 165 minors of size $2 \times 2$ and $3 \times 3$ in $I_w$ form a Gröbner basis for $I_w$. □

In keeping with the convention of this monograph to state results in the main body only for the gradings we're considering, we assume the ideals $I$ and $J$ in what follows are $\mathbb{Z}^n$-graded in the coordinate ring of $M_n$. However, the result holds for any multigrading on any polynomial ring that refines the usual $\mathbb{Z}$-grading.

**Lemma 2.1.5** Suppose $I$ is a multigraded ideal and $J$ is an equidimensional square-free monomial ideal contained in the initial ideal $\operatorname{in}(I)$ of $I$ for some term order. If the zero sets of $I$ and $J$ have the same dimension and their multidegrees $\mathcal{C}(k[z]/I; x) = \mathcal{C}(k[z]/J; x)$ agree, then $\operatorname{in}(I) = J$, and $I$ is reduced.

*Proof.* By Proposition 1.2.5, the multidegree of $k[z]/J$ is a sum of monic monomials, one for each irreducible component in its zero set. Since $J \subseteq \operatorname{in}(I)$, any maximal dimensional irreducible component of the scheme defined by $\operatorname{in}(I)$ is contained in one of these subspaces, and hence equal to it (and thus reduced) by comparing dimensions. Proposition 1.2.7 says that the multidegree of $k[z]/\operatorname{in}(I)$ is therefore also a sum of monomials, one for each irreducible component of $J$ that happens also to be an irreducible component of $\operatorname{in}(I)$. But the first sentence of Corollary 1.2.6 with $y = 0$ says that the multidegree of $k[z]/\operatorname{in}(I)$ equals that of $k[z]/I$. By hypothesis, the multidegrees of $k[z]/\operatorname{in}(I)$ and $k[z]/J$ coincide, and we conclude that $\operatorname{in}(I) \subseteq J$.

Since $I$ degenerates to $J$, and $J$ (being squarefree) is reduced, $I$ is reduced too. □

The $\mathbb{Z}$-graded version of this lemma, along with the ensuing conclusion that a candidate Gröbner basis actually is one, appears also in [Mar01], for a different ideal.

*Proof of Theorem 2.1.2 [using the rest of Section 2]*. The hardest step in the proof is Corollary 2.4.4, which shows that the Hilbert series for $\{k[z]/J_w\}_{w \in S_n}$ satisfy the Demazure recursion defining the Grothendieck polynomials. The Hilbert numerator $\mathcal{K}(k[z]/J_w; x)$ is easily seen to be $G_{w_0}(x)$, so Corollary 2.4.4 and downward induction on length($w$) imply that the Hilbert numerator of $k[z]/J_w$ is $G_w(x)$. Therefore the Hilbert series of $k[z]/J_w$ matches the right hand side of (7).

Lemma 1.5.6 immediately implies that the multidegree of $k[z]/J_w$ is the Schubert polynomial $G_w(x)$, which agrees with the multidegree of the corresponding matrix Schubert variety by Theorem 1.4.3.

Next we show that $J_w$ is equidimensional of dimension $\dim \overline{X}_w$. This is completed in Proposition 2.5.6, which applies some of the combinatorics arising in the calculation of the Hilbert series of $k[z]/J_w$ to the facets of its Stanley–Reisner simplicial complex.
At this point we can apply our Lemma 2.1.5, concluding that \( \text{in}(I_w) = J_w \). Therefore \( k[z]/I_w \) has the advertised Hilbert series, since \( k[z]/J_w \) does. We already knew that \( I_w \) determines \( \overline{X}_w \) as a set; since we now know (again by Lemma 2.1.5) that \( I_w \) is reduced, we conclude that \( I_w = I(\overline{X}_w) \).

Having already proved that the minors form a Gröbner basis, so that \( J_w \) is the initial ideal, the \( \mathbb{Z}^{2n} \)-graded Hilbert series calculation only requires checking that the numerator of \( H(k[z]/J_w; x, y) \) equals the double Grothendieck polynomial. This is Corollary 2.4.4, the base case \( w = w_0 \) being trivial.

We could avoid checking the equidimensionality of \( J_w \) by consulting an oracle to find that the Grothendieck polynomials are the equivariant \( K \)-classes of the matrix Schubert varieties (this result is not very far from that in [LS82b]). Indeed, then the Hilbert series of \( k[z]/I_w \) would immediately match that of \( k[z]/J_w \). Interestingly, we could then conclude the purity of \( J_w \) from its squarefreeness, because the geometrically isolated (i.e. non-embedded) components of initial ideals of prime ideals automatically have the same dimension; see [KS95]. However, the analysis of \( J_w \)’s components is worthwhile in any case for our combinatorial applications, such as Theorem 4.3.1.

**Remark 2.1.6** The Gröbner basis in Theorem 2.1.2 defines a flat deformation over any ring, because all of the coefficients of the minors in \( I_w \) are integers, and the leading coefficients are all \( \pm 1 \). Indeed, each loop of the division algorithm in Buchberger’s criterion [Eis95, Theorem 15.8] works over \( \mathbb{Z} \) and therefore over any ring. The only reason for our restricting to fields \( k \) is to make sense of Hilbert series.

The reader is advised to skim the rest of Section 2 upon first reading. Although the main results will be applied in later sections, the methods will find direct applications in relatively few places. The only substantial exception is Section 4.2, which translates the Stanley–Reisner theory for \( J_w \) into the language of re-graphs via the combinatorics of antidiagonals.

### 2.2 Antidiagonals and mutation

In this subsection we begin investigating the combinatorial properties of the monomials outside \( J_w \) and the antidiagonals generating \( J_w \). For the rest of this subsection, fix a permutation \( w \) and a transposition \( s_i \) satisfying \( \text{length}(ws_i) < \text{length}(w) \).

Define the **rank matrix** \( \text{rk}(w) \) to have \((q, p)\) entry equal to \( \text{rank}(w_{[q, p]}) \). There are two standard facts we’ll need concerning rank matrices, both proved simply by looking at the picture of \((ws_i)^T\) in (4).

**Lemma 2.2.1** Suppose the \((i, j)\) entry of \( \text{rk}(ws_i) \) is \( r \).

1. If \( j \geq w(i + 1) \) then the \((i - 1, j)\) entry of \( \text{rk}(ws_i) \) is \( r - 1 \).
2. If \( j < w(i + 1) \) then the \((i + 1, j)\) entry of \( \text{rk}(ws_i) \) is \( r \).

In what follows, a **rank condition** refers to a statement requiring “\( \text{rank}(Z_{[q, p]}) \leq r \)” for some \( r \geq 0 \). Most often, \( r \) will be either \( \text{rank}(w^T_{[q, p]}) \) or \( \text{rank}((ws_i)^T_{[q, p]}) \), thereby
making the entries of \( \text{rk}(w) \) and \( \text{rk}(ws_i) \) into rank conditions. We say that a rank condition \( \text{rk}(Z_{[q,p]}) \leq r \) causes an antidiagonal \( a \) of the generic matrix \( Z \) if \( Z_{[q,p]} \) contains \( a \) and the number of variables in \( a \) is strictly larger than \( r \). For instance, when the rank condition is in \( \text{rk}(w) \), the antidiagonals it causes are precisely those \( a \in J_w \) that are contained in \( Z_{[q,p]} \). Although antidiagonals in \( Z \) (i.e. antidiagonals of square submatrices of the generic matrix \( Z \)) are by definition monomials, we routinely identify each antidiagonal with its support: the subset of the variables dividing it in \( k[z] \).

**Lemma 2.2.2** Antidiagonals in \( J_w \setminus J_{w,i} \) are subsets of \( Z_{[i,w(i)]} \) and intersect row \( i \).

**Proof.** If an antidiagonal in \( J_w \) is either contained in \( Z_{[i-1,w(i)]} \) or not contained in \( Z_{[i,w(i)]} \), then some rank condition causing it is in both \( \text{rk}(w) \) and \( \text{rk}(ws_i) \). Indeed, it is easy to check that the rank matrices \( \text{rk}(ws_i) \) and \( \text{rk}(w) \) differ only in row \( i \) between the columns \( w(i+1) \) and \( w(i) - 1 \), inclusive. \( \square \)

Though simple, the next lemma is the key combinatorial observation. Note that the permutation \( w \) there is arbitrary; in particular, we will frequently apply the lemma in the context of antidiagonals for a permutation called \( ws_i \).

**Lemma 2.2.3** Suppose \( a \in J_w \) is an antidiagonal and \( a' \subset Z \) is another antidiagonal.

(W) If \( a' \) is obtained by moving west one or more of the variables in \( a \), then \( a' \in J_w \).

(E) If \( a' \in k[z] \) is obtained by moving east any variable except the northeast one in \( a \), then \( a' \in J_w \).

(N) If \( a' \) is obtained by moving north one or more of the variables in \( a \), then \( a' \in J_w \).

(S) If \( a' \in k[z] \) is obtained by moving south any variable except the southwest one in \( a \), then \( a' \in J_w \).

**Proof.** Every rank condition causing \( a \) also causes any of the antidiagonals \( a' \). \( \square \)

**Example 2.2.4** Parts (W) and (E) of Lemma 2.2.3 together imply that the type of motion depicted in the following diagram preserves the property of an antidiagonal being in \( J_w \). The presence of the northeast * justifies moving the southwest * east.

\[
\begin{array}{cccccc}
\ast & \cdots & \ast \\
\end{array} \quad \in J_w \quad \Rightarrow \quad \begin{array}{cccccc}
\ast & \leftrightarrow & \ast \\
\end{array} \quad \in J_w
\]

The two rows could also be separated by some other rows—possibly themselves containing elements of the original antidiagonal—as long as the indicated motion preserves the fact that we have an antidiagonal. \( \square \)
**Definition 2.2.5** Let \( \mathbf{b} \) be an array \( \mathbf{b} = (b_{i,j}) \) of nonnegative integers, i.e. \( \mathbf{b} \) is an exponent array for a monomial \( z^\mathbf{b} \in k[z] \). Let

\[
\text{west}_q(\mathbf{b}) := \min(\{p \mid b_{q,p} \neq 0\} \cup \{\infty\})
\]

be the column of the leftmost (most “western”) nonzero entry in row \( q \). Define the mutation of \( \mathbf{b} \),

\[
\mu_i(\mathbf{b}) := \text{the exponent array of } (z_{i,p}/z_{i+1,p})z^\mathbf{b} \text{ for } p = \text{west}_{i+1}(\mathbf{b}).
\]

For ease of notation, we write \( \mu_i(z^\mathbf{b}) \) for \( z^{\mu_i(\mathbf{b})} \).

If one thinks of the array \( \mathbf{b} \) as a chessboard with some coins stacked in each square, then \( \mu_i \) is performed by taking a coin off the western stack in row \( i + 1 \) and putting it onto the stack due north of it in row \( i \).

**Example 2.2.6** Suppose \( \mathbf{b} \) is the left array below and \( i = 3 \). In Fig. 1 we list (reading left to right as usual) 7 mutations of \( \mathbf{b} \), namely \( \mathbf{b} = (\mu_3)^0(\mathbf{b}) \) through \( (\mu_3)^6(\mathbf{b}) \) (after that it involves the dots we left unspecified). Here, the empty boxes denote entries equal to 0, and the nonzero mutated entries at each step are in boldface. To make things easier to look at, the entries on or below the main antidiagonal are represented by dots, each of which may be zero or not (independently of the others). The 3 and 4 at left are labels for rows \( i = 3 \) and \( i + 1 = 4 \).

Having chosen our permutation \( w \) and row index \( i \), various entries of a given \( \mathbf{b} \in \mathbb{Z}^{n^2} \) play special roles. To begin with, we call the union of rows \( i \) and \( i + 1 \) the gene\(^2 \) of \( \mathbf{b} \). For exponent arrays \( \mathbf{b} \) such that \( z^\mathbf{b} \notin J_w \), the spot in row \( i \) and column

\[
\text{start}_i(\mathbf{b}) := \min\{p \mid z_{i,p}z^\mathbf{b} \notin J_w\}
\]

is called the start codon of \( \mathbf{b} \). The minimum defining \( \text{start}_i(\mathbf{b}) \) is taken over a nonempty set because \( J_w \subseteq J_{w_0} = I_{w_0} = (z_{qp} \mid q + p \leq n) \), so that \( z_{in}z^\mathbf{b} \) remains outside of \( J_w \). Of course, \( z_{ip}z^\mathbf{b} \notin J_w \) whenever \( z_{ip} \) divides \( z^\mathbf{b} \notin J_w \) because \( J_w \) is generated by squarefree monomials. Therefore,

\[
\text{start}_i(\mathbf{b}) \leq \text{west}_i(\mathbf{b}).
\]

For completeness, set \( \text{start}_i(\mathbf{b}) = 0 \) if \( z^\mathbf{b} \in J_w \).

Also of special importance is the promoter \( \text{prom}(\mathbf{b}) \), consisting of the rectangular \( 2 \times (\text{start}_i(\mathbf{b}) - 1) \) array of locations in the gene of \( \mathbf{b} \) that are strictly west of \( \text{start}_i(\mathbf{b}) \). Again, we omit the explicit reference to \( i \) and \( w \) in the notation because these are fixed for the discussion. The sum of all entries in the promoter of \( \mathbf{b} \) is

\[
|\text{prom}(\mathbf{b})| = \sum_{j<\text{start}_i(\mathbf{b})} b_{i+1,j}.
\]

\(^2\)All of the unusual terminology in what follows comes from genetics. Superficially, our diagrams with two rows of boxes look like geneticists’ schematic diagrams of the DNA double helix; but there is a much more apt analogy that will become clear only in Section 2.4, where the biological meanings of the terms can be found in another footnote.
**Example 2.2.7** Let \( b \) be the left array in Example 2.2.6, \( i = 3 \), and \( w = 13865742 \), the permutation displayed in Example 2.1.4. Then the gene of \( b \) consists of rows \( i = 3 \) and \( i + 1 = 4 \), and we claim \( \text{start}_i(b) = 5 \).

To begin with, we have 6 choices for an antidiagonal \( a \in J_w \) dividing \( z_{31} z_b \): we must have \( z_{31} \in a \), but other than that we are free to choose one element of \( \{ z_{23}, z_{24}, z_{25} \} \) and one element of \( \{ z_{16}, z_{17} \} \). (This gives an example of the \( a \) produced in the first paragraph of the proof of Lemma 2.3.3, below.) Even more varied choices are available for \( z_{32} z_b \), such as \( z_{41} z_{32} z_{23} \) or \( z_{41} z_{32} z_{13} \). We can similarly find lots of antidiagonals in \( J_w \) dividing \( z_{33} z_b \), and \( z_{34} z_b \). On the other hand, \( z_{35} \) already divides \( z_b \), and one can verify that \( z_b \) isn’t divisible by the antidiagonals of any of the \( 2 \times 2 \) or \( 3 \times 3 \) minors defining \( I_w \) (see Example 2.1.4). Therefore \( z_{35} z_b \nsubseteq J_w \), so \( \text{start}_i(b) = 5 \).

The promoter \( \text{prom}(b) \) consists of the \( 2 \times 4 \) block

\[
\begin{array}{cccc}
3 & & & \\
4 & 2 & 2 & 2 \\
\end{array}
\]

at the western end. In particular, \( |\text{prom}(b)| = 6 \).

Nothing in this example depends on the values chosen for the dots on or below the main antidiagonal.
2.3 Lifting Demazure operators

Now we need to understand the Hilbert series of $k[z]/J_w$ for varying $w$. Since $J_w$ is a monomial ideal, it makes sense to consider its $\mathbb{Z}^{n_2}$-graded Hilbert series $H(k[z]/J_w; z)$, which we express in the variables $(z_{ij})$. Observe that $H(k[z]/J_w; z)$ is simply the sum of all monomials outside of $J_w$. Using the combinatorics of the previous subsection, we construct operators $\varepsilon^w_i$ defined on monomials and taking the power series $H(k[z]/J_w; z)$ to $H(k[z]/J_{w_s}; z)$ whenever $\text{length}(ws_i) < \text{length}(w)$. In other words, the sum of all monomial outside $J_{ws_i}$ is obtained from the sum of monomials outside $J_w$ by replacing $z^b \not\in J_w$ with $\varepsilon^w_i(z^b)$. It is worth keeping in mind that we will eventually show (in Section 2.4) how $\varepsilon^w_i$ refines the usual $\mathbb{Z}^n$-graded Demazure operator $\overline{\sigma}$, when these operators are applied to the variously graded Hilbert series of $k[z]/J_w$.

Again, fix for the duration of this subsection a permutation $w$ and a transposition $s_i$ satisfying $\text{length}(ws_i) < \text{length}(w)$.

**Definition 2.3.1** The **lifted Demazure operator** corresponding to $w$ and $i$ is a map of abelian groups $\varepsilon^w_i : \mathbb{Z}[z][w] \rightarrow \mathbb{Z}[z][w]$ determined by its action on monomials:

$$\varepsilon^w_i(z^b) := \sum_{d=0}^{|\text{prom}(b)|} \mu_i^d(z^b).$$

Here, $\mu_i^d$ means take the result of applying $\mu_i$ a total of $d$ times, and $\mu_i^0(b) = b$.

**Example 2.3.2** If $b$ is the array in Examples 2.2.6 and 2.2.7, then $\varepsilon^w_3(z^b)$ is the sum of the 7 monomials whose exponent arrays are displayed in Example 2.2.6.

Observe that $\varepsilon^w_i$ replaces each monomial by a homogeneous polynomial of the same total degree, so the result of applying $\varepsilon^w_i$ to a power series is actually a power series.

In preparation for Proposition 2.3.11, we need a few lemmas detailing the effects of mutation on monomials and their genes. The first of these implies that $\varepsilon^w_i$ takes monomials outside $J_w$ to sums of monomials outside $J_{ws_i}$, given that $\mu_i^0(b) = b$.

**Lemma 2.3.3** If $z^b \not\in J_w$ and $1 \leq d \leq |\text{prom}(b)|$ then $\mu_i^d(z^b) \in J_w \setminus J_{ws_i}$.

**Proof.** We may as well assume $|\text{prom}(b)| \geq 1$, or else the statement is vacuous. By definition of $\text{prom}(b)$ and $\text{start}_i(b)$, some antidiagonal $a \in J_w$ divides $z_{ip}z^b$, where here (and for the remainder of this proof) $p = \text{west}_{i+1}(b)$. Since $a$ doesn’t divide $z^b$, we find that $z_{ip} \not\in a$, whence $a$ cannot intersect row $i + 1$, which is zero to the west of $z_{ip}$. Thus $a$ also divides $\mu_i(z^b)$, and hence $\mu_i^d(z^b)$ for all $d$ (including $d > |\text{prom}(b)|$, but we won’t need this).

It remains to show that $\mu_i^d(z^b) \not\in J_{ws_i}$ when $d \leq |\text{prom}(b)|$. Let’s start with $d \leq b_{i+1,p}$. Any antidiagonal $a$ dividing $\mu_i^d(z^b)$ does not continue southwest of $z_{ip}$; this is by Lemma 2.2.3(S) and the fact that $z^b \not\in J_w$ (we could move $z_{ip}$ south). Suppose for contradiction that $a \in J_{ws_i}$, and consider the smallest northwest submatrix $Z_{[i,j](a)}$ containing $a$. If $j(a) \geq w(i + 1)$ then the antidiagonal $a' = a/z_{ip}$ obtained by omitting
$z_{ip}$ from $a$ is still in $J_{w_{si}}$, being caused by the entry of $rk(w_{si})$ at $(i-1,j(a))$ as per Lemma 2.2.1.1. On the other hand, if $j(a) < w(i+1)$, then $a'' = (z_{i+1,p}/z_{ip})a$ is still in $J_{w_{si}}$, being caused by the entry of $rk(w_{si})$ at $(i+1,j(a))$ as per Lemma 2.2.1.2. Since both $a'$ and $a''$ divide $z^b$ by construction, we find that $z^b \in J_{w_{si}} \subset J_w$, the desired contradiction. It follows that $\mu_i^d(z^b) \not\in J_{w_{si}}$ for $d \leq b_{i+1,p}$.

Assuming the result for $d \leq \sum_{j=p}^{p'} b_{i+1,j}$, where $p' < \text{start}_i(b) - 1$, we now demonstrate the result for $d \leq \sum_{j=p}^{p'+1} b_{i+1,j}$. Again, any antidiagonal $a \in J_w$ dividing $\mu_i^d(z^b)$ must end at row $i$, for the same reason as in the previous paragraph. But now if $a \in J_{w_{si}}$, then moving its southwest variable to $z_{ip}$ creates an antidiagonal that is in $J_{w_{si}}$ (by Lemma 2.2.3(W)) and divides $\mu_i(z^b)$, which we have seen is impossible. □

Now we show that mutation of monomials outside $J_w$ can’t produce the same monomial more than once, as long we stop after $|\text{prom}|$ many steps.

**Lemma 2.3.4** Suppose $z^b, z^{b'} \not\in J_w$ and that $d, d' \in \mathbb{Z}$ satisfy $1 \leq d \leq |\text{prom}(b)|$ and $1 \leq d' \leq |\text{prom}(b')|$. If $b \neq b'$ then $\mu_i^d(b) \neq \mu_i^d(b')$.

**Proof.** The inequality $d \leq |\text{prom}(b)|$ guarantees that the mutations of $b$ only alter the promoter of $b$, which is west of west$_i(b)$ by (9). Therefore, assuming (by switching $b$ and $b'$ if necessary) that west$_i(b') \leq$ west$_i(b)$, we reduce to the case where $b$ and $b'$ differ only in their genes, in columns strictly west of west$_i(b)$.

Let $c = \mu_i^d(b)$ and $c' = \mu_i^d(b')$. Mutating preserves the sums

$$b_{i+1,j} = c_{ij} + c_{i+1,j} \quad \text{and} \quad c'_{ij} + c'_{i+1,j} = b_{ij} + b_{i+1,j}$$

for $j < \text{west}_i(b)$, and we may as well assume these are equal for every $j$, or else $c \neq c'$ is clear. The westernmost column where $b$ and $b'$ disagree is now necessarily $p = \text{west}_i(b')$. It follows that $z_{ip}z^b \not\in J_w$, because $b$ agrees with $b'$ strictly to the west of column $p$ as well as strictly to the north of row $i$, and any antidiagonal $a \in J_w$ dividing $z_{ip}z^b$ must be contained in this region (since it contains $z_{ip}$). In particular, $\text{start}_i(b) \leq p$. We conclude that mutating $b$ and $b'$ fewer than $|\text{prom}(b)|$ or $|\text{prom}(b')|$ times cannot alter the column $p$ where $b$ and $b'$ differ. Thus $c$ differs from $c'$ in column $p$. □

**Example 2.3.5** If we apply $\mu_i$ more than $|\text{prom}(b)|$ times to some array $b$, it is possible to reach $\mu_i^{d'}(b')$ for some $b' \neq b$ and $d' \leq |\text{prom}(b')|$. Take $b$, $i$, and $w$ as in Examples 2.2.6, 2.2.7, and 2.3.2, and set the dot in $b$ at position $(4,5)$ equal to 3. If $z^{b'} = (z_{35}/z_{45})z^b$, then we have $|\text{prom}(b)| = |\text{prom}(b')| = 6$, but the entries of $b$ and $b'$ in column 5 of their genes are $\frac{1}{3}$ and $\frac{1}{2}$, respectively. Mutating $b$ and $b'$ up to 6 times yields 7 arrays each, all distinct because of the $\frac{1}{3}$ and $\frac{2}{2}$ in column 5. However, mutating $b$ to an 8th array $(\mu_3)^7(b)$ changes the $\frac{1}{3}$ to $\frac{2}{2}$, and outputs $(\mu_3)^6(b')$. □

If $c$ is the result of applying $\mu_i$ to $b$ some number of times, we can recover $b$ from $c$ by reversing certain entries of $c$ from row $i$ back to row $i + 1$. Formally, reversing an entry $c_{ij}$ of $c$ means making a new array that agrees with $c$ except at
$(i, j)$ and $(i + 1, j)$. In those spots, the new array has $(i, j)$ entry 0 and $(i + 1, j)$ entry $c_{ij} + c_{i+1,j}$. (In terms of the stacks-of-coins picture, we revert only entire stacks of coins, not single coins.) Even if we’re just given $c$ without knowing $b$, we still have a criterion to determine when a certain reversion of $c$ yields a monomial $z^b \not\in J_w$.

**Claim 2.3.6** Suppose $z^c \in J_w \setminus J_{w_s}$. If $b$ is obtained from $c$ by reverting all entries of $c$ in row $i$ that are west of or at column west$_{i+1}(c)$, then $z^b \not\in J_w$.

**Proof.** Suppose $z^b \in J_w$, and let’s try to produce an antidiagonal witness $a \in J_w$, dividing it. Either $a$ ends at row $i$, or not. In the first case, $a$ divides $z^c$, because the nonzero entries in row $i$ of $b$ are the same as the corresponding entries of $c$. Thus we can replace $a$ by the result $a'$ of tacking on $z_{i+1,p}$ to $a$, where $p = \text{west}_{i+1}(c)$. This new $a'$ is in $J_w$ because $a \in J_w$ divides $a'$. It follows from Lemma 2.2.2 that $a' \in J_{w_s}$. Furthermore, $a'$ divides $z^c$ by construction, and thus contradicts our assumption that $z^c \not\in J_{w_s}$. Therefore, we may assume for the remainder of the proof of this lemma that $a$ doesn’t end at row $i$, so $a \in J_{w_s}$, by Lemma 2.2.2.

We now prove that $z^b \not\in J_w$ by showing that if $a \in J_{w_s}$ and $a$ divides $z^b$, then from $a$ we can synthesize $a' \in J_{w_s}$ dividing $z^c \in J_{w_s}$, again contradicting our running assumption $z^c \in J_w \setminus J_{w_s}$. There are three possibilities (an illustration for (ii) is described in Example 2.3.7):

(i) The antidiagonal $a \in J_{w_s}$ intersects row $i$ but doesn’t end there.

(ii) The antidiagonal $a \in J_{w_s}$ skips row $i$ but intersects row $i + 1$.

(iii) The antidiagonal $a \in J_{w_s}$ skips both row $i$ as well as row $i + 1$.

In case (i) either $a$ already divides $z^c$ or we can move east the row $i + 1$ variable in $a$, into the location $(i + 1, \text{west}_{i+1}(c))$. The resulting antidiagonal $a'$ divides $c$ by construction and is in $J_{w_s}$, by Lemma 2.2.3(E). In case (iii) the antidiagonal already divides $z^c$ because $b$ agrees with $c$ outside of their genes.

This leaves case (ii). If $a$ doesn’t already divide $z^c$, then the intersection $z_{i+1,j}$ of $a$ with row $i + 1$ is strictly west of $\text{west}_{i+1}(c)$. The antidiagonal $a' = (z_{ij}/z_{i+1,j})a$ then divides $z^c$ by construction, and is in $J_{w_s}$, by Lemma 2.2.3(N). □

**Example 2.3.7** Here is an example of what happens in case (ii) from the proof of Claim 2.3.6. Let $b$ and $c$ be the first and last arrays from Example 2.2.6, and consider what happens when we fiddle with their $(5, 1)$ entries. The antidiagonals $z_{51}z_{42}z_{23}$ and $z_{51}z_{42}z_{24} \in J_{w_s}$, both divide $z_{51}z^b$. Using Lemma 2.2.3(N) we can move the $z_{42}$ north to $z_{32}$ to get $a' \in \{z_{51}z_{32}z_{23}, z_{51}z_{32}z_{24}\}$ in $J_{w_s}$, dividing $z_{51}z^c$. (It almost goes without saying, of course, that $z_{51}z^c$ is no longer in $J_w \setminus J_{w_s}$, so it doesn’t satisfy the hypothesis of Claim 2.3.6; we were, after all, looking for a contradiction.) □

Any array $c$ whose row $i$ begins west of its row $i + 1$ can be expressed as a mutation of some array $b$. By Claim 2.3.6, we even know how to make sure $z^b \not\in J_w$ whenever $z^c \in J_w \setminus J_{w_s}$. But we also want each $z^c \in J_w \setminus J_{w_s}$ to appear in $\varepsilon_i^w(z^b)$ for some $z^b \not\in J_w$, and this involves making sure start$_i(b)$ isn’t too far west.
Example 2.3.8 If \( \text{west}_i(c) \) is sufficiently smaller than \( \text{west}_{i+1}(c) \), then it might be hard to determine which entries in row \( i \) of \( c \) to revert while still assuring that \( z^c \) appears in \( \varepsilon^w_w(z^b) \). For example, let \( c \) be the last array in Example 2.2.6, that is, \( c = (\mu_3)^0(b) \). Suppose further that the dot at \((4,5)\), is really blank—i.e. zero. Without a priori knowing \( b \), how are we to know not to revert the 1 in position \((3,5)\)? Well, let’s suppose we did revert this entry, along with all of the entries west of it in row 3. Then we’d end up with an array \( b' \) such that \( z^b' \not\in J_w \) all right, as per Claim 2.3.6, but also such that \( z_{335}z^b' \not\in J_w \). This latter condition is intolerable, since \( 5 \geq \text{start}_i(b) \) implies that our original \( z^c \) won’t end up in the sum \( \varepsilon^w_w(z^b') \).

Thus the problem with trying to revert the 1 in position \((3,5)\) is that it’s too far east. On the other hand, we might also try reverting only the row 3 entries in columns 1 and 2, but with dire consequences: we end up with an array \( b'' \) such that \( z^{b''} \) is divisible by \( z_{42}z_{34}z_{25} \in J_w \). (This is an example of the antidiagonal \( a \) to be produced after the displayed equation in the proof of Lemma 2.3.10.) We are left with only one choice: revert the boldface \( 2 \) in position \((3,4) \) and all of its more western brethren. □

In general, as in the previous example, the critical column for \( z^c \in J_w \setminus J_{ws} \) is
\[
\text{crit}(c) := \min\{p \leq \text{west}_{i+1}(c) \mid z_{ip} \text{ divides } z^c \text{ and } z_{i+1,p}z^c \not\in J_{ws}\}.
\]

Claim 2.3.9 If \( z^c \in J_w \setminus J_{ws} \), then:
1. the set used to define \( \text{crit}(c) \) is nonempty;
2. reverting \( c_{i,\text{crit}(c)} \) creates an array \( c' \) such that \( z^{c'} \not\in J_{ws} \); and
3. if \( \text{west}_i(c) < \text{crit}(c) \), then the monomial \( z^{c'} \) from part 2 remains in \( J_w \).

Proof. Claim 2.3.6 implies \( \text{west}_i(c) \leq \text{west}_{i+1}(c) \), so \( p' = \max\{p \leq \text{west}_{i+1}(c) \mid c_{ip} \neq 0\} \) is well-defined. If \( a \) is an antidiagonal dividing the monomial whose exponent array is the result of reverting \( c_{ip} \), then \( a \) divides either \( z^c \) or the monomial \( z^b \) from Claim 2.3.6, and neither of these is in \( J_{ws} \). Thus \( a \not\in J_{ws} \) and part 1 is proved. Part 2 is by definition, and part 3 follows from it by Lemmas 2.2.2 and 2.2.3(W). □

Lemma 2.3.10 Suppose \( z^c \in J_w \setminus J_{ws} \) and that \( b \) is obtained by reverting all row \( i \) entries of \( c \) west of or at \( \text{crit}(c) \). Then \( z^b \not\in J_w \), and \( \text{crit}(c) < \text{start}_i(b) \).

Proof. The proof that this \( z^b \) is not in \( J_w \) has two cases. In the first case we have \( \text{crit}(c) = \text{west}_{i+1}(c) \), and Claim 2.3.6 immediately implies the result. In the second case we have \( \text{crit}(c) < \text{west}_{i+1}(c) \), and we can apply Claim 2.3.6 to the monomial \( z^{c'} \in J_w \setminus J_{ws} \), from Claim 2.3.9.

Now we need to show \( z_{ip}z^b \in J_w \) for two kinds of \( p \): for \( p \leq \text{west}_i(c) \) and \( \text{west}_i(c) < p \leq \text{crit}(c) \). (Of course, when \( \text{west}_i(c) = \text{crit}(c) \) the second of these cases is vacuous.) The case \( p \leq \text{west}_i(c) \) is a little easier, so we treat it first.

There is some antidiagonal in \( J_w \) ending on row \( i \) and dividing \( z^c \), by Lemma 2.2.2. When \( p \leq \text{west}_i(c) \), we get the desired result by appealing to Lemma 2.2.3(W).

Next we treat \( \text{west}_i(c) < p \leq \text{crit}(c) \). These inequalities mean precisely that
\[
j = \max\{p' < p \mid c_{ip} \neq 0\}
\]

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is well-defined, and that $z_{i+1,j}^c \in J_{w_{s_i}}$. Any antidiagonal $a \in J_{w_{s_i}}$ dividing $z_{i+1,j}^c$ must contain $z_{i+1,j}$ because $a$ doesn’t divide $z^c$, and the fact that $z^b \notin J_w$ implies that $a$ also doesn’t divide $z^b$. It follows that $a$ intersects row $i$ at some spot in which $c$ is nonzero strictly east of column $j$. This spot is necessarily east of or at $(i, p)$ by construction. Without changing whether $a \in J_{w_{s_i}}$, Lemma 2.2.3(W) says that we may assume $a$ contains $z_{ip}$ itself. This $a$ divides $z_{ip}z^b$, whence $z_{ip}z^b \in J_{w_{s_i}}$. □

The next proposition is the main result of Section 2.3, pinpointing, at the level of individual standard monomials, the relation between $J_w$ and $J_{w_{s_i}}$.

**Proposition 2.3.11** \(H(k[z]/J_{w_{s_i}}; z) = \varepsilon^w_i H(k[z]/J_w; z)\) if length(\(w_{s_i}\)) < length(\(w\)).

**Proof.** We need the sum $H(k[z]/J_{w_{s_i}}; z)$ of monomials outside $J_{w_{s_i}}$ to be obtained from the sum of monomials outside $J_w$ by replacing $z^b \notin J_w$ with $\varepsilon^w_i(z^b)$. We know by Lemma 2.3.3 that $\varepsilon^w_i H(k[z]/J_w)$ is a sum of monomials outside $J_{w_{s_i}}$. Furthermore, no monomial $z^c$ is repeated in this sum: if $z^c \notin J_w$ appears in $\varepsilon^w_i(b)$, then $b$ must equal $c = \mu_i(\varepsilon^w_i(b))$ by Lemma 2.3.3; and if $z^c \in J_w$ then Lemma 2.3.4 applies.

It remains to demonstrate that each monomial $z^c \notin J_{w_{s_i}}$ is equal to $\mu_i d(z^b)$ for some monomial $z^b \notin J_w$ and $d \leq |\text{prom}(b)|$. This is easy if $z^c$ isn’t even in $J_w$: we take $z^b = \mu_i(\varepsilon^w_i(z^b)) = z^c$. Since we can now assume $z^c \in J_w \setminus J_{w_{s_i}}$, the result follows from Lemma 2.3.10, once we notice that the inequality $\text{crit}(c) < \text{start}_i(b)$ there is equivalent to the inequality $d \leq |\text{prom}(b)|$. □

### 2.4 Coarsening the grading

As in Section 2.3, fix a permutation $w$ and an index $i$ such that length(\(w_{s_i}\)) < length(\(w\)). Our goal in this subsection is to prove (in Corollary 2.4.4) that the set of \(\mathbb{Z}^{n}\)-graded Hilbert series $H(k[z]/J_w;x)$ for varying $w$ is closed under Demazure operators. The idea is to combine lifted Demazure operators $\varepsilon^w_i$ with the specialization map $\mathcal{X} : \mathbb{Z}[z] \to \mathbb{Z}[x]$ that sends $z_{qp} \mapsto x_q$, which we also call coarsening the grading from $\mathbb{Z}^{2n}$ to $\mathbb{Z}$. As usual, we present the proof in the “single” case, for ease of notation, but indicate which changes of notation make the arguments work for the $\mathbb{Z}^{2n}$-graded Hilbert series $H(k[z]/J_w;x,y)$, with the specialization $\mathcal{X}_y : \mathbb{Z}[z] \to \mathbb{Z}[x,y^{-1}]$ sending $z_{qp} \mapsto x_q/y_p$.

At the outset, we could hope that $\mathcal{X}_y \circ \varepsilon^w_i = \mathcal{D} \circ \mathcal{X}$ monomial by monomial. However, although this works in some cases (see (11), below) it fails in general. The next lemma will be used to take care of the general case. Its proof is somewhat involved and irrelevant to its application, so we postpone the proof until after Proposition 2.4.3. (In fact, there’s really no reason for anyone to go through the proof of the lemma on their first reading.) Denote by std($J_w$) the set of **standard exponent arrays**: the exponents on monomials not in $J_w$.

**Lemma 2.4.1** There is an involution $\tau : \text{std}(J_w) \to \text{std}(J_w)$ such that $\tau^2 = 1$ and:

1. $\tau b$ agrees with $b$ outside their genes;
2. prom($\tau b$) = prom($b$);
3. if $\mathcal{X}(z^b) = x_{i+1}^\ell x^a$ with $\ell = |\text{prom}(b)|$, then $\mathcal{X}(z^r b) = x_{i+1}^\ell s_i(x^a)$; and

4. $r$ preserves column sums. In other words, if $b' = r b$, then $\sum q b_{qp} = \sum q b'_{qp}$ for any fixed column index $p$.

In particular, $\mathcal{X}(\varepsilon_i^w z^r b) = \overline{\delta}(x_{i+1})(s_i x^a)$.

**Remark 2.4.2** The squarefree monomials outside $J_w$ for a Grassmannian permutation $v$ (that is, a permutation having a unique descent) are in natural bijection with the semistandard Young tableaux of the appropriate shape and content. (This follows from Definition 4.2.5 and Theorem 4.2.11, below, along with the bijection in [Kog00] between $\gamma$-graphs and semistandard Young tableaux.) Under this natural bijection, intron mutation reduces to an operation that arises in a well-known combinatorial proof of the symmetry of the Schur function $\mathcal{S}_v$ associated to $v$. \hfill \Box

Our next proposition justifies the term ‘lifted Demazure operator’ for $\varepsilon_i^w$.

**Proposition 2.4.3** Specializing $z$ to $x$ in $\varepsilon_i^w H(k[z]/J_w; z)$ yields $\overline{\delta}(H(k[z]/J_w; x))$. More generally, specializing $z_{qp}$ to $x_{qp}$ yields $\overline{\delta}(H(k[z]/J_w; z))$.

**Proof.** Suppose $z^b \notin J_w$ specializes to $\mathcal{X}(z^b) = x_{i+1}^\ell x^a$, where $\ell = |\text{prom}(b)|$. The definition of $\varepsilon_i^w z^b$ implies that

$$
\mathcal{X}(\varepsilon_i^w z^b) = \sum_{d=0}^\ell x_{i+1}^{\ell-d} x^a
= \frac{x_{i+1}^{\ell} x_{i+1}^{-1}}{x_{i+1}^{\ell} x_{i}^{-1}} x^a
= \overline{\delta}(x_{i+1}^{\ell}) x^a.
$$

If it happens that $s_i x^a = x^a$, so $x^a$ is symmetric in $x_i$ and $x_{i+1}$, then

$$
\mathcal{X}(\varepsilon_i^w z^b) = \overline{\delta}(x_{i+1}^{\ell}) x^a = \overline{\delta}(x_{i+1}^{\ell}) x^a = \overline{\delta}(\mathcal{X}(z^b)). \tag{11}
$$

Of course, there will in general be lots of $z^b \notin J_w$ whose $x^a$ isn’t fixed by $s_i$. We overcome this difficulty using Lemma 2.4.1, which says how to pair each $z^b \notin J_w$ with a partner so that their corresponding $\mathcal{X} \circ \varepsilon_i^w$ sums add up nicely. Using the notation of the Lemma, notice that if $r b = b$, then $s_i x^a = x^a$ and $\mathcal{X}(\varepsilon_i^w z^b) = \overline{\delta}(\mathcal{X}(z^b))$, as in (11). On the other hand, if $r b \neq b$, then the Lemma implies

$$
\mathcal{X}(\varepsilon_i^w (z^b + z^r b)) = \overline{\delta}(x_{i+1}^{\ell})(x^a + s_i x^a)
= \overline{\delta}(x_{i+1}^{\ell}(x^a + s_i x^a))
= \varepsilon_i^w(\mathcal{X}(z^b + z^r b))
$$

because $x^a + s_i x^a$ is symmetric in $x_i$ and $x_{i+1}$. This proves the $\mathbb{Z}^n$-graded statement.

The $\mathbb{Z}^{2n}$-graded version of the argument works mutatis mutandis by the preservation of column sums under mutation (Definition 2.2.5 and part 4 of Lemma 2.4.1), which allows us to replace $\mathcal{X}$ by $\mathcal{X}_Y$ and $x^a$ by a monomial in the $x$ variables and the inverses of the $y$ variables. \hfill \Box

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Propositions 2.3.11 and 2.4.3 imply the result used in the proof of Theorem 2.1.2.

**Corollary 2.4.4** \( H(k|z|/J_w; x) = \overline{H}(k|z|/J_w; x) \) if \( \text{length}(w_i) < \text{length}(w) \). More generally, \( H(k|z|/J_w; x, y) = \overline{H}(k|z|/J_w; x, y) \) if \( \text{length}(w_i) < \text{length}(w) \).

Before constructing this magic involution \( \tau \), we introduce some necessary notation and provide examples. Recall that the union of rows \( i \) and \( i+1 \) is the **gene** of \( b \) (we view the row index \( i \) as being fixed for the discussion). Order the boxes in columns east of \( \text{start}_i(b) \) in the gene of \( b \) as in the diagram, using the notation \( \text{start}_i(b) \) from (8) on page 29:

\[
\begin{array}{c|c|c|c|c|c}
\text{start}_i(b) & 1 & 3 & 5 & 7 & \ldots \\
\hline
i & \ldots & 2 & 4 & 6 & 8 & \ldots \\
i+1 & & & & & & \\
\end{array}
\]

Now define five different kinds of blocks in the gene of \( b \), called the promoter, the start codon, exons, introns, and the stop codon.\(^3\) In the following, \( k, \ell \in \mathbb{N} \).

- **promoter**: the rectangle consisting of unnumbered boxes at the left end
- **start codon**: the box numbered 1, which lies at \( (i, \text{start}_i(b)) \)
- **stop codon**: the last numbered box, which lies at \( (i+1, n) \)
- **exon**: any sequence \( 2k, \ldots, 2\ell+1 \) (with \( k \leq \ell \)) of consecutive boxes satisfying:
  1. the entries of \( b \) in the boxes corresponding to \( 2k, \ldots, 2\ell+1 \) are all zero;
  2. either box \( 2k+1 \) is the start codon, or box \( 2k \) has a nonzero entry in \( b \); and
  3. either box \( 2\ell \) is the stop codon, or box \( 2\ell+1 \) has a nonzero entry in \( b \)
- **intron**: any rectangle of consecutive boxes \( 2\ell+1, \ldots, 2k \) (with \( \ell < k \)) satisfying:
  1. the rectangle contains no exons;
  2. box \( 2\ell+1 \) is either the start codon or the last box in an exon; and
  3. box \( 2k \) is either the stop codon or the first box in an exon

Roughly speaking, the nonzero entries in gene(\( b \)) are partitioned into the promoter and introns, the latter being contiguous rectangles having nonzero entries in their northwest and southeast corners. Exons connect adjacent introns via bridges of zeros.

**Example 2.4.5** Suppose we're given a permutation \( w \), an array \( b \) such that \( z^b \notin J_w \), and a row index \( i \) such that \( \text{start}_i(b) = 4 \) and \( b \) has the gene in Figure 2. The gene of \( b \) breaks up into promoter, start codon, exons, introns, and stop codon as indicated. We will say something more about the mutated gene \( \tau b \) in Example 2.4.7. \( \square \)

---

\( ^3 \) All of these are terms from genetics. The DNA sequence for a single gene is not necessarily contiguous. Instead, it sometimes comes in blocks called **exons**. The intervening DNA sequences whose data are excised are called **introns** (note that the structure of the gene of an exon array is determined by its exons, not its introns). The **promoter** is a medium-length region somewhat before the gene that signals the transcriptase enzyme where to attach to the DNA, so that it may begin transcribing the DNA into RNA. The **start codon** is a short sequence signaling the beginning of the actual gene; the **stop codon** is a similar sequence signaling the end of the gene.
For ease of comparison, we dissect here the mutated gene $\tau b$ of $b$.

Figure 2: Intron mutation

If $c$ is an array having two rows filled with nonnegative integers, then let $\mathcal{C}$ be the rectangle obtained by rotating $c$ through an angle of $180^\circ$. When $c$ is an intron, the rows of $c$ are identified as rows $i$ and $i + 1$ in a gene, and we view $c$ as an $n \times n$ array that happens to be zero outside of its $2 \times k$ rectangle.

**Definition 2.4.6 (Intron mutation)** Let $c_i$ and $c_{i+1}$ be the sums of the entries in the top and bottom nonzero rows of an intron $c$, and set $d = |c_i - c_{i+1}|$. Then

$$\tau c = \begin{cases} 
\mu^d(\mathcal{C}) & \text{if } c_i > c_{i+1} \\
\mu^d(c) & \text{if } c_i < c_{i+1}
\end{cases}$$

is the mutation of $c$. Define the **intron mutation** $\tau b$ of an exponent array $b$ by

- adding 1 to the start and stop codons of $b$;
- mutating every intron in the gene of the resulting exponent array; and then
- subtracting 1 from the boxes that were the start and stop codons of $b$.  

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Intron mutation pushes the entries of each intron either upward from left to right or downward from right to left—whichever initially brings the row sums in that particular intron closer to agreement.

**Example 2.4.7** Although the "look" of $b$ in Example 2.4.5 completely changes when it is mutated into $\tau b$, the columns of $\tau b$ containing a nonzero entry are exactly the same as those in $b$, and the column sums are preserved. Note that mutating the gene of $\tau b$ yields back the gene of $b$, as long as $z^c \notin J_w$ and the location of the start codon hasn't changed. The proof of Lemma 2.4.1 shows why $\tau$ always works this way. \hfill \square

**Lemma 2.4.8** *Intron mutation outputs an exponent array (that is, the entries are nonnegative).* Assume, for the purpose of defining exons in $\tau b$, that the start codon of $\tau b$ lies at the same location as the start codon in $b$. The boxes occupied by exons of $\tau b$ thus defined coincide with the boxes occupied by exons of $b$ itself.

*Proof.* The definitions ensure that any intron not containing the start or stop codon has nonzero northwest and southeast corners. After adding 1 to the start and stop codons, every intron has this property. Mutation of such an intron leaves strictly positive entries in the northwest and southeast corners (this is crucial—it explains why we have to add and subtract the 1’s from the codons), so subtracting 1 preserves nonnegativity. Furthermore, intron mutation does not introduce any new exons, because the nonzero entries in an intron both before and after mutation follow a snake pattern that drops from row $i$ to row $i + 1$ precisely once. \hfill \square

*Proof of Lemma 2.4.1.* First we show that $\tau b \in \text{std}(J_w)$, or equivalently that $z^{\tau b} \in J_w \Rightarrow z^b \in J_w$. Observe that $z^b \in J_w$ if and only if $z_{ip}z_{i+1,n}^b \in J_w$, where $p = \text{start}_i(b)$, by definition of $\text{start}_i(b)$ and the fact that $z_{i+1,n}$ is a nonzerodivisor modulo $J_w$ for all $w$. Therefore, it suffices to demonstrate how an antidiagonal $a \in J_w$ dividing $z^{\tau b}$ gives rise to a possibly different antidiagonal $a' \in J_w$ dividing $z_{ip}z_{i+1,n}^b$, where $p = \text{start}_i(b)$. There are five cases:

(i) $a$ intersects neither row $i$ nor row $i + 1$;
(ii) the southwest variable in $a$ is in row $i$;
(iii) $a$ intersects row $i$ and continues south, but skips row $i + 1$;
(iv) $a$ intersects both row $i$ and row $i + 1$; or
(v) $a$ skips row $i$ but intersects row $i + 1$.

In each of these cases, $a'$ is constructed as follows. Outside the gene of the generic matrix $Z$, the new $a'$ will agree with $a$ in all five cases, since $b$ and $\tau b$ agree outside of their genes. Inside their genes, we may need some adjustments.

(i) Leave $a' = a$ as is.
(ii) Move the variable in row $i$ west to $z_{ip}$, using Lemma 2.2.3(W).
(iii) The gene of $b$ has nonzero entries in precisely the same columns as the gene of $\tau b$, by definition. Either $a$ already divides $z_{ip}z^b$, or moving the variable in row $i$ due south to row $i + 1$ yields $a'$ by Lemma 2.2.3(S).
(iv) Use Example 2.2.4 to make $a'$ contain the nonzero entries in some exon of $b$ (see Lemma 2.4.8).

(v) Same as (iii), except that either $a$ already divides $z_{i+1,n} z^b$ or Lemma 2.2.3(N) says we can move the variable due north from row $i+1$ to row $i$.

Now that we know $\tau b \in \text{std}(J_w)$, we find that

$$\text{start}_{\tau}(\tau b) = \text{start}_{i}(b).$$

Indeed, when $j \leq \text{start}_{i}(b)$, we have $z_{ij} \tau^b \in J_w$ if and only if $z_{ij} z^b \in J_w$, because any antidiagonal containing $z_{ij}$ interacts with $\tau b$ north of row $i$ and west of column $\text{start}_{i}(b)$, where $\tau b$ agrees with $b$. It follows that $\text{prom}(\tau b) = \text{prom}(b)$, so $\tau b$ has the same exons as $b$ by Lemma 2.4.8. We conclude that $\tau b$ also has the same introns as $b$. The statement $\tau^2 = b$ identity holds because the partitions of the genes of $b$ and $\tau b$ into promoter and introns are the same, and mutation on each of these blocks in the partition has order 1 or 2. Part 3 follows intron by intron, except that the added and subtracted 1's in the first and last introns cancel. $\square$

2.5 The initial complexes $L_w$: equidimensionality

Let $L_w$ be the Stanley-Reisner simplicial complex of $J_w$, with vertex set $|n|^2 = \{(q, p) \mid 1 \leq q, p \leq n\}$. That is, $L_w$ consists of the subsets of $|n|^2$ containing no antidiagonal in $J_w$. Faces of $L_w$ may be identified with coordinate subspaces in $\text{Spec}(k[z]/J_w) \subseteq M_n$, as follows. Denote by $E_{qp}$ the elementary matrix whose nonzero entry is in row $q$ and column $p$. We identify the vertices with the variables $z_{qp}$ in the generic matrix $Z$, so that a coordinate subspace

$$L = \{z_{qp} = 0 \mid (q, p) \in D_L \subseteq |n|^2\} = \text{Span}(E_{qp} \mid (q, p) \notin D_L) \quad (12)$$

is identified with the subset $|n|^2 \setminus D_L$, when $L$ is being regarded as a face of $L_w$. We demonstrate here that the facets of $L_w$ all have the same dimension.

The monomials $z^b$ that are nonzero in $k[z]/J_w$ are the so-called standard monomials for $J_w$, and are precisely those with support sets

$$\text{supp}(z^b) := \{z_{qp} \in Z \mid z_{qp} \text{ divides } z^b\}$$

in the complex $L_w$. In particular, the maximal support sets of standard monomials are the facets of $L_w$. (We will see in Theorem 4.2.11 that the subsets $D_L \subseteq |n|^2$ for facets $L \in L_w$ are the rc-graphs for the permutation $w$.) The next four items (Examples 2.5.1–Lemma 2.5.3) serve as preparation for the task of verifying that these facets all have the same cardinality. (The next item isn’t really an example of anything, but we have to give it a number so we can refer to it.)

**Example 2.5.1** Suppose $v = w_0 s_1 s_2 \cdots s_i$, where $s_1 s_2 \cdots s_i$ is the lexicographically first reduced expression for $w_0 v$ in which $s_1 > s_2 > \cdots > s_{n-1}$. For the identity permutation $v = 12345 \in S_5$ the lex first expression for $w_0 v$ is $s_1 s_2 s_1 s_3 s_2 s_1 s_4 s_3 s_2 s_1$.
and if $v = 31524$ then $w_0v = s_2s_1s_3s_4s_3s_2$ is lex first. In general, one finds the lex first expression for $w_0v$ by transforming the $n \times n$ antidiagonal matrix $w_0^T$ into the permutation matrix $v^T$, beginning with sliding the 1 in column $n-1$ north into the correct position relative to the 1 in column $n$, then sliding the 1 in column $n-2$ north into the correct position relative to the 1's in columns $n$ and $n-1$, etc. Because the 1's in columns to the west of the sliding 1 are as far south as possible, the entry of $\text{rk}(w) = \text{rk}(v s_i)$ at position $(i+1, w(i+1))$ is 1.

**Lemma 2.5.2** If $z^b$ divides $z^c \notin J_w$ and $d \leq |\text{prom}(b)|$, then $\mu_i^d(z^b)$ divides $\mu_i^e(z^c)$ for some $e \leq |\text{prom}(c)|$. If $\text{supp}(z^b) \in \mathcal{L}_w$ is not a facet and $d \leq |\text{prom}(b)|$, then $\text{supp}(\mu_i^d(z^b)) \in \mathcal{L}_{w s_i}$ is not a facet.

**Proof.** If $z^b$ divides $z^c$, then $\text{start}_i(b) \leq \text{start}_i(c)$ by definition ($z_{ip}z^b \in J_w \Rightarrow z_{jq}z^c \in J_w$). Therefore, if the $d^{th}$ mutation of $b$ is the $k^{th}$ occurring in column $j < \text{start}_i(b)$, we can choose $e$ so that the $e^{th}$ mutation of $c$ is also the $k^{th}$ occurring in column $j$. If, in addition, $z_{qp} \in \text{supp}(z^c) \setminus \text{supp}(z^b)$, then either $z_{qp}$ or $z_{q-1,p}$ ends up in $\text{supp}(\mu_i^e(z^c)) \setminus \text{supp}(\mu_i^d(z^b))$, depending on whether or not $(q,p) \in \text{prom}(c)$ and $p < j$. Note that $\mu_i^d(z^b)$ and $\mu_i^e(z^c)$ are not in $J_{w s_i}$ by Proposition 2.3.11, so their supports are in $\mathcal{L}_{w s_i}$.

**Lemma 2.5.3** Suppose $w \in S_n$ and $\text{rk}(w)_{i+1,w(i+1)} = 1$. Then:

1. length($w s_i$) < length($w$); and
2. $z^b \notin J_w$ implies $b_{qp} = 0$ if $(q,p) \leq (i+1, w(i+1) - 1)$ or $(q,p) \leq (i, w(i) - 1)$.

Every permutation $v \in S_n$ aside from $w_0$ may be written as $v = ws_i$ for some $w$ and $i$ satisfying these hypotheses.

**Proof.** The condition on $\text{rk}(w)$ implies part 1 because $(i, w(i))$ is necessarily northeast of $(i+1, w(i+1))$. Part 2 is then immediate from the definitions of $\text{rk}(w)$ and $J_w$. Example 2.5.1 implies the final statement.

**Claim 2.5.4** Assume the conditions of Lemma 2.5.3, and that $z_{i+1,w(i+1)}$ divides $z^b$ whenever $\text{supp}(z^b) \in \mathcal{L}_w$ is a facet. If $z^b \notin J_w$ has maximal support then $\text{west}_{i+1}(b) = w(i+1) < \text{start}_i(b) = \text{west}_i(b)$.

**Proof.** Maximality of support and part 2 of Lemma 2.5.3 imply $\text{start}_i(b) = \text{west}_i(b)$. That $w(i+1) < \text{start}_i(b)$ is then by part 1 of Lemma 2.5.3. Part 2 of Lemma 2.5.3 implies $w(i+1) \leq \text{west}_{i+1}(b)$, so $w(i+1) = \text{west}_{i+1}(b)$ by the hypothesis on $z_{i+1,w(i+1)}$.

**Lemma 2.5.5** Under the conditions of Lemma 2.5.3, $z_{i+1,w(i+1)}$ divides $z^b$ when $\text{supp}(z^b) \in \mathcal{L}_w$ is a facet.

**Proof.** Since this is obvious for $w = w_0$, we may prove it by downward induction on length($w$). More precisely, we need to show that if the statement

$$\text{rk}(w)_{k,w(k)} = 1 \Rightarrow z_{k,w(k)} \in F \text{ for all facets } F \in \mathcal{L}_w$$

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is true, then the statement holds with \( ws_i \) in place of \( w \) when \( \text{rk}(w)_{i+1, w(i+1)} = 1 \).

By Proposition 2.3.11 and Lemma 2.5.2, every facet of \( \mathcal{L}_{ws_i} \) is \( \text{supp}(\mu_i^d(z^b)) \) for some \( b \) such that \( \text{supp}(z^b) \) is a facet of \( \mathcal{L}_w \). Since mutation only affects rows \( i \) and \( i + 1 \) (the gene of \( b \)), we find that \( z_{k, w(k)} \in F \) for all facets \( F \in \mathcal{L}_w \) implies \( z_{k, ws_i(k)} \in F \) for all facets \( F \in \mathcal{L}_{ws_i} \), as long as \( k \neq i, i + 1 \), in which case \( w(k) = ws_i(k) \). On the other hand, \( ws_i(i + 1) = w(i) \) and \( \text{rk}(ws_i)_{i+1, w(i)} \geq 2 \) (the two 1's in the right-hand picture in (4) on p. 17), so we needn't worry about \( k = i + 1 \). It remains to show that \( z_{i, ws_i(i+1)} \in F \) for all facets \( F \in \mathcal{L}_{ws_i} \), given the conclusion of the Lemma.

Let \( z^b \notin J_w \) have maximal support. By Lemma 2.5.2, we may assume that \( b_{i+1, ws_i(i+1)} \geq 2 \). Observe that \( \text{supp}(z^b) \) is not itself a facet of \( \mathcal{L}_{ws_i} \), because \( \text{supp}(z^b) \not\subseteq \text{supp}(\mu_i^d(z^b)) \). Therefore, every facet of \( \mathcal{L}_{ws_i} \) can be expressed as \( \text{supp}(\mu_i^d(z^b)) \) for some non-zero \( d \leq |\text{prom}(b)| \), and Claim 2.5.4 says that \( z_{i, ws_i(i+1)} \) divides all of these. \( \Box \)

If we knew a priori that \( J_w \) were the initial ideal of \( I(X_w) \), the following Proposition would follow from [KS95] (except for embedded components). Since we are working in the opposite direction, we have to prove it ourselves.

**Proposition 2.5.6** The simplicial complex \( \mathcal{L}_w \) is pure, each facet having \( \dim X_w = n^2 - \text{length}(w) \) vertices; i.e. \( \text{Spec}(k[z]/J_w) \) is equidimensional of dimension \( \dim X_w \).

**Proof.** \( \mathcal{L}_{w_0} \) has only one facet, with \( \binom{n+1}{2} = n^2 - \binom{n}{2} \) vertices \( \{z_{qp} \in Z \mid q + p > n\} \), so we may again resort to downward induction on \( \text{length}(w) \). In proving purity of \( \mathcal{L}_{ws_i} \) from that of \( \mathcal{L}_w \), we may assume by Lemma 2.5.3 that \( w \) and \( i \) satisfy the conditions put forth there. Proposition 2.3.11 implies that the facets of \( \mathcal{L}_{ws_i} \) are supports of monomials \( \mu_i^d(z^b) \) for \( z^b \notin J_w \) and \( d \leq |\text{prom}(b)| \). By Lemma 2.5.2, we may restrict our attention to square monomials \( z^{2b} \) with maximal support.

By definition, repeated mutation can only increase the support size of a monomial by 0 or 1. It suffices therefore to demonstrate that if the cardinalities of \( \text{supp}(\mu_i^d(z^{2b})) \) and \( \text{supp}(z^{2b}) \) are equal for some \( d \leq |\text{prom}(2b)| \), then there is some \( e \leq |\text{prom}(2b)| \) such that \( \text{supp}(\mu_i^d(z^{2b})) \not\subseteq \text{supp}(\mu_i^e(z^{2b})) \). If \( d = 0 \), then we may take \( e = 1 \) by Lemma 2.5.5. If \( d > 0 \), then mutation will have just barely pushed a row \( i + 1 \) entry of \( 2b \) up to row \( i \), and we may take \( e = d - 1 \). Containment requires the appropriate entry \( b_{ij} \) to be \( \geq 2 \); strict containment is automatic, by checking cardinalities. \( \Box \)

**Example 2.5.7** The ideal \( J_{1432} \) is generated by the antidiagonals of the five \( 2 \times 2 \) minors contained in the union of the northwest \( 2 \times 3 \) and \( 3 \times 2 \) submatrices of \( (z_{ij}) \):

\[
J_{1432} = \langle z_{12}z_{21}, z_{13}z_{21}, z_{13}z_{22}, z_{12}z_{31}, z_{22}z_{31} \rangle
\]

\[
= \langle z_{12}, z_{13}, z_{22} \rangle \cap \langle z_{12}, z_{21}, z_{22} \rangle \cap \langle z_{21}, z_{22}, z_{31} \rangle \cap \langle z_{13}, z_{21}, z_{31} \rangle \cap \langle z_{12}, z_{13}, z_{31} \rangle.
\]

\( \mathcal{L}_{1432} \) is the join of a pentagon with a simplex having 11 vertices \( \{z_{11}\} \cup \{z_{rs} \mid r + s \geq 5\} \) (note \( n = 4 \) here). Each facet of \( \mathcal{L}_w \) therefore has \( 13 = 4^2 - \text{length}(1432) \) vertices. \( \Box \)

We will have much more to say about the simplicial complexes \( \mathcal{L}_w \) throughout Section 4, including better reasons why Proposition 2.5.6 is true. In particular, the reader is referred to Proposition 4.2.2, Theorem 4.2.11, and Corollary 4.5.7.
3 Applications of the Gröbner basis

Sections 3.1–3.2 apply the Gröbner basis Theorem 2.1.2 to Schubert and Grothendieck polynomials. Sections 3.3–3.5 relation of our Gröbner basis to some others in the literature. Along the way, Section 3.3 contains a statement concerning minimal Gröbner bases, and Section 3.4 provides a closed formula for the degrees of all one-sided ladder determinantal varieties. Finally, Section 3.6 outlines the connection between matrix Schubert varieties and degeneracy loci for maps between vector bundles.

3.1 Positive formulae for Schubert polynomials

The original definition of Schubert polynomials was by Lascoux and Schützenberger, via the divided difference operator recursion (our Theorem 1.4.3; see also Corollary 1.5.2). Since this formula involves negation, it is quite nonobvious from it that the coefficients of $S_w(x)$ are in fact positive. This was first proven in [BJS93, FS94] combinatorially, and [BS00, Kog00] geometrically.

We prove it now as an equation on multidegrees:

**Theorem 3.1.1** The formula

$$[X_w] = \sum_{L \in \mathcal{L}_w} [L]$$

holds for $\mathbb{Z}^n$-graded and $\mathbb{Z}^{2n}$-graded multidegrees, where $[L]_{\mathbb{Z}^n} = \prod_{(q,p) \in D_L} x_p$, and $[L]_{\mathbb{Z}^{2n}} = \prod_{(q,p) \in D_L} (x_p - y_q)$. For the $\mathbb{Z}^n$-grading this is a formula for Schubert polynomials with positive coefficients. For the $\mathbb{Z}^{2n}$-grading, it is a formula for double Schubert polynomials with positive coefficients, if written in the variables $x$ and $-y$.

**Proof.** Apply Theorem 2.1.2 to the second part of Corollary 1.2.6: the multidegree of $X_w$ in any grading equals the multidegree of the zero set $\mathcal{L}_w$ of $J_w$. The specific formulae for the multidegrees of coordinate subspaces is (2) in Example 1.2.3, along with the specified weights earlier in Section 1.2.

While it is nice to see the positivity demystified, this formula will become much more effective when we have a better description of the facets of $\mathcal{L}_w$, in Section 4. At that point our formula for Schubert polynomials will be seen to be equivalent to that in [BJS93, FS94, BB93], while our formula for double Schubert polynomials will follow from [FK96b, p. 134–5]. (This bypasses the “double re-graphs” of [BB93].) 

**Remark 3.1.2** The version of this positivity in algebraic geometry is the notion of “effective homology class”, meaning that it is representable by a subscheme.

On the flag manifold, a homology class is effective exactly if it is a nonnegative combination of Schubert classes. (Proof: one direction is a tautology. For the other, if $X$ is a subscheme of the flag manifold, let $B$ act on the Hilbert scheme containing the point $X$, and look at the closure of the $B$-orbit through $X$. This will be projective.
because the Hilbert scheme is, so by Borel's theorem there will be a fixed point, necessarily a union of Schubert varieties, perhaps nonreduced.) In particular the classes of monomials in the $x_i$ (the first Chern classes of the standard line bundles; see Appendix A.5) are not usually effective.

We work instead on $M_n$, where a class is effective exactly if it is a nonnegative combination of monomials. (Proof: instead of using $B$ to degenerate a subscheme $X$, use a 1-parameter subgroup of the $n^2$-dimensional torus. Algebraically, this exactly amounts to picking a Gröbner basis.)

3.2 Grothendieck polynomials and $K$-theory

This subsection recovers a geometric result from our algebraic treatment of matrix Schubert varieties: the Grothendieck polynomials represent the $K$-classes of ordinary Schubert varieties in the flag manifold. We use standard facts about flag variety and notions from equivariant algebraic $K$-theory, on which background material can be found in Appendix A.1 and Appendix A.5.

Recall that the $K$-cohomology ring $K^0(\mathcal{F}_\ell_n)$ is the quotient of $\mathbb{Z}[x]$ by the ideal

$$K_n = \langle e_d(x) - \binom{n}{d} \mid d \leq n \rangle,$$

where $e_d$ is the $d$th elementary symmetric function. These relations hold in $K^0(\mathcal{F}_\ell_n)$ because the exterior power $\bigwedge^d \mathbb{C}^n$ of the trivial rank $n$ bundle is itself trivial, of rank $\binom{n}{d}$. Furthermore, there can be no more relations, because $\mathbb{Z}[x]/K_n$ is an abelian group of rank $n!$. Indeed, substituting $\bar{x}_k = 1 - x_k$, we find that $\mathbb{Z}[\bar{x}]/K_n \cong \mathbb{Z}[\bar{x}]/\bar{K}_n$, where $\bar{K}_n = \langle e_d(\bar{x}) \mid d \leq n \rangle$, and this quotient has rank $n!$ because it is the familiar cohomology ring of $\mathcal{F}_\ell_n$.

Thus it makes sense to say that a polynomial in $\mathbb{Z}[x]$ “represents a class” in $K^0(\mathcal{F}_\ell_n)$. Lascoux and Schützenberger, based on work of Bernstein-Gel’fand-Gel’fand [BGG73] and Demazure [Dem74], realized that the classes $[\mathcal{O}_{X_w}] \in \mathbb{Z}[x]/K_n$ of (structure sheaves of) Schubert varieties could be represented independently of $n$. To make a precise statement, let $\mathcal{F}_{\ell N} = B/\text{GL}_N$ be the manifold of flags in $\mathbb{C}^N$ for $N \geq n$, so $B$ is understood to consist of $N \times N$ lower triangular matrices. Let $X_{w,N} \subseteq \mathcal{F}_{\ell N}$ be the Schubert variety for the permutation $w \in S_n$ considered as an element of $S_N$ that fixes $n+1, \ldots, N$.

**Corollary 3.2.1 ([LS82, Las90])** The Grothendieck polynomial $\mathcal{G}_w(x)$ represents the class $[\mathcal{O}_{X_{w,N}}] \in K^0(\mathcal{F}_{\ell N})$ for all $N \geq n$.

The main point is Theorem 2.1.2, but we do still need a couple more lemmas. Note that $\mathcal{G}_w(x)$ is expressed without reference to $N$. Here is the reason why.

**Lemma 3.2.2** The Grothendieck polynomial $\mathcal{G}_w(x)$ in $n$ variables equals the Grothendieck polynomial $\mathcal{G}_{w_N}(x_1, \ldots, x_N)$, whenever $w_N$ agrees with $w$ on $1, \ldots, n$ and fixes $n+1, \ldots, N$. 

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Proof. The ideal $I_{w,N}$ in the polynomial ring $k[z_{ij} \mid i, j = 1, \ldots, N]$ is extended from the ideal $I_w$ in the multigraded polynomial subring $k[z]$. Therefore $I_{w,N}$ has the same multigraded Betti numbers as $I_w$, so their Hilbert numerators agree. □

Lemma 3.2.3 The natural map $K_B^0(M_n) \to K_T^0(M_n) \cong \mathbb{Z}[x^{\pm 1}]$ is an isomorphism.

Proof. Suppose $E$ is a $B$-equivariant vector bundle on $M_n$. The global sections $\Gamma E$ form a free $\mathbb{Z}^n$-graded $k[z]$-module because $T \subset B$ (see Proposition A.1.8). Any minimal generator of $\Gamma E$ invariant under the unipotent matrices $N \subset B$ generates a $B$-submodule $\Gamma E'$ that is split as a $T$-submodule. Since $|E|_B = |E'|_B + |E/E'|_B$, it follows by induction on rank that $|E|_B$ is a sum of classes $|E'|_B$ for line bundles $E'$. But $B$-equivariant line bundles are the same as $T$-equivariant line bundles, both being uniquely determined by the caracter of $T$ on the $k$-vector space spanned by a generator of the global section module. □

Proof of Corollary 3.2.1. In view of Lemma 3.2.2, we may as well assume $N = n$, since $\mathcal{G}_w$ doesn’t care anyway. Let us justify the following diagram:

$$
\begin{array}{c}
X_w \\
\cap
\end{array} \quad 
\begin{array}{c}
\overline{X}_w \\
\cap
\end{array} 
\quad 
\begin{array}{c}
B \backslash GL_n \\
\leftrightarrow \\
GL_n \\
\hookrightarrow \\
M_n \\
K^0(B \backslash GL_n) \\
\cong \\
K_B^0(GL_n) \\
\hookleftarrow \\
K_T^0(M_n)
\end{array}
$$

Pulling back vector bundles under the quotient map $B \backslash GL_n \leftrightarrow GL_n$ induces the isomorphism $K^0(B \backslash GL_n) \to K_B^0(GL_n)$. The inclusion $GL_n \hookrightarrow M_n$ induces a surjection $K_B^0(GL_n) \twoheadrightarrow K_B^0(M_n)$ because the classes of (structure sheaves of) algebraic cycles generate both of the equivariant $K$-homology groups $K_B^0(M_n)$ and $K_T^0(GL_n)$.

Now let $\tilde{X}_w = \overline{X}_w \cap GL_n$. Any $B$-equivariant resolution of $\mathcal{O}_{\tilde{X}_w} = k[z]/I_w$ by vector bundles on $M_n$ pulls back to a $B$-equivariant resolution $E_\cdot$ of $\mathcal{O}_{\tilde{X}_w}$ on $GL_n$. Viewing a vector bundle on $GL_n$ as a geometric object (i.e. as the scheme $\text{Spec}(\text{Sym}^* \mathcal{E}^\vee)$ rather than its sheaf of sections $\mathcal{E}$), the quotient $B \backslash E_\cdot$ is a resolution of $\mathcal{O}_{\tilde{X}_w}$ by vector bundles on $B \backslash GL_n$. Thus $[\mathcal{O}_{\tilde{X}_w}]_B \in K_B^0(M_n)$ maps to $[\mathcal{O}_{X_w}] \in K^0(B \backslash GL_n)$. The result follows by identifying the $B$-equivariant class $[\mathcal{O}_{\tilde{X}_w}]_B$ as the $T$-equivariant class $[\mathcal{O}_{\tilde{X}_w}]_T$ as in Lemma 3.2.3, and identifying the $T$-equivariant class as the Hilbert numerator $\mathcal{K}(k[z]/I_w; x) = \mathcal{G}_w(x)$ by Proposition A.1.8 applied to Theorem 2.1.2. □

Remark 3.2.4 The same line of reasoning recovers the double version of Corollary 3.2.1, in which $K^0(F\ell_N)$ is replaced by the equivariant $K$-theory $K^0_B(F\ell_N)$ for the action of the invertible upper triangular matrices on the right. □

3.3 Minimal Gröbner bases and essential sets

Our main purpose in the next three subsections is to relate our Gröbner bases and connections with Schubert varieties to those of other authors. Before we can say anything meaningful, however, we need some preliminary results and notation. Therefore, this
subsection and the next contain some refinements and special cases of Theorem 2.1.2 that are interesting in their own right, but also serve to facilitate the comparative discussion in Section 3.5. Here, we determine a minimal Gröbner basis for \( I_w \).

Define a **one-sided ladder**\(^4\) to be an order ideal in the poset \( \mathbb{Z}_{>0} \times \mathbb{Z}_{>0} \)—that is, a subset \( \mathbb{L} \) such that

\[
(q, p) \in \mathbb{L} \Rightarrow (q', p') \in \mathbb{L} \quad \text{for all } q' \leq q \text{ and } p' \leq p.
\]

Although we usually drop the adjective ‘one-sided’ in what follows, so that ‘ladder’ without qualification always means ‘one-sided ladder’, it is there to distinguish these ladders from the ‘two-sided ladders’ that appear in the literature on determinantal ideals. The **southeast corners** of a ladder \( \mathbb{L} \) are those locations \( (q, p) \) such that \((q', p')\) lies outside \( \mathbb{L} \) whenever \( q' > q \) or \( p' > p \). The southeast corners are called **outside corners** in the literature, but often appear graphically as northeast corners due to reorientation of the ladder.

The entries of the rank matrix \( \text{rk}(w) \) that are bounded above by \( r \) form a ladder. These **rank ladders** are nested inside the \( n \times n \) array \( [n]^2 \) for increasing \( r \):

\[
\Lambda_0(w) \subseteq \Lambda_1(w) \subseteq \cdots \subseteq \Lambda_r(w) = [n]^2, \quad \text{where } \Lambda_r(w) = \{(q, p) \mid \text{rk}(w)_{qp} \leq r\}.
\]

Setting \( \Lambda_{-1} = \emptyset \) by convention, define the **essential set** of \( w \) to be the set

\[
\mathcal{E}ss(w) = \bigcup_{r=0}^{n-1} \{\text{southeast corners } (q, p) \in \Lambda_r \mid (q - 1, p - 1) \notin \Lambda_{r-1}\} \quad (13)
\]

of corners whose immediate northwest neighbor doesn’t lie in a smaller rank ladder.

**Example 3.3.1** Let \( w = 13865742 \) as in Example 2.1.4. The ladder \( \Lambda_0(w) \) is empty, whereas \( \Lambda_1(w) \) and \( \Lambda_2(w) \) are depicted in Example 2.1.4. The essential set of \( w \) has four elements: the single southwest corner \((7, 2)\) of \( \Lambda_1(w) \), and the three southwest corners \((6, 4)\), \((4, 5)\), and \((3, 7)\) of \( \Lambda_2(w) \). Observe that some southeast corners of \( \Lambda_2(w) \) don’t make it into \( \mathcal{E}ss(w) \), and that none of the southeast of corners of \( \Lambda_i(w) \) for \( 3 \leq i \leq 7 \) make it in.

This alternate definition of essential set agrees with the original definition due to Fulton, because of the sentence after [Ful92, Eq. (3.8)], which in our language reads: “A point \((q, p)\) is in \( \mathcal{E}ss(w) \) if no entry 1 in the permutation matrix \( w^T \) lies due west or due north of or at \((q, p)\), and no entry 1 in \( w^T \) lies due east of or due south of or at \((q + 1, p + 1)\).” The first condition says that the entries of \( \text{rk}(w) \) in locations \((q - 1, p)\), \((q, p - 1)\), and therefore \((q - 1, p - 1)\), are equal to the entry at \((q, p)\). The second condition says that \((q, p)\) is a southeast corner of its rank ladder. The reader is urged to consult Fulton’s paper for numerous examples, illustrations, and other ways of thinking about \( \mathcal{E}ss(w) \).

\(^4\)Most people recognize these as **partitions**, but apparently partitions occur too often in other guises in the literature on determinantal ideals, so these had to be called something else. Also, these are only special cases of more general ladders that aren’t partitions and have more the feel of rungs.
Before getting to the main result in this section, here is a preliminary characterization of the minimal generators of the antidiagonal ideal \( J_w \). Recall from Section 2.2 what it means for a rank condition to cause an antidiagonal. Here, we say that an antidiagonal is caused by a subset \( D \subseteq [n]^2 \) if it is caused by a rank condition \( \text{rk}(w)_{qp} \) for some \((q, p) \in D\).

**Lemma 3.3.2** An antidiagonal \( a \in k[z] \) of degree \( r + 1 \) is a minimal generator of \( J_w \) if and only if it is caused by \( \Lambda_r \) but not by \( \Lambda_{r'} \) for \( r' < r \).

**Proof.** The antidiagonal \( a \) lies in \( J_w \) if and only if \( a \) is caused by some rank condition \( \text{rk}(w)_{qp} \). By definition, \( \text{rk}(w)_{qp} \leq r \), so the remaining condition ensures that \( a \) is not divisible by some smaller antidiagonal in \( J_w \). \( \square \)

Let \( \mathcal{E}ss(w)_r \subseteq [n]^2 \) be the set of locations \((q, p) \in \mathcal{E}ss(w)\) at which the rank matrix \( \text{rk}(w) \) has entry equal to \( r \).

**Proposition 3.3.3** There is a unique set \( A_w \) of minors in \( I_w \) forming a minimal Gröbner basis for every antidiagonal term order. The minors of size \( r + 1 \) in \( A_w \) are precisely those with antidiagonals caused by \( \mathcal{E}ss(w)_r \), but not by \( \mathcal{E}ss(w)_{r'} \) for \( r' < r \).

**Proof.** The uniqueness of a minimal Gröbner basis consisting of minors follows from Theorem 2.1.2 and the fact that a minor is determined by its antidiagonal. If \( \text{rk}(w)_{qp} \) causes a degree \( r + 1 \) antidiagonal \( a \) but \((q, p) \notin \mathcal{E}ss(w)_r \), then chopping off the northeast or southwest end of \( a \) yields an antidiagonal caused by \( \Lambda_{r-1}(w) \) and dividing \( a \). Therefore every minimal generator of \( J_w \) is caused by \( \mathcal{E}ss(w) \).

When \( a \in J_w \) is a minimal generator, the criterion of Lemma 3.3.2 clearly implies that \( \mathcal{E}ss(w)_{r'} \subseteq \Lambda_{r'}(w) \) can’t cause \( a \) for \( r' < r \). On the other hand, if \( a \in J_w \) is not a minimal generator, then \( \Lambda_{r'}(w) \) causes \( a \) for some \( r' < r \) by Lemma 3.3.2. By definition, \( |a \cap Z_{[q,p]}| > r' \) for some \((q, p) \in \Lambda_{r'}(w) \). Some minimal generator of \( J_w \) divides the antidiagonal \( a \cap Z_{[q,p]} \), and this minimal generator is caused by \( \mathcal{E}ss(w)_{r''} \) for some \( r'' \leq r' \). Therefore \( \mathcal{E}ss(w)_{r''} \) causes \( a \cap Z_{[q,p]} \), and hence \( a \) as well. \( \square \)

**Example 3.3.4** Although the 165 minors of size 2 and 3 in Example 2.1.4 do form a Gröbner basis, the minimal Gröbner basis \( A_w \) has only 102 elements. It consists of the 21 minors of size 2 in \( \Lambda_1(w) \) and 81 of the minors of size 3 in \( \Lambda_2(w) \). \( \square \)

The minimal Gröbner basis \( A_w \) in Proposition 3.3.3 is rarely a reduced Gröbner basis, because a minimal generator \( a \in J_w \) can divide a monomial \( m \neq a \) in some minor in \( A_w \).

**Example 3.3.5** Label each point in the essential set of \( w = 13254 \in S_5 \) by \(* \), so that

\[
\begin{bmatrix}
1 & * & 1 \\
* & 1 & 1 \\
1 & * & 1 \\
\end{bmatrix}, \quad \text{and} \quad A_w = \left\{ \begin{bmatrix}
\bar{z}_{11} & \bar{z}_{12} & \bar{z}_{13} & \bar{z}_{14} \\
\bar{z}_{21} & \bar{z}_{22} & \bar{z}_{23} & \bar{z}_{24} \\
\bar{z}_{31} & \bar{z}_{32} & \bar{z}_{33} & \bar{z}_{34} \\
\bar{z}_{41} & \bar{z}_{42} & \bar{z}_{43} & \bar{z}_{44}
\end{bmatrix} \right\}
\]

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consists of $\det(Z_{[2,2]})$ and $\det(Z_{[4,4]})$. Although these two minors happen to generate $I_w$ minimally, the minimal Gröbner basis consists of $\det(Z_{[2,2]})$ along with the non-minor $\det(Z_{[4,4]}) - \det(Z_{[2,2]}) \cdot (z_{33} z_{44} - z_{34} z_{43})$.

For the record, observe that the minors in $I_{13254}$ never form a Gröbner basis for a diagonal term order (in which the leading term of every minor is its diagonal term), because the main diagonal of $Z_{[2,2]}$ divides that of $Z_{[4,4]}$. In contrast, the antidiagonal terms are relatively prime, so the Gröbner basis condition is immediate. \(\square\)

For later reference, here is the reason why Fulton introduced the essential set in the first place [Ful92, Lemma 3.10(a)].

**Corollary 3.3.6** The minors of size $1 + \text{rk}(w)_{qp}$ for $(q, p) \in \mathcal{E}(w)$ generate $I_w$.

3.4 Degrees of ladder determinantal varieties

We continue our preparation for comparison to the literature in the next subsection with a formula for the degrees of some popular determinantal ideals. A modicum of comparison of various authors’ work also occurs here.

Let $\mathbb{L}$ be a ladder, and regard $\mathbb{L}$ as a subset of a sufficiently large (to be made more precise in Proposition 3.4.1) generic $n \times n$ matrix $Z$ of variables $z_{ij}$. The boundary of $\mathbb{L}$ is a ribbon strip proceeding from the western edge of the square to its northern edge. Consider a sequence of boxes $(b_1, a_1), \ldots, (b_k, a_k)$ lying along the boundary of $\mathbb{L}$, so that

$$a_1 \leq a_2 \leq \cdots \leq a_k \quad \text{and} \quad b_1 \geq b_2 \geq \cdots \geq b_k. \quad (14)$$

Fill the boxes $(b_t, a_t)$ with nonnegative integers $r_t$ satisfying

$$0 < a_1 - r_1 < a_2 - r_2 < \cdots < a_k - r_k \quad \text{and} \quad b_1 - r_1 > b_2 - r_2 > \cdots > b_k - r_k > 0. \quad (15)$$

Define the ladder determinantal ideal $I(\mathbf{a}, \mathbf{b}, \mathbf{r})$ to be generated by the minors of size $r_\ell$ in the northwest $a_\ell \times b_\ell$ corner of $Z$ for all $\ell \in 1, \ldots, k$.

Since ‘ladders’ are just another name for ‘partitions’ one might also call these ‘partition determinantal ideals’, but as we shall argue, the better name is actually ‘vexillary determinantal ideals’. Indeed, for reasons that will become clear shortly, we refer to the data $(\mathbf{a}, \mathbf{b}, \mathbf{r})$ as a vexillary essential set. The condition in (15) on the vexillary essential set simply ensures that the vanishing of the minors of size $r_\ell$ in the northwest $b_\ell \times a_\ell$ submatrix does not imply the vanishing of the minors of size $r_{\ell'}$ in the northwest $b_{\ell'} \times a_{\ell'}$ submatrix when $\ell \neq \ell'$.

The polynomials generating $I(\mathbf{a}, \mathbf{b}, \mathbf{r})$ involve only the variables inside $\mathbb{L}$, so that $I(\mathbf{a}, \mathbf{b}, \mathbf{r})$ could be considered as an ideal in the polynomial ring $k[z_{ij} \mid (i, j) \in \mathbb{L}]$. There is little harm, however, in allowing extra variables outside the ladder; nothing substantial about the algebra (free resolutions, homology) changes, and there results only a product with an affine space at the level of varieties.

The ladder determinantal ideals in [GL00] have vexillary essential sets whose ranks $\mathbf{r}$ weakly increase from southwest to northeast (in our language). In contrast, those treated in [GM00] have vexillary essential sets with no two boxes in the same
row or column—that is, with strict inequalities in (14)—but no restriction other than (15) on the numbers \( r \) in those boxes. Gonciulea and C. Miller in fact mention that arbitrary vexillary essential sets can be treated with their methods [GM00, p. 106–107]. Fulton considered many aspects of ladder determinantal ideals in the generality presented here, [Ful92, Section 9]. The connection is through **vexillary permutations** (also known as 2143-avoiding and single-shaped permutations), and Fulton’s Proposition 9.6:

**Proposition 3.4.1 ([Ful92])** Given a vexillary essential set \((a, b, r)\), there exists an integer \( n_0 \leq a_k + b_1 \) such that for all \( n \geq n_0 \), a unique permutation \( w \in S_n \) satisfies

\[
\varepsilon ss(w) = \{(b_1, a_1), (b_2, a_2), \ldots, (b_k, a_k)\} \quad \text{and} \quad \text{rk}(w)_{b_{ai}} = r_{\ell} \quad \text{for} \ 1 \leq \ell \leq k.
\]

The permutation \( w \) is vexillary, and every vexillary permutation has a vexillary essential set. The length of \( w \), which equals the codimension of \( I(a, b, r) \), is \( m_1p_1 + m_2p_2 + \ldots + m_kp_k \), where the integers \( p_1, \ldots, p_k \) and \( m_1, \ldots, m_k \) are defined by

\[
p_1 = a_k - r_k, \ p_2 = a_{k-1} - r_{k-1}, \ldots, p_k = a_1 - r_1 \quad \text{and} \quad m_1 = b_k - r_k, \ m_2 = (b_{k-1} - r_{k-1}) - (b_k - r_k), \ldots, m_k = (b_1 - r_1) - (b_2 - r_2).
\]

Briefly then, Proposition 3.4.1 along with Corollary 3.3.6 says that the class of one-sided ladder determinantal ideals \( I(a, b, r) \) coincides with the class of vexillary Schubert determinantal ideals \( I_w \), and provides their codimensions. See Fulton’s paper and [Mac91, Chapter 1] for details concerning vexillary permutations, and ways of visualizing the numbers \( p_{\ell} \) and \( m_{\ell} \). These numbers appear implicitly in the codimension formulae of [GL00] and [GM00]. Note (as Fulton does) that the probability of a permutation being vexillary decreases exponentially to zero as \( n \) approaches infinity, so the class of Schubert determinantal ideals is significantly more general than ladder determinantal ideals. In fact, [Ful92, Section 3] says that Schubert determinantal ideals are the largest possible class of prime determinantal ideals defined by rank conditions of the form \( \text{rank}(Z_{[q,p]}) \leq r_{qp} \).

Now we provide a formula for the degree of a ladder determinantal ideal.

**Theorem 3.4.2** Let \((a, b, r)\) be a vexillary essential set and \( \lambda = (p_1^{m_1}, p_2^{m_2}, \ldots, p_k^{m_k}) \) be the partition determined by the integers from (16), so that

\[
\lambda = \left( \underbrace{p_1, \ldots, p_1}_{m_1}, \underbrace{p_2, \ldots, p_2}_{m_2}, \ldots, \underbrace{p_k, \ldots, p_k}_{m_k} \right).
\]

Given \( q \leq m_1 + m_2 + \ldots + m_k \), define \( \ell(q) \) by \( \lambda_q = p_\ell(q) \), so that we have \( \ell(q) = \min\{\ell \mid q \leq m_1 + \ldots + m_\ell\} \). The degree of the ladder determinantal ideal \( I(a, b, r) \) is

\[
\deg(I(a, b, r)) = \det \left[ \begin{array}{c}
(b_{\ell(q)} + \lambda_q - q + p - 1) \\
\end{array} \right]^{m_1 + \cdots + m_k}_{q,p=1}.
\]

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Proof. With \( w \) as in Proposition 3.4.1 and \( \lambda \) as in (17), the double Schubert polynomial \( G_w(x, y) \) is the multi-Schur polynomial [Ful92, Eq. (9.4)] because \( w \) is vexillary; this is [Ful92, Proposition 9.6(d)]. Setting the \( y \) variables to 0 and the \( x \) variables to 1 in this determinantal expression yields the desired formula, by Proposition A.3.10. \( \square \)

The proof of Theorem 3.4.2 actually points out a much stronger statement: the \( \mathbb{Z}^{2n} \)-graded multidegree of a vexillary determinantal ideal is a multi-Schur polynomial, which has an explicit determinantal expression. (Those wishing to see the determinantal expression in its full glory can check [Ful92] for a brief introduction, or [Mac91] for much more.) It would be desirable to make the Hilbert series in Theorem 2.1.2 just as explicit. Is there an analogously “nice” formula\(^5\) for vexillary double Grothendieck polynomials \( G_w(x, y) \), or an ordinary version for \( G_w(x) \), or even for \( G_w(t, \ldots, t) \)?

### 3.5 Between Schubert varieties and determinantal ideals: comparison with known results

Broadly speaking, the study of generic determinantal ideals has progressed in recent years well beyond its classical origins, which concerned ideals generated by equal size minors in rectangular matrices and subvarieties of Grassmannians. Recent trends involve collections of variously sized minors in more general types of regions inside generic matrices, all of which go under the name ‘ladder’ with some preprended adjectives, such as ‘one-sided’, ‘two-sided’, ‘one-corner’, or ‘wide’ (there are many others). The varieties defined by each class of ladder determinantal ideals come with some connection to Schubert varieties in type \( A \) partial flag manifolds, frequently via opposite big cells. Although we review the bare necessities below, we refer the reader to Gonciulea and C. Miller’s paper [GM00] for a well-written and nearly self-contained (as noted in Mathematical Reviews, MR 2001d:14055) entry point into the literature on this popular subject. In particular, their Introduction contains a brief historical account.

Let \( \mathbb{L} \subset [n]^2 \) be a ladder (as before, ‘ladder’ without qualification here always means ‘one-sided ladder’). Define \( \mathbb{A}_l^c \) to be the vector subspace of the \( n \times n \) matrices \( M_n \) consisting of matrices whose nonzero entries lie in \( \mathbb{L} \), and let \( w_0^T + \mathbb{A}_l^c \) be the translate of the subspace \( \mathbb{A}_l^c \) by the unit antidiagonal matrix \( w_0^T \). If \( \mathbb{L} \) does not intersect the main antidiagonal of \([n]^2\), so that \( \mathbb{L} \) lies strictly within the upper-left triangle, then \( w_0^T + \mathbb{A}_l^c \) is an affine subspace of \( M_n \) that happens to be contained in \( GL_n \). In other words, \( w_0^T + \mathbb{A}_l^c \) is contained in the subset \( w_0^T N \) of \( GL_n \), where \( N \) is the subgroup of \( B \) (the lower triangular Borel group) having 1’s along the main diagonal. The set \( w_0^T N \) consists of the matrices with 1’s along the main antidiagonal and zeros below it. Since \( w_0^T N \) intersects \( B w_0^T \) precisely at the long word \( w_0^T \) in \( GL_n \),

\(^5\) In the introductions to [Abb88] and its second chapter, Abhyankar writes of formulae he first presented at a conference at the University of Nice, in France. Although his formulae enumerate certain kinds of tableaux, his results were used to obtain formulae for degrees and Hilbert series of determinantal ideals. Authors since then have been looking for “nice” (uncapitalized, and always in quotes) formulae for Hilbert series of determinantal ideals; cf. [HT92, p. 3] and [AK89, p. 55].
the variety \( w_0^T N \) (and hence also \( w_0^T + A_n \)) maps isomorphically to its image in \( B \setminus GL_n \), which is called the **opposite big cell** \( \Omega^p \) in the full flag variety. We summarize this discussion in a lemma.

**Lemma 3.5.1** If \( L \) is a ladder that does not intersect the main antidiagonal, then \((w_0^T + A_n) \times A_L \subset GL_n \) maps isomorphically to the opposite big cell \( \Omega^p \) in the full flag variety \( B \setminus GL_n \), where \( A_L \) is the vector space of matrices whose nonzero entries lie strictly above the main antidiagonal but outside of \( L \).

Given \( w \in S_n \), let \( L = L(w) \) be the smallest ladder in \([n]^2\) containing the essential set \( Ess(w) \), and set \( k[L] \coloneqq k[z_{ij} \mid (i, j) \in L] \), the coordinate ring of \( A_n^L \cong w_0^T + A_n \). By Corollary 3.3.6, the ideal \( \overline{I}_w \coloneqq I_w \cap k[L] \) cuts out a variety in \( A_n^L \cong w_0^T + A_n \) whose product with \( A_{[n]^2 \setminus L} \) coincides with \( \overline{X}_w \), where \( A_{[n]^2 \setminus L} \) is the vector space of matrices in \( M_n \) having zeros inside \( L \). In other words, \( \overline{I}_w k[z] = I_w \), so it is natural to consider \( \overline{I}_w \) instead of \( I_w \subset k[z] \). There exist in the literature several different but closely related ways to pass from the Schubert variety \( X_w \subset B \setminus GL_n \) to the determinantal ideal \( I_w \). All of them first involve taking the preimage \( \widetilde{X}_w \subset GL_n \) of \( X_w \).

**Proposition 3.5.2** Given \( w \in S_n \) and the determinantal locus \( \widetilde{X}_w \subset GL_n \), assume that \( L \supset L(w) \) misses the main antidiagonal in \([n]^2\). The ideals of the varieties obtained by the following four procedures all coincide with \( I_w \subset k[L] \):

1. Project \( \widetilde{X}_w \) to \( A_n^L \).
2. Intersect \( \widetilde{X}_w \) with \( w_0^T + A_n^L \).
3. Intersect \( \widetilde{X}_w \) with \( Bw_0^T B \) and then project to \( w_0^T + A_n^L \).
4. Take the closure of \( \widetilde{X}_w \) inside of \( M_n \), and then project to \( A_n^L \).

**Proof.** Procedure 4 yields \( \overline{I}_w \) by definition.

The ideal of \( \widetilde{X}_w \) is the extension \( \overline{I}_w k[z][\det^{-1}] \) of \( \overline{I}_w \) from \( k[L] \). Procedure 2 results algebraically from the map \( k[z][\det^{-1}] \to k[L] \) sending \( z_{ij} \) to 0 whenever \( i + j \neq n + 1 \) and \((i, j) \notin L \) and sending \( z_{i,n+1-i} \) to 1 for all \( i \). Since none of the generators of \( \overline{I}_w \) involve these variables, the image of \( \overline{I}_w k[z][\det^{-1}] \) in \( k[L] \) is again \( \overline{I}_w \).

Due to the stability of both \( \widetilde{X}_w \) and \( Bw_0^T B \) under multiplication by \( B \) on the left, the intersection in procedure 3 commutes with projection to \( B \setminus GL_n \). Procedures 2 and 3 therefore yield the same result by Lemma 3.5.1, because \( Bw_0^T B \) is the preimage in \( GL_n \) of the opposite big cell \( \Omega^p \).

Finally, the variety produced by procedure 1 is contained in the variety produced by procedure 4, but contains the variety produced by procedure 3. \( \square \)

Of the four procedures in Proposition 3.5.2, only the last works even when \( L \) intersects the main antidiagonal. We postpone the discussion of the origins of these procedures until later in this subsection. First, we use the equivalence to express matrix Schubert varieties in \( M_n \) as open subsets of honest Schubert varieties in higher dimensional flag varieties.
Corollary 3.5.3 Given \( w \in S_n \), consider \( w \) as an element of \( S_{2n} \) fixing each of \( n + 1, \ldots, 2n \). The product \( \overline{X}_w \times k^{n^2-n} \) of the matrix Schubert variety \( \overline{X}_w \subseteq M_n \) with a vector space of dimension \( n^2-n \) is isomorphic to the intersection of the Schubert variety \( X_{w,2n} \subseteq B\backslash GL_{2n} \) with the opposite big cell in \( B\backslash GL_{2n} \).

Proof. Replace \( n \) by \( 2n \) in Lemma 3.5.1 and Proposition 3.5.2, and take \( \mathbb{L} \) to be the whole square \([n]^2\) considered as a ladder in \([2n]^2\). The point is that the northwest \( n \times n \) submatrix of a \( 2n \times 2n \) matrix misses the main antidiagonal, and leaves \( 2 \cdot \binom{n}{2} = n^2-n \) empty spots above the main antidiagonal.

In fact one can do somewhat better than the corollary, if one is only interested in the commutative algebra of the determinants generating \( I_w \): the product of the zero set of \( I_w \) with some vector space is isomorphic to the intersection of the Schubert variety \( X_{w,N} \) in \( B\backslash GL_N \) with the opposite big cell in \( B\backslash GL_N \), where we choose \( N = \max\{q + p \mid (q,p) \in \mathcal{E}(w)\} \), so that \( N < 2n \).

The point of the above results is a rather general equivalence principle that has been applied many times in the literature. By a local condition, we mean a condition that holds for a variety whenever it holds on each subvariety in some open cover.

Theorem 3.5.4 Let \( \mathcal{C} \) be a local condition that holds for a variety \( X \) whenever it holds for the product of \( X \) with any vector space. Then \( \mathcal{C} \) holds for every Schubert variety in every flag variety if and only if \( \mathcal{C} \) holds for all matrix Schubert varieties.

Proof. If \( \mathcal{C} \) holds for Schubert varieties, then it holds for matrix Schubert varieties by Corollary 3.5.3. On the other hand, if \( \mathcal{C} \) holds for the matrix Schubert variety \( \overline{X}_w \subseteq M_n \), then it holds for \( \overline{X}^G_w = \overline{X}_w \cap GL_n \). Therefore \( \mathcal{C} \) holds for the Schubert variety \( X_w \subseteq B\backslash GL_n \), because \( X_w \) is locally isomorphic to the product of \( X_w \) with \( B \), which is an open subset of a vector space.

The local conditions that are of primary interest include normality, the Cohen–Macaulay property, and rational singularities. Since all of these are known for Schubert varieties [Ram85, RR85] and can be deduced for Schubert determinantal ideals using an argument of Fulton [Ful92, Section 3], our inclusion Theorem 3.5.4 is mostly for the record; we know of no new facts that can be derived from it. On the other hand, new proofs become available: Section 4.5 derives Cohen–Macaulayness of Schubert varieties by working directly with initial ideals of Schubert determinantal ideals. We purposely avoid deriving facts about Schubert determinantal ideals from the corresponding facts about Schubert varieties, to keep this exposition self-contained.

Procedure 1 in Proposition 3.5.2 is Fulton’s construction that appears in [Ful92, Section 3]. His argument there essentially proves the ‘only if’ part of Theorem 3.5.4. The idea behind procedure 2 seems to be due originally to Mulay [Mul89], and has since then been used explicitly only in the ladder determinantal case (as far as we know). Most recently, it appeared in [GL00] and [GM00], although the constructions in these papers and [Mul89] are not exactly the same as procedure 2. Instead, their methods involve constructing a parabolic subgroup \( Q \subseteq GL_n \) from the vexillary essential set, and then associating a coset of the Weyl group \( W_Q \) to the ladder.
determinantal ideal. Translated into our language (that is, using block lower triangular parabolic subgroups $Q$ instead of their upper-triangular transposes, permutations $w_0w$ in place of what they call $w$, and partial flag varieties $Q\backslash GL_n$), the preimage in $GL_n$ of their constructed Schubert variety in $Q\backslash GL_n$ is $\tilde{X}_w$, where $w$ is the shortest representative of its coset. Intersecting this $\tilde{X}_w$ with the opposite unipotent radicals of $B$ and $Q$ yields two varieties in different affine spaces whose ideals are both extended from $T_w$. Our main point from this perspective is that one can always find an appropriate opposite big Schubert cell in $B'\backslash GL_n$, without passing to $Q\backslash GL_n$.

The original connection between determinantal ideals and geometry of partial flag manifolds came from the Schubert subvarieties on Grassmann varieties. Granted the many appearances of partitions in this context, it seems natural in hindsight that Sturmfels was able to make his fundamental application of the Knuth–Robinson–Schensted correspondence to prove results on ideals generated by same-size determinants in rectangular matrices [Stu90]. It also seems natural from this point of view that Herzog and Trung were able to extend Sturmfels’ methods to ideals cogenerated by fixed minors [HT92], because these correspond to the open cells of Schubert varieties in Grassmannians.

The application of KRS techniques to the algebra of determinantal rings has since become quite an industry, to the point where alternative methods seem somewhat scarce (see the Introduction). However, we found ourselves unable to apply KRS techniques to the Schubert determinantal ideals; perhaps others will succeed where we have failed. That being said, note that the term orders driving the KRS methods differ in one fundamental respect from ours: in our language, diagonal term orders rather than antidiagonal term orders permeate the papers of Sturmfels, Herzog–Trung, and others building on their work. As noted in Example 3.3.5, the minors in a determinantal ideal fail to be Gröbner bases for diagonal term orders as soon as the rank conditions become nested, whereas nested minors have relatively prime antidiagonal terms. Since ladder determinantal ideals are precisely those whose rank conditions are not nested (Proposition 3.4.1), this leads us to believe that diagonal term orders nearly approach their limits in treatments such as [GL00, GM00].

On the other hand, our use of antidiagonal term orders does not allow us to draw any new conclusions about ideals generated by determinants of submatrices whose entries lie on a two-sided ladder. By definition, a two-sided ladder contains any square submatrix as soon as it contains the main diagonal of the submatrix. However, we can use the elimination methods of Herzog and Trung [HT92] to prove similar results for the analogous two-sided antiladders, which contain a square submatrix as soon as they contain the main antidiagonal. We have left such generalizations open because this paper is already quite lengthy, and because we have no obvious comments on the intrinsic interest of such ideals. For instance, it seems unlikely that there is a clear connection to the geometry of Schubert varieties, although their initial ideals (for antidiagonal term orders, of course) might be combinatorially interesting.
3.6 Degeneracy loci

We recall here Fulton’s theory of degeneracy loci, and explain its relation to equivariant cohomology (see also the Appendix). This was our initial interest in this subject (and while not strictly an “application of the Gröbner basis,” it does give another motivation for wanting formulae for double Schubert polynomials). Since completing this work, however, we learned of the papers [FR01, Kaz97] taking essentially the same viewpoint, and we refer to them for detail.

Given a flagged vector bundle $E_\ast = (E_1 \hookrightarrow E_2 \hookrightarrow \cdots \hookrightarrow E_n)$ and a co-flagged vector bundle $F_\ast = (F_n \twoheadrightarrow F_{n-1} \twoheadrightarrow \cdots \twoheadrightarrow F_1)$ over the same base $X$, a generic map $\sigma : E_n \to F_n$, and a permutation $w$, define the degeneracy locus $\Omega_w$ as the subset

$$\Omega_w = \{ x \in X \mid \text{rank}(E_q \to E_n) - \text{rank}(w^T_{q,p}) \}$$

for all $q, p$. The principal goal in Fulton’s paper [Ful92] was to provide “formulae for degeneracy loci” as polynomials in the Chern classes of the vector bundles. In terms of the Chern roots $\{ c_1(E_p/E_{p-1}), c_1(\ker F_q \to F_{q-1}) \}$, Fulton found that the desired polynomials were actually the double Schubert polynomials.

It is initially surprising that there is a single formula, for all $X, E, F$ and not really depending on $\sigma$. This follows from a classifying space argument, at least when $k = \mathbb{C}$, as follows.

The group of automorphisms of a flagged vector space is the lower triangulars $B$, and so the classifying space $BB$ of $B$-bundles carries a universal flagged vector bundle. The classifying space of interest to us is thus $BB \times BB_+$, which carries a pair of universal vector bundles $E$ and $F$, the first flagged and the second co-flagged. We write $\mathcal{H}om(E, F)$ for the bundle whose fiber at $(x, y) \in BB \times BB_+$ is $\mathcal{H}om(E_x, F_y)$.

Define the universal degeneracy locus $U_w \subseteq \mathcal{H}om(E, F)$ as the subset

$$U_w = \{ (x, y, \phi) \mid \text{rank}(\phi) \leq \text{rank}(w^T_{q,p}) \}$$

where $x \in BB$, $y \in BB_+$, and $\phi : E_x \to F_y$. In other words, the homomorphisms in the fiber of $U_w$ at $(x, y)$ lie in the corresponding matrix Schubert variety.

The name is justified by the following. Recall that our setup is a space $X$, a flagged vector bundle $E$ on it, a coflagged vector bundle $F$, and a ‘generic’ vector bundle map $\sigma : E \to F$; we will soon see what ‘generic’ means. Pick a classifying map $\chi : X \to BB \times BB_+$, which means that $E, F$ are isomorphic to pullbacks of the universal bundles. (Classifying maps exist uniquely up to homotopy.) Over the target we have the universal Hom-bundle $\mathcal{H}om(E, F)$, and the vector bundle map $\sigma$ is a choice of a way to factor the map $\chi$ through a map $\tilde{\sigma} : X \to \mathcal{H}om(E, F)$. The degeneracy locus $\Omega_w$ is then $\tilde{\sigma}^{-1}(U_w)$, and it is natural to request that $\tilde{\sigma}$ be transverse to each $U_w$—this will be the notion of $\sigma$ being generic.

What does this say cohomologically? The closed subset $U_w$ defines a class in Borel-Moore homology (and thus ordinary cohomology) of the Hom-bundle $\mathcal{H}om(E, F)$ (see Appendix A.A for one proof of this). If $\sigma$ is generic, then

$$[\Omega_w] = [\tilde{\sigma}^{-1}(U_w)] = \tilde{\sigma}^*[U_w]$$

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and this is the sense in which there is a universal formula \([U_w] \in H^*(\text{Hom}(\mathcal{E}, \mathcal{F}))\). The cohomology ring of this Hom-bundle is the same as that of the base \(BB \times BB_+\) (to which it retracts), namely a polynomial ring in the \(2n\) first Chern classes, so one knows a priori that the universal formula should be expressible as a polynomial in these \(2n\) variables.

We can rephrase this using Borel’s mixing space definition of equivariant cohomology. Given a space \(S\) carrying an action of a group \(G\), and a contractible space \(EG\) upon which \(G\) acts freely, define the equivariant cohomology \(H_G^*(S)\) of \(S\) as

\[
H_G^*(S) := H^*((S \times EG)/G)
\]

where the quotient is respect to the diagonal action. Note that the ‘mixing space’ \((S \times EG)/G\) is a bundle over \(EG/G =: BG\), with fibers \(S\). In particular \(H_G^*(S)\) is automatically a module over \(H^*(BG)\), thereby called the ‘base ring’ of \(G\)-equivariant cohomology.

For us, the relevant group is \(B \times B_+\), and we have two spaces \(S\); the space of matrices \(M_n\) under left and right multiplication, and inside it the matrix Schubert variety \(\overline{X}_w\). Applying the mixing construction to the pair \(M_n \supseteq \overline{X}_w\), it can be shown that we recover the bundles \(\text{Hom}(\mathcal{E}, \mathcal{F}) \supseteq U_w\). As such, the universal formula \([U_w] \in H^*(\text{Hom}(\mathcal{E}, \mathcal{F}))\) we seek can be viewed instead as the class defined in \((B \times B_+)\)-equivariant cohomology by \(\overline{X}_w\) inside \(M_n\). As we prove in Theorem 3.1.1 (in the setting of multidegrees; see Theorem A.5.5 for the direct equivariant cohomological version), these are the double Schubert polynomials.

We point out the (few) differences between this approach and that of Fulton in [Ful92]. In the algebraic category, where Fulton worked, some pairs \((E, F)\) of algebraic vector bundles may have no algebraic generic maps \(\sigma\). The derivation given above works more generally in the topological category, where no restriction on \((E, F)\) is necessary.

Secondly, we don’t even need to know, a priori, which polynomials represent the cohomology classes of matrix Schubert varieties to show that these classes are the universal degeneracy locus classes. This contrasts with methods relying on divided differences.

### 4 Mitosis, rc-graphs, and subword complexes

Sections 4.1 and 4.2 translate some of the more combinatorial parts of Section 2 into a language compatible with rc-graphs (which don’t fully appear until Definition 4.2.5). Section 4.3 uses the translation to produce an inductive algorithm for generating rc-graphs, while Section 4.4 provides an independent fully combinatorial proof. Section 4.5 introduces a new family of simplicial complexes (subword complexes, in Definition 4.5.1), with the initial complexes \(L_w\) as special cases, and exploits their shellability to give a new proof of Cohen–Macaulayness for Schubert varieties. After proving that subword complexes are balls or spheres in Section 4.6, the final Section 4.7 employs Stanley-Reisner theory for subword complexes to elucidate the combinatorics of Grothendieck polynomials.

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4.1 Pipe dreams and mitosis

Consider a square grid \( \mathbb{Z}_{>0} \times \mathbb{Z}_{>0} \) extending infinitely south and east, with the box in row \( i \) and column \( j \) labeled \((i, j)\), as in an \( \infty \times \infty \) matrix. If each box in the grid is covered with a square tile containing either \(+\) or \(-\), then one can think of the tiled grid as a network of pipes.

**Definition 4.1.1** A pipe dream\(^6\) is a finite subset of \( \mathbb{Z}_{>0} \times \mathbb{Z}_{>0} \), identified as the set of crosses in a tiling by crosses and elbow joints. Given an \( n \times n \) exponent array \( \mathbf{b} \), let \( D(\mathbf{b}) = [n]^2 \setminus \text{supp}(\mathbf{z}^\mathbf{b}) \) be the pipe dream associated to \( \mathbf{b} \).

To readers familiar with rc-graphs, which we will recall in Definition 4.2.5, we point out that not all pipe dreams are rc-graphs; for instance, the \( n = 5 \) pipe dreams in Example 4.1.2 are not rc-graphs. Unless otherwise stated, all pipe dreams are assumed to be contained in the triangular region \( \{(q, p) \in [n]^2 \mid q + p \leq n\} \) strictly above the main antidiagonal. Whenever we draw pipe dreams, we fill the boxes with crossing tiles by ‘+’. However, we often leave the elbow tiles blank, or denote them by dots for ease of notation.

**Example 4.1.2** If \( \mathbf{z}^\mathbf{b} = \prod_{q+p\leq n} z_{qp} \), then \( D(\mathbf{b}) \) is a pipe dream called \( D_0 \), with crosses strictly above the main antidiagonal and elbow joints elsewhere. Here are two rather arbitrary pipe dreams with \( n = 5 \):

\[
\begin{array}{c}
+ + + \\
+ + \\
\end{array} = \begin{array}{ccc}
\cdot & \\
\cdot & \\
\cdot & \\
\end{array}
\quad \text{and} \quad
\begin{array}{c}
+ + + \\
+ + \\
\end{array} = \begin{array}{ccc}
\cdot & \\
\cdot & \\
\cdot & \\
\end{array}
\]

Another (slightly less arbitrary) example, with \( n = 8 \), is \( D = \)

\[
\begin{array}{c}
+ + + \\
+ + \\
\end{array} = \begin{array}{cccc}
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\end{array}
\]

in which the first diagram represents \( D \) as a subset of \([8]^2\), whereas the second demonstrates how the tiles fit together. Since no cross in \( D \) occurs on or below the 8th

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\(^6\)In the game Pipe Dream, the player is supposed to guide water flowing out of a spigot at one edge of the game board to its destination at another edge by laying down given square tiles with pipes going through them. Definition 4.2.5 interprets the spigot placements and destinations.
antidiagonal, the pipe entering row $i$ exits column $w_i = w(i)$ for some permutation $w \in S_n$. In this case, $w = 13865742$. For clarity, we omit the square tile boundaries as well as the “sea” of elbows below the main antidiagonal in the right pipe dream. We also use the thinner symbol $w_i$ instead of $w(i)$ to make the widths come out right. \(\square\)

For the rest of this subsection we think of the row index $i$ as being fixed, just as we did in Sections 2.2–2.5. Given a pipe dream in $[n]^2$, define the column index

$$start_i(D) = \min \{ j \mid (i, j) \notin D \} \cup \{ n + 1 \}$$ (18)

of the westernmost empty box in row $i$ of $D \subseteq [n]^2$. The similarity with the notation in Section 2.3 is explained by the following lemma, whose proof is immediate from the definitions.

Lemma 4.1.3 \(\text{start}_i(b) = \text{west}_i(b) = \text{start}_i(D(b))\) if $a^b \notin J_w$ has maximal support.

Most of the rest of Sections 4.1–4.3 concerns the relation between mutation (Definition 2.2.5) and mitosis,\(^7\) which we now define.

**Definition 4.1.4** Given a pipe dream $D \subseteq [n]^2$ and a row index $i$, construct a set $\text{mitosis}_i(D)$ of pipe dreams contained in $[n]^2$ as follows. Set

$$\mathcal{J}(D) = \{ j < \text{start}_i(D) \mid (i, j) \notin D \};$$

$$\mathcal{J}_{i, \leq p} = \{ (i, j) \mid j \in \mathcal{J}(D) \text{ and } j \leq p \} \text{ for } p \in \mathcal{J}(D);$$

$$\mathcal{J}_{i+1, < p} = \{ (i + 1, j) \mid j \in \mathcal{J}(D) \text{ and } j < p \} \text{ for } p \in \mathcal{J}(D).$$

Define the offspring $D_p = D \cup \mathcal{J}_{i+1, < p} \setminus \mathcal{J}_{i, \leq p}$ and $\text{mitosis}_i(D) = \{ D_p \mid p \in \mathcal{J}(D) \}$. Given a set $\mathcal{P}$ of pipe dreams, write $\text{mitosis}_i(\mathcal{P}) = \bigcup_{D_p \in \mathcal{P}} \text{mitosis}_i(D_p)$.

For a pipe dream $D$, the offspring $D_p$ is obtained by first deleting the cross at $(i, p)$ in $D$, and then moving the crosses west of it from row $i$ south to empty boxes in row $i + 1$ (mitosis only cares about the columns $\mathcal{J}(D)$, which are empty in row $i + 1$). In the region of $D$ that is west of $(i, p)$, row $i$ is filled solidly with crosses. Observe that $\text{mitosis}_i(D)$ is an empty set whenever $\mathcal{J}(D)$ is empty.

Proposition 4.1.6 presents another verbal description of mitosis—this one a little more algorithmic—using a certain local transformation on pipe dreams that was discovered by Bergeron and Billey.

**Definition 4.1.5** ([BB93]) A **chutable rectangle** is a connected $2 \times k$ rectangle $C$ inside a pipe dream $D$ such that $k \geq 2$ and all but the following 3 locations in $C$ are crosses: the northwest, southwest, and southeast corners. Applying a **chute move**\(^8\) to $D$ is accomplished by placing a ‘+’ in the southwest corner of a chutable rectangle $C$ and removing the ‘+’ from the northeast corner of the same $C$.

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\(^7\)The term mitosis is biological lingo for cell division in multicellular organisms.

\(^8\)The transpose of a chute move is called a **ladder move** in [BB93].
Heuristically, a chute move therefore looks like:

![Diagram of a chute move](image)

**Proposition 4.1.6** Let \( D \) be a pipe dream, and suppose \( j \) is the smallest column index such that \((i+1,j) \notin D\) and \((i,p) \in D\) for all \( p \leq j\). Then \( D_p \in \text{mitosis}_i(D) \) is obtained from \( D \) by

1. removing \((i,j)\), and then
2. performing chute moves from row \( i \) to row \( i+1 \), each one as far west as possible, so that \((i,p)\) is the last ‘+’ removed.

**Proof.** Immediate from Definitions 4.1.5 and 4.1.4. \( \square \)

**Example 4.1.7** Fig. 3 depicts the algorithmic ‘chute’ form of mitosis in Proposition 4.1.6, with \( i = 3 \). The dot in each pipe dream represents the last ‘+’ removed. \( \square \)

Proposition 4.1.6 for mitosis has the following analogue for mutation.

**Claim 4.1.8** Suppose that \( \text{length}(ws_i) < \text{length}(w) \), and let \( z^b \notin J_w \) be a squarefree monomial of maximal support. If \( D = D(b) \) then

\[
|\text{prom}(b)| = |J(D)|.
\]

If \( 0 \leq d < |\text{prom}(b)| \), then \( D(\mu_i^{2d+1}(2b)) \) is obtained from \( D \) by

1. removing \((i,j)\), where \( j \) is as in Proposition 4.1.6, and then
2. performing \( d \) chute moves from row \( i \) to row \( i+1 \), each as far west as possible.

**Proof.** By Lemma 4.1.3, the columns in \( J(D) \) are in bijection with the nonzero entries in the promoter of \( b \), each of which is a 1 in row \( i+1 \). The final statement follows easily from the definitions. \( \square \)

**Example 4.1.9** The array \( b \) in Example 2.2.6 has maximal support among exponent arrays on monomials not in \( J_w \), where \( w = 13865742 \) as in Examples 2.2.7 and 2.1.4. Substituting ‘+’ for each blank space and then removing the numbers and dots yields the pipe dream \( D = D(b) \) in Fig. 3. Applying the same makeover to the middle column of Fig. 1 results in the offspring \( D(\mu_3^5(b)) \), \( D(\mu_3^5(b)) \), and \( D(\mu_3^5(b)) \). \( \square \)

**Lemma 4.1.10** If \( \text{length}(ws_i) < \text{length}(w) \) and \( z^b \notin J_w \) is squarefree of maximal support, then \( \text{mitosis}_i(D(b)) = \{D(\mu_i^{2d+1}(2b)) \mid 0 \leq d < |\text{prom}(b)|\} \).

**Proof.** Compare Claim 4.1.8 with Proposition 4.1.6. \( \square \)
Figure 3: Mitosis

4.2 Facets of $\mathcal{L}_w$ and rc-graphs

Proposition 4.2.2, in which $D_L = [n]^2 \setminus L$ for $L \in \mathcal{L}_w$ as per the beginning of Section 2.5, will conclude the translation of mutation on monomials into mitosis on facets. Its full potential will be realized in Theorem 4.2.11 and especially the algorithm in Theorem 4.3.1. The translation requires an intermediate result.

**Lemma 4.2.1** If $\text{length}(w_{s_i}) < \text{length}(w)$, then $\{D_L \mid L \text{ is a facet of } \mathcal{L}_{w{s_i}}\}$ is the set of pipe dreams $D(\mu_i^{2d+1}(2b))$ such that $z^b \notin J_w$ is square-free of maximal support and $0 \leq d < |\text{prom}(b)|$.

**Proof.** By Proposition 2.3.11 and Lemma 2.5.2, every facet of $\mathcal{L}_{w_{s_i}}$ is the support of a mutation $\mu_i^d(z^{2b})$ for some monomial $z^{2b} \notin J_w$ and $d \leq |\text{prom}(2b)|$. Furthermore, it is clear from Lemma 2.5.2 and the definition of mutation that we may assume $z^b$ is a square-free monomial (so the entries of $2b$ are all 0 or 2). In this case, the supports of odd mutations $\mu_i^{2d+1}(z^{2b})$ for $0 \leq d < |\text{prom}(b)|$ are facets of $\mathcal{L}_{w_{s_i}}$ by Proposition 2.5.6 because they each have cardinality $n^2 - \text{length}(w) + 1 = n^2 - \text{length}(w_{s_i})$, while the supports of even mutations $\mu_i^{2d}(z^{2b})$ for $d \leq |\text{prom}(b)|$ aren’t facets, each having cardinality $n^2 - \text{length}(w)$. \hfill \Box
Proposition 4.2.2  If $\text{length}(w_s) < \text{length}(w)$, then
\[
\{D_L \mid L \text{ is a facet of } \mathcal{L}_w\} = \text{mitosis}_i(\{D_L \mid L \text{ is a facet of } \mathcal{L}_w\}).
\]
Moreover, $\text{mitosis}_i(D_L) \cap \text{mitosis}_i(D_{L'}) = \emptyset$ if $L \neq L'$ are facets of $\mathcal{L}_w$.

Proof. The displayed equation is a consequence of Lemma 4.2.1 and Lemma 4.1.10, so we concentrate on the final statement. Let $z^b \notin J_w$ be a squarefree monomial of maximal support $L = \text{supp}(b)$, and let $0 \leq d < |\text{prom}(b)|$. The entries of the array $\mu_i^{2d+1}(2b)$ are all either 0 or 2, except for precisely two 1’s, both in the same column $p$ (the boldface entries in Fig. 1, middle column). By Lemma 4.1.3, $p$ is the westernmost column of $D(\mu_i^{2d+1}(2b))$ in which neither row $i$ nor row $i + 1$ has a cross.

Now suppose $z^{b'} \notin J_w$ is another squarefree monomial of maximal support $L'$, and let $0 \leq d' < |\text{prom}(b')|$. If $D(\mu_i^{2d+1}(2b)) = D(\mu_i^{2d+1}(2b'))$, then the argument in the first paragraph of the proof implies that $\mu_i^{2d'+1}(2b) = \mu_i^{2d'+1}(2b')$, since they have the same entries equal to 1 as well as the same support, and all of their other nonzero entries equal 2. We conclude that $b = b'$ by Lemma 2.3.4. Using Lemma 4.2.1, we have proved that $\text{mitosis}_i(D_L) \cap \text{mitosis}_i(D_{L'}) \neq \emptyset$ implies $L = L'$.

Example 4.2.3  The pipe dream $D$ in Example 4.1.9 and Fig. 3 is $D_L$ for a facet of $\mathcal{L}_{13685742}$. By Proposition 4.2.2, the three pipe dreams in the right column of Fig. 3 can be expressed as $D_{L'}$ for facets $L' \in \mathcal{L}_{13685742}$, where $13685742 = 13865742 \cdot s_3$.

Next is a result whose proof connects chuting with antidiagonals. It will be used in the proof of Theorem 4.2.11.

Lemma 4.2.4  The set $\{D_L \mid L \in \text{facets} (\mathcal{L}_w)\}$ is closed under chute moves.

Proof. A pipe dream $D$ is equal to $D_L$ for some (not necessarily maximal) $L \in \mathcal{L}_w$ if and only if $D$ meets every antidiagonal in $J_w$, which by definition of $\mathcal{L}_w$ equals $\bigcap_{L \in \mathcal{L}_w} \{z^p \mid (q, p) \in D_L\}$. Supposing that $C$ is a chuteable rectangle in $D_L$ for $L \in \mathcal{L}_w$, it is therefore enough to show that the intersection $a \cap D_L$ of any antidiagonal $a \in J_w$ with $D_L$ does not consist entirely of the single cross in the northeast corner of $C$. Indeed, the purity of $\mathcal{L}_w$ (Proposition 2.5.6) will then imply that chuting $D_L$ in $C$ yields $D_{L'}$ for some facet $L'$ whenever $L \in \text{facets} (\mathcal{L}_w)$.

To prove the claim concerning $a \cap D_L$, we may assume $a$ contains the cross in the northeast corner $(q, p)$ of $C$, and split into cases:

(i) $a$ does not continue south of row $q$.

(ii) $a$ continues south of row $q$ but skips row $q + 1$.

(iii) $a$ intersects row $q + 1$, but strictly east of the southwest corner of $C$.

(iv) $a$ intersects row $q + 1$ at or west of the southwest corner of $C$.

Letting $(q + 1, t)$ be the southwest corner of $C$, construct new antidiagonals $a'$ that are in $J_w$ (and hence intersect $D_L$) by replacing the cross at $(q, p)$ with a cross at:

(i) $(q, t)$, using Lemma 2.2.3(W);
(ii) \((q + 1, p)\), using Lemma 2.2.3(S);
(iii) \((q, p)\), so \(a = a'\) trivially; or
(iv) \((q + 1, t)\), using Lemma 2.2.3(E).

Observe that in case (iii), \(a\) already shares a spot in row \(q + 1\) where \(D_L\) has a cross.
Each of the other antidiagonals \(a'\) intersects both \(a\) and \(D_L\) in some spot that isn’t
\((q, p)\), since the location of \(a' \setminus a\) has been constructed not to be a cross in \(D_L\).  □

**Definition 4.2.5** ([FK96b]) An **rc-graph** is a pipe dream in which each pair of
pipes crosses at most once. If \(D\) is an rc-graph and \(w \in S_n\) is the permutation such
that the pipe entering row \(i\) exits from column \(w(i)\), then \(D\) is said to be an **rc-graph**
for \(w\). The set of rc-graphs for \(w\) is denoted by \(RC(w)\).

The name ‘rc-graph’ was coined by Bergeron and Billey [BB93], although Fomin
and Kirillov introduced objects represented by rc-graphs (rotated by 135°) in special cases. The “rc” stands for “reduced-compatible”, and is justified using our next lemma. Given a pipe dream \(D\), say that a ‘+’ at \((q, p)\) sits on the \(i^{th}\) **antidiagonal** if \(q + p - 1 = i\). Let \(Q(D)\) be the ordered list of simple reflections \(s_i\) corresponding
to the antidiagonals on which the crosses sit, starting from the northeast corner of \(D\)
and reading **right to left** in each row, snaking down to the southwest corner.

**Lemma 4.2.6** If \(D\) is a pipe dream, then multiplying the reflections in \(Q(D)\) yields
the permutation \(w\) such that the pipe entering row \(i\) exits column \(w(i)\). Furthermore,
\(|D| \geq \text{length}(w)\), with equality if and only if \(D \in RC(w)\).

**Proof.** For the first statement, use induction on the number of crosses: adding a ‘+’
in the \(i^{th}\) antidiagonal at the end of the list switches the destinations of the pipes
beginning in rows \(i\) and \(i + 1\). Each inversion in \(w\) contributes at least one crossing in
\(D\), whence \(|D| \geq \text{length}(w)\). The expression \(Q(D)\) is reduced when \(D\) is an rc-graph
because each inversion in \(w\) contributes \(\leq 1\) crossings in \(D\).  □

Thus \(Q(D)\) gives a reduced expression for \(w\) if and only if \(D \in RC(w)\). The
ordered list of row indices for the crosses in \(D\) (taken in the same order as before) is
called a “compatible sequence” for the expression \(Q(D)\) (we won’t need this concept
from [BJS93]).

**Example 4.2.7** The upper-left triangular pipe dream \(D_0 \subset [n]^2\), with crosses strictly
above the main antidiagonal and elbows elsewhere, is in (and in fact equals) \(RC(w_0)\).
The \(8 \times 8\) pipe dream \(D\) in Example 4.1.2 is in \(RC(13865742)\); this is the same pipe
dream \(D\) appearing in Examples 4.1.9 and 4.2.3. We will show in Theorem 4.2.11
that the three pipe dreams in the right hand column of Fig. 3 are in fact rc-graphs
for \(13685742 = 13865742 \cdot s_3\).

In general, notice how pipe dreams whose crosses lie strictly above the main antidiagonal
in \([n]^2\) are naturally **subwords** of \(Q(D_0)\), while rc-graphs are naturally **reduced**
subwords. This point of view will take center stage in Section 4.5.  □
Lemma 4.2.6 implies the following criterion for when removing a ‘+’ from a pipe
dream \( D \in \mathcal{RC}(w) \) yields a pipe dream in \( \mathcal{RC}(ws_i) \). Specifically, it concerns the
removal of a cross at \((i, j)\) from configurations that look like

\[
\begin{array}{ccccccc}
  & & & & & & \\
 1 & \ldots & 1 & \ldots & 1 & \ldots & 1 \\
 1 & & & & & & \\
 i & & & & & & \\
 i+1 & & & & & & \\
\end{array}
\]

at the west end of rows \( i \) and \( i+1 \) in \( D \).

**Lemma 4.2.8** Let \( D \in \mathcal{RC}(w) \) and \( j \) be a fixed column index with \((i+1, j) \notin D\),
but \((i, p) \in D\) for all \( p \leq j \), and \((i+1, p) \in D\) for all \( p < j \). Then \( \text{length}(ws_i) < \text{length}(w) \),
and if \( D' = D \setminus (i, j) \) then \( D' \in \mathcal{RC}(ws_i) \).

Proof. Removing \((i, j)\) only switches the exit points of the two pipes starting in rows
\( i \) and \( i+1 \), so the pipe starting in row \( k \) of \( D' \) exits out of column \( ws_i(k) \) for each \( k \).
The result follows from Lemma 4.2.6. \( \square \)

The connection between the complexes \( \mathcal{L}_w \) and rc-graphs requires certain facts
proved by Bergeron and Billey [BB93]. The next lemma consists mostly of the combinatorial
parts (a), (b), and (c) of [BB93, Theorem 3.7], their main result. Its proof relies exclusively
on elementary properties of rc-graphs.

**Lemma 4.2.9 ([BB93])**

1. The set \( \mathcal{RC}(w) \) of rc-graphs for \( w \) is closed under chute operations.
2. There is a unique **top rc-graph** for \( w \) such that every cross not in the first row
   has a cross due north of it.
3. Every rc-graph for \( w \) can be obtained by applying a sequence of chute moves to
   the top rc-graph for \( w \).

**Lemma 4.2.10** The top rc-graph for \( w \) is \( D_L \) for a facet \( L \in \mathcal{L}_w \).

Proof. The unique rc-graph for \( w_0 \), whose crosses are at \( \{(q, p) \mid q+p \leq n\} \), is also \( D_L \)
for the unique facet of \( \mathcal{L}_{w_0} \). Therefore, if \( \text{length}(ws_i) < \text{length}(w) \) we employ downward
induction to prove the result for \( ws_i \) by assuming it for \( w \). Using Lemma 2.5.3
(particularly part 2) and Lemma 2.5.5, we assume the top rc-graph \( D \in \mathcal{RC}(w) \) satisfies
the conditions of Lemma 4.2.8, with \( j = w(i+1) \). By definition, the column
index \( j \) is the minimal one in \( J(D) \). Thus \( D \setminus (i, j) \) equals the first offspring \( D_j \) from
Definition 4.1.4 (or Proposition 4.1.6) as well as the rc-graph \( D' \in \mathcal{RC}(ws_i) \) from
Lemma 4.2.8. That \( D' \) is the top rc-graph for \( ws_i \) uses part 2 of Lemma 4.2.9, while
\( D_j = D_L \) for some facet \( L \in \mathcal{L}_{ws_i} \) by Proposition 4.2.2. \( \square \)

We come now to our first main theorem of Section 4. It ascribes a truly geometric
origin to rc-graphs, identifying them with the subspaces of \( M_n \) resulting from flat
deformation of matrix Schubert varieties (Section 2).
Theorem 4.2.11 \( \mathcal{RC}(w) = \{ D_L \mid L \text{ is a facet of } \mathcal{L}_w \} \), where \( D_L = [n]^2 \setminus L \). In other words, rc-graphs for \( w \) are complements of maximal supports of monomials \( \not\in J_w \).

Proof. Lemmas 4.2.10 and 4.2.4 imply that \( \mathcal{RC}(w) \subseteq \{ D_L \mid L \in \text{facets}(\mathcal{L}_w) \} \), given Lemma 4.2.9. Since the opposite containment \( \{ D_L \mid L \in \text{facets}(\mathcal{L}_w) \} \subseteq \mathcal{RC}(w) \) is obvious for \( v = w_0 \), it suffices to prove it for \( v = ws_i \) by assuming it for \( v = w \).

A pair \((i + 1, j)\) as in Lemma 4.2.5 exists in \( D \) if and only if the set \( J(D) \) in Definition 4.1.4 nonempty, and this occurs if and only if \( \text{mitosis}_i(D) \neq \emptyset \). In this case, the pipe dream produced from \( D \) by the first stage of \( \text{mitosis}_i \) (as in Proposition 4.1.6) is \( D' = D \setminus (i, j) \in \mathcal{RC}(ws_i) \). The desired containment follows from Proposition 4.2.2 and part 1 of Lemma 4.2.9.

The next formula was our motivation for relating \( \mathcal{L}_w \) to \( \mathcal{RC}(w) \). It was originally proved by Billey, Jockusch, and Stanley, but Fomin and Stanley shortly thereafter gave a better combinatorial proof.

Corollary 4.2.12 ([BJS93, FS94]) \( G_w(x) = \sum_{D \in \mathcal{RC}(w)} x^D \).

Proof. Theorem 3.1.1 expresses \( G_w(x) \) as a sum of monomials \( [L]_T = x^{D_L} \) for \( D_L \) in \( \{ [n]^2 \setminus L \mid L \text{ is a facet of } \mathcal{L}_w \} \). Theorem 4.2.11 identifies this set as \( \mathcal{RC}(w) \).

Here is the double version of the previous result; its proof is the same.

Corollary 4.2.13 ([FK96b]) \( G_w(x, y) = \sum_{D \in \mathcal{RC}(w)} \prod_{(q, p) \in D} (x_q - y_p) \).

4.3 The mitosis algorithm for rc-graphs

The lifted Demazure operators from Section 2.3, in their mitosis avatar, produce an algorithm for obtaining coefficients on monomials in Schubert polynomials by induction on the weak Bruhat order. This already follows from Proposition 4.2.2 and Theorem 3.1.1. However, we express the algorithm in Theorem 4.3.1 in terms of rc-graphs—which represent \( \mathbb{Z}^n^2 \)-graded data that is more refined than \( \mathbb{Z}^n \)-graded coefficients—by way of Theorem 4.2.11. The Schubert polynomial interpretation of Theorem 4.3.1 via Corollary 4.2.12 serves as a geometrically motivated replacement for a conjecture of Kohnert [Mac91, Appendix to Chapter IV, by N. Bergeron] concerning an algorithm for Schubert coefficients that is similarly inductive, but on Rothe diagrams (Section 3.5) instead of rc-graphs.

Theorem 4.3.1 \( \mathcal{RC}(ws_i) \) is the disjoint union \( \bigcup_{D \in \mathcal{RC}(w)} \text{mitosis}_i(D) \). Therefore, if \( s_{i_1} \cdots s_{i_k} \) is a reduced expression for \( w_0w \) and \( D_0 \) is the unique rc-graph for \( w_0 \), then

\[
\mathcal{RC}(w) = \text{mitosis}_{i_k} \cdots \text{mitosis}_{i_1}(D_0).
\]

Proof. Theorem 4.2.11 and Proposition 4.2.2.
Remark 4.3.2 Bergeron and Billey used Lemma 4.2.9 and Corollary 4.2.12 to give an algorithm for generating rc-graphs using chute moves [BB93, Theorem 3.7(d)]. Though motivated by and related to Kohnert’s conjecture, their algorithm is not inductive on the weak Bruhat order, instead remaining inside $\mathcal{RC}(w)$ by starting from the top rc-graph for $w$ (Lemma 4.2.9). Some rc-graphs are produced twice in the process. □

The algorithm in Theorem 4.3.1 for generating $\mathcal{RC}(w)$ is irredundant, in the sense that each rc-graph appears exactly once in the implicit union on the right hand side of (19). However, the efficiency of the algorithm can vary widely with the reduced expression $s_{i_1} \cdots s_{i_k}$ for $w_0 w$. For instance, the set $\mathcal{RC}(\text{id}_n)$ for the identity permutation consists of one element, $\emptyset \subset [n]^2$, even though the repeated mitosis in (19) for $w = \text{id}_n$ can pass through $\mathcal{RC}(w)$ for any permutation $w$. As a consequence, huge numbers of rc-graphs must be killed along the way, without producing any offspring.

Definition 4.3.3 A reduced expression for $w_0 w$ is called poptotic\(^9\) if every rc-graph along the way to $\mathcal{RC}(w)$ has at least one offspring via Theorem 4.3.1. More precisely, the reduced expression $w_0 w = s_{i_1} \cdots s_{i_k}$ is poptotic if $\text{mitosis}_{i_i}(D)$ is nonempty whenever $1 \leq \ell \leq k$ and $D \in \mathcal{RC}(w_0 s_{i_1} \cdots s_{i_{\ell - 1}})$.

Example 4.3.4 The three pipe dreams from the second column of Fig. 3 are all rc-graphs for $v = 13685742$ (see Example 4.2.7). Setting $i = 4$ and either inspecting the inversions of $v$ or applying Lemma 4.2.8 to the last rc-graph in Fig. 3, we find that $	ext{length}(v s_4) < \text{length}(v)$. On the other hand, mitosis$_4$ kills the first two of the three rc-graphs, whereas the last has two offspring. Thus any reduced expression for $w_0 v$ ending $s_4$ is necessarily apoptotic.

On the other hand, the lex first reduced expression of Example 2.5.1, which is $w_0 v = s_2 s_1 s_3 s_5 s_4 s_8 s_3 s_7 s_5 s_1 s_3 s_2 s_1$ in the present case, is always poptotic by Lemmas 2.5.3 and 4.2.8. In particular, the lex first reduced expression for $w_0 = w_0 \text{id}_n$ passes through permutations with exactly one rc-graph (each is a dominant permutation, whose unique rc-graph is shaped like a Young diagram). □

Theorem 4.3.1 makes the set $\mathcal{RC}_n = \bigcup_{w \in S_n} \mathcal{RC}(w)$ of rc-graphs for permutations of $n$ elements into a poset determined by

$$D' \prec i \quad \text{if} \quad D' \in \text{mitosis}_{i_i}(D) \text{ for some } i.$$

If this condition holds and $D \in \mathcal{RC}(w)$, then automatically $D' \in \mathcal{RC}(w_{si})$, where $\text{length}(w s_i) < \text{length}(w)$ by Lemma 4.2.8. Therefore the poset $\mathcal{RC}_n$, which is ranked by length = cardinality, fibers over the weak Bruhat order on $S_n$, with the preimage of $w \in S_n$ being $\mathcal{RC}(w)$.

---

\(^9\)The word ‘apoptosis’ is a biological term referring to “programmed cell death”, in which some cell in a multicellular organism in effect commits suicide for the greater good of the organism. Thus poptotic is an apt term for a total order in which some rc-graph dies along the way. In analogy with the way ‘crepant’ is formed from ‘discrepancy’, we take the opposite of ‘apoptotic’ to be ‘poptotic’.
A reduced decomposition for \( w_0w \) can be thought of as a decreasing path in the weak Bruhat order on \( S_n \), beginning at \( w_0 \) and ending at \( w \). The preimage in \( \mathcal{RC}_n \) of such a path is a tree having \( \mathcal{RC}(w) \) among its leaves. The path is poptotic, as in Definition 4.3.3, when the leaves are precisely \( \mathcal{RC}(w) \).

**Example 4.3.5** Here is the Hasse diagram for \( \mathcal{RC}_3 \):

![Hasse diagram](image)

The right hand path from 321 to 123 is poptotic because there's only one rc-graph at each stage. The left hand path is apoptotic because the first rc-graph for 132 has no offspring under mitosis.

Whether or not a path from \( w_0 \) to \( w \) is poptotic, breadth-first search on the preimage tree (ordering the mitosis offspring as in Proposition 4.1.6) yields a total order on \( \mathcal{RC}(w) \). It can be shown that poptotic total orders by breadth-first search are linear extensions of the partial order on rc-graphs determined by chute operations. Through heuristic arguments, computer calculations in small symmetric groups, and the fact that \( \mathcal{L}_w \) is shellable (Section 4.5), we are convinced of the following.

**Conjecture 4.3.6** Poptotic total orders on \( \mathcal{RC}(w) \) are shellings of \( \mathcal{L}_w \).

To emphasize: shellability is not in question – we will give shellings in Section 4.5. The conjecture would just give some more intuitive shellings than those we know. It is conceivable that all of the apoptotic total orders are shellings, too, but we hold out less hope for these.

### 4.4 A combinatorial approach to mitosis

It is possible to give a complete proof of Theorem 4.3.1 based entirely on the BJS formula in Corollary 4.2.12 and the characterization of Schubert polynomials by divided differences, along with elementary combinatorial properties of rc-graphs. Before getting to a sketch of this proof, let us recast the definition of intron from Section 2.4 directly in the language of rc-graphs. We emphasize that no reference to Section 2.4 is necessary here: these reformulations are independent. For visualization purposes, recall that an elbow may be denoted by a \( \cdot^{-1} \), a dot, or an empty box in the diagrams.
**Definition 4.4.1** Let $D$ be a pipe dream and $i$ a fixed row index. Order the boxes in the gene of $D$ (that is, rows $i$ and $i+1$) as in the following diagram:

\[
\begin{array}{cccccc}
1 & 2 & 3 & 4 & \cdots \\
1 & 2 & 3 & 4 & \cdots \\
1 & 2 & 3 & 4 & \cdots \\
1 & 2 & 3 & 4 & \cdots \\
\end{array}
\]

An **intron** in this gene is a $2 \times k$ rectangle $C$ such that

1. the first and last boxes in $C$ (the northwest and southeast corners) are elbows;
2. no elbow in $C$ is northeast or southwest of another elbow (but due north or due south is okay); and
3. the elbow with largest index before $C$ (if there is one) resides in row $i+1$, and the elbow with smallest index after $C$ (if there is one) resides in row $i$.

In the language of Section 2.4, conditions 1 and 3 guarantee that $C$ is flanked by two exons (or possibly the start and/or stop codon), while the middle condition says that $C$ contains no exons. The intron mutation of Definition 2.4.6 becomes here the self-evident Lemma 4.4.2, below. Although it is irrelevant for the present purpose, it may be worthwhile to note that the reason why Lemma 4.4.2 agrees with intron mutation (as per Definition 2.4.6) restricted to maximal-degree squarefree monomials $\not \in J_w$ comes from Theorem 4.2.11 and Proposition 2.5.6.

**Lemma 4.4.2** Given an intron $C$, there is a unique intron $\tau_i C$ such that

1. the sets of columns with exactly two crosses are the same in $C$ and $\tau_i C$, and
2. the number of crosses in row $i$ of $C$ equals the number of crosses in row $i+1$ of $\tau_i C$, and conversely.

The involution $\tau_i$ is called **intron mutation.** □

For an example, simply replace every empty box by ‘+’ and every nonzero number by an elbow in each intron from Example 2.4.5. Another arbitrary example is:

\[
\begin{array}{c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c}
& + & + & + & + & + & + \\
\hline
\tau_i &  &  &  &  &  & \downarrow & + & + & + & + & + \\
\hline
\tau_i &  &  &  &  &  & \downarrow & + & + & + & + & + \\
\end{array}
\]

Here now is the re-graph analogue of Lemma 2.4.1.

**Lemma 4.4.3** For each $i$ there is an involution $\tau_i : \mathcal{RC}(w) \rightarrow \mathcal{RC}(w)$ such that $\tau_i^2 = 1$, and for all $D \in \mathcal{RC}(w)$:

1. $\tau_i D$ agrees with $D$ outside rows $i$ and $i+1$.
2. $\text{start}_i(\tau_i D) = \text{start}_i(D)$, and $\tau_i D$ agrees with $D$ strictly west of this column.
3. $\ell_i(\tau_i D) = \ell_{i+1}(D)$,
where $\ell_i^r(-)$ is the number of crosses in row $r$ that are east of or in column $\text{start}_i(-)$.

**Proof.** Let $D \in \mathcal{RC}(w)$. The union of all columns in the gene of $D$ that are east of coincide with column $\text{start}_i(D)$ can be written as a disjoint union of $2 \times k$ rectangles, each of which is either an intron or completely filled with crosses. Indeed, this follows from (18) and Definition 4.4.1. Therefore the lemma comes down to verifying that intron mutation preserves the property of being in $\mathcal{RC}(w)$.

More accurately, mutation of a single intron in the gene of $D$ yields a pipe dream in $\mathcal{RC}(w)$. To verify this statement, one need only check the routes of pipes intersecting the intron, and this is straightforward: the numbers of crosses traversed horizontally by each pipe are the same in $C$ and $\tau_iC$; similarly for the vertically traversed crosses. □

**Remark 4.4.4** The intron mutation illustrated in (20) results from a sequence of chute moves. This phenomenon is general, whenever the pipe dream is an rc-graph: in any single intron of an rc-graph, either the mutation or its inverse results from a sequence of chute moves. The proof is omitted, but rests on the fact that the $2 \times 2$ configuration \[\begin{array}{cc} & \downarrow \\ \downarrow & \end{array}\] is disallowed in rc-graphs, so every intron in an rc-graph contains a column with two elbows. □

**Sketch of combinatorial proof of Theorem 4.3.1.** The argument employs the description of mitosis in Proposition 4.1.6, from which Lemma 4.2.8 and part 1 of Lemma 4.2.9 imply that mitosis$_i(D)$ consists of rc-graphs for $ws_i$ whenever $D \in \mathcal{RC}(w)$. It follows directly from the definitions that mitosis$_i(D) \cap \text{mitosis}_i(D') = \emptyset$ if $D \neq D'$ are rc-graphs for $w$. Thus it suffices to prove that mitosis$_i(\mathcal{RC}(w))$ has the same cardinality as $\mathcal{RC}(ws_i)$.

Using Lemma 4.4.3, one proves that

$$
\sum_{E \in \text{mitosis}_i(D)} x^E + \sum_{E' \in \text{mitosis}_i(\tau_iD)} x^{E'} = \partial_i(x^D + x^{\tau_iD}).
$$

This is the same trick we applied in the context of lifted Demazure operators $e_i^w$ (Proposition 2.4.3). Pairing off the elements of $\mathcal{RC}(w)$ not fixed by $\tau_i$, as we did in the proof of Proposition 2.4.3, conclude that

$$
\sum_{E \in \text{mitosis}_i(\mathcal{RC}(w))} x^E = \partial_i\left( \sum_{D \in \mathcal{RC}(w)} x^D \right) = \partial_i(\mathfrak{S}_w(x)) = \mathfrak{S}_{ws_i}(x) = \sum_{E \in \mathcal{RC}(ws_i)} x^E
$$

by Corollary 4.2.12 and the definition of Schubert polynomial by divided differences. Plugging in $(1, \ldots, 1)$ for $x$ implies that $|\text{mitosis}_i(\mathcal{RC}(w))| = |\mathcal{RC}(ws_i)|$, as desired. □

### 4.5 Shellability of subword complexes in Coxeter groups

Our goal in this subsection is Corollary 4.5.7: the complexes $\mathcal{L}_w$ are shellable, and hence Cohen–Macaulay. Together with the Gröbner basis theorem, this provides a new proof that Schubert varieties are Cohen–Macaulay (Corollary 4.5.8). We prove these results by introducing a new class of vertex-decomposable simplicial complexes,
the subword complexes of Definition 4.5.1 and Theorem 4.5.6, and identifying the complexes \( \mathcal{L}_w \) as special cases.

This subsection and the next deal with an arbitrary Coxeter system \((\Pi, \Sigma)\) consisting of a group \(\Pi\) and a set \(\Sigma\) of generators. See [Hum90] for background and definitions; the applications to rc-graphs concern only the case where \(\Pi = S_n\) and \(\Sigma\) consists of the simple reflections switching \(i\) and \(i + 1\) for \(1 \leq i \leq n - 1\).

**Definition 4.5.1** A word of size \(m\) is an ordered list \(Q = (\sigma_1, \ldots, \sigma_m)\) of elements of \(\Sigma\). An ordered sublist \(P\) of \(Q\) is called a subword of \(Q\).

1. \(Q\) represents \(\pi \in \Pi\) if the ordered product of the simple reflections in \(Q\) is a reduced decomposition for \(\pi\).
2. \(Q\) contains \(\pi \in \Pi\) if some sublist of \(Q\) represents \(\pi\).

The subword complex \(\Delta(Q, \pi)\) is the set of subwords \(P \subseteq Q\) whose complements \(Q \setminus P\) contain \(\pi\).

Sometimes we abuse notation and say that \(Q\) is a word in \(\Pi\). Note that \(Q\) need not itself be a reduced expression. The following lemma is immediate from the definitions and the fact that all reduced expressions for \(\pi \in \Pi\) have the same length.

**Lemma 4.5.2** \(\Delta(Q, \pi)\) is a pure simplicial complex whose facets are the subwords \(Q \setminus P\) such that \(P \subseteq Q\) represents \(\pi\). \(\square\)

Subword complexes place rc-graphs in a context very close to that of [FS94] and [FK96b].

**Proposition 4.5.3** \(\mathcal{L}_w\) is the join of a simplex with \(\Delta(Q, w)\), where \(\Pi = S_n\) and

\[
Q = s_{n-1} \cdots s_3 s_2 s_1 s_{n-1} \cdots s_3 s_2 \cdots s_n s_{n-2} s_{n-1},
\]

the lexicographically last reduced expression for \(w_0\), in which \(s_1 > s_2 > \cdots > s_{n-1}\) (this is the same order as in Example 2.5.1). In other words, an rc-graph for \(w\) is a reduced expression for \(w\) that is a subword of the lex last long word.

**Proof.** Use Lemma 4.2.6, Example 4.2.7, and Theorem 4.2.11. The simplex has vertices corresponding to the elbow joints on and below the main antidiagonal in \([n]^2\). \(\square\)

**Definition 4.5.4** Let \(\Delta\) be a simplicial complex and \(F \in \Delta\) a face.

1. The deletion of \(F\) from \(\Delta\) is \(\text{del}(F, \Delta) = \{G \in \Delta \mid G \cap F = \emptyset\}\).
2. The link of \(F\) in \(\Delta\) is \(\text{link}(F, \Delta) = \{G \in \Delta \mid G \cap F = \emptyset\ \text{and} \ G \cup F \in \Delta\}\).

\(\Delta\) is vertex-decomposable if \(\Delta\) is pure and either (1) \(\Delta = \{\emptyset\}\), or (2) for some vertex \(v \in \Delta\), both \(\text{del}(v, \Delta)\) and \(\text{link}(v, \Delta)\) are vertex-decomposable. A shelling of \(\Delta\) is an ordered list \(F_1, F_2, \ldots, F_t\) of its facets such that \(\bigcup_{j < i} F_j \cap F_i\) is a union of codimension 1 faces of \(F_i\) for each \(i \leq t\). We say \(\Delta\) is shellable if it is pure and has a shelling.
**Remark 4.5.5** The empty set \( \emptyset \) is a perfectly good face of \( \Delta \), representing the empty set of vertices. Thus the empty complex \( \Delta = \{ \emptyset \} \), whose unique face has dimension \(-1\), is to be distinguished from the void complex \( \Delta = \{ \} \) consisting of no subsets of the vertex set. Writing out their reduced chain complexes, the empty complex has integral reduced homology \( \mathbb{Z} \) in dimension \(-1\), while the void complex has entirely zero homology. This distinction reveals itself most prominently when taking the link of a facet (maximal face) of \( \Delta \): the result is the empty complex, not the void complex. The link of \( \emptyset \) in \( \Delta \) is just \( \Delta \) itself, including \( \emptyset \in \Delta \). □

Provan and Billera [BP79] introduced the notion of vertex-decomposability and proved that it implies shellability (proof: use induction on the number of vertices by first shelling \( \text{del}(v, \Delta) \) and then shelling the cone from \( v \) over \( \text{link}(v, \Delta) \) to get a shelling of \( \Delta \)). It is well-known that shellability implies Cohen–Macaulayness [BH93, Theorem 5.1.13]. Here is our central observation concerning subword complexes.

**Theorem 4.5.6** The subword complex \( \Delta(Q, \pi) \) is vertex-decomposable. Therefore subword complexes are Cohen–Macaulay and even shellable.

**Proof.** Supposing that \( Q = (\sigma, \sigma_2, \sigma_3, \ldots, \sigma_m) \), it suffices to show that both the link and the deletion of \( \sigma \) from \( \Delta(Q, \pi) \) are subword complexes. By definition, both consist of subwords of \( Q' = (\sigma_2, \ldots, \sigma_m) \). The link is naturally identified with the subword complex \( \Delta(Q', \pi) \). For the deletion, there are two cases. If \( \sigma \pi \) is longer than \( \pi \), then the deletion of \( \sigma \) equals its link because no reduced expression for \( \pi \) begins with \( \sigma \). On the other hand, when \( \sigma \pi \) is shorter than \( \pi \), the deletion is \( \Delta(Q', \sigma \pi) \). □

**Corollary 4.5.7** \( \mathcal{L}_w \) is vertex-decomposable, and hence shellably Cohen–Macaulay.

**Proof.** Theorem 4.5.6 and Proposition 4.5.3. □

Among the known vertex decomposable simplicial complexes are the dual greedoid complexes [BKL85], which include the matroid complexes. Although subword complexes strongly resemble dual greedoid complexes, the exchange axioms defining greedoids seem to be slightly stronger than the exchange axioms for facets of subword complexes imposed by Coxeter relations. In particular, the naïve ways to correspond these complexes to dual greedoid complexes do not work, and we conjecture that they are not in general isomorphic to dual greedoid complexes.

There seems to be little direct relation between our shellings of subword complexes and the lexicographic shellings of intervals in the Bruhat order by Björner and Wachs [BW82], even though the results look superficially quite similar (see Section 4.6, where we prove that subword complexes are balls or spheres). Their results concern the Bruhat order, so their simplicial complexes are independent of the reduced expressions involved, although their shellings depend on such choices. In contrast, our results concern the weak Bruhat order, where the reduced expressions involved form the substance of the simplicial complexes, and fewer choices are left over for the shellings. Nonetheless, comparison of the main results suggest that the Bruhat and weak Bruhat orders “feel” somewhat similar.

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Fulton proved that \( \overline{X}_w \) is Cohen–Macaulay in [Ful92], but he used the fact that Schubert varieties are Cohen–Macaulay [Ram85] to do it. Here we can turn the tables, deriving a new proof of the Cohen–Macaulayness in of Schubert varieties in flag manifolds from an independent proof of Fulton’s result.

**Corollary 4.5.8** Every matrix Schubert variety \( \overline{X}_w \), and hence every Schubert variety \( X_w \subseteq B\setminus GL_n \), is Cohen–Macaulay.

**Proof.** The Cohen–Macaulayness of \( \mathcal{L}_w \) (Corollary 4.5.7) implies that of \( \overline{X}_w \) by Theorem 2.1.2 and the flatness of Gröbner deformation. Now use Theorem 3.5.4. □

**Remark 4.5.9** The vertex decomposition we use for matrix Schubert varieties has direct analogues in the Gröbner degenerations and formulae for Schubert polynomials. Consider the following sequence \( >_1, >_2, \ldots, >_{n^2} \) of partial term orders, where \( >_i \) is lexicographic in the first \( i \) matrix entries snaking from northeast to southwest one row at a time, and treats all remaining variables equally. The order \( >_{n^2} \) is a total order; this total order is antidiagonal, and hence degenerates \( \overline{X}_w \) to the subword complex by Theorem 2.1.2. Each \( >_i \) gives a degeneration of \( \overline{X}_w \) to a bunch of components. Each of these components degenerates at \( >_{n^2} \) to its own subword complex.

If we study how a component at stage \( i \) degenerates into components at stage \( i + 1 \), by degenerating both using \( >_{n^2} \), we recover the vertex decomposition for the corresponding subword complex.

Note that these components are *not* always matrix Schubert varieties; the set of rank conditions involved does not necessarily involve only upper left submatrices. We do not know how general a class of determinantal ideals can be tackled by partial degeneration of matrix Schubert varieties, using antidiagonal partial term orders.

However, if we degenerate using the partial order \( >_n \) (order just the first row of variables), the components are *are* matrix Schubert varieties, except for the fact that the minors involved are all shifted down one row. This gives an inductive formula for Schubert polynomials, which already appears in Section 1.3 of [BJS93]. □

### 4.6 Subword complexes are balls or spheres

Knowing now that subword complexes in Coxeter groups are shellable, we are able to prove a much more precise statement. Our proof technique requires a certain deformation of the group algebra of a Coxeter group.

**Definition 4.6.1** Let \( R \) be a commutative ring, and \( D \) a free \( R \)-module with basis \( \{ e_\pi \mid \pi \in \Pi \} \). Defining a multiplication on \( D \) by

\[
e_\pi e_\sigma = \begin{cases} e_{\pi\sigma} & \text{if } \text{length}(\pi\sigma) > \text{length}(\pi) \\ e_\pi & \text{if } \text{length}(\pi\sigma) < \text{length}(\pi) \end{cases}
\]

(21)

for \( \sigma \in \Sigma \) yields the **Demazure algebra** of \( (\Pi, \Sigma) \) over \( R \). Define the **Demazure product** \( \delta(Q) \) of the word \( Q = (\sigma_1, \ldots, \sigma_m) \) by \( e_{\sigma_1} \cdots e_{\sigma_m} = \delta(Q) \).

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Example 4.6.2 When $\Pi = S_n$ and $\Sigma$ is the set of simple reflections $s_1, \ldots, s_{n-1}$, the algebra $\mathcal{D}$ is generated over $R$ by the usual Demazure operators $\partial_i$ (hence the name “Demazure algebra”). In this case, the fact that $\mathcal{D}$ is an associative algebra with given free $R$-basis follows from the considerations in Section 1.5.

In general, the fact that the equations in (21) define an associative algebra is the special case of [Hum90, Theorem 7.1] where all of the ‘$a$’ variables equal 1 and all of the ‘$b$’ variables are zero. Observe that the ordered product of a word equals the Demazure product if the word is reduced. Here are some basic properties of Demazure products, in which the ‘$\geq$’ and ‘$>'$ signs denote the Bruhat partial order on $\Pi$ [Hum90, Section 5.9].

Lemma 4.6.3 Let $P$ be a word in $\Pi$ and let $\pi \in \Pi$.

1. The Demazure product $\delta(P)$ is $\geq \pi$ if and only if $P$ contains $\pi$.
2. If $\delta(P) = \pi$, then every subword of $P$ containing $\pi$ has Demazure product $\pi$.
3. If $\delta(P) > \pi$, then $P$ contains a word $T$ representing an element $\tau > \pi$ satisfying $|T| = \text{length}(\tau) = \text{length}(\pi) + 1$.

Proof. If $P \subseteq P$ and $P'$ contains $\pi$, then $P'$ contains $\delta(P')$ and $\pi = \delta(P) \geq \delta(P') \geq \pi$, proving part 2 from part 1. Choosing any $\tau \in \Pi$ such that length($\tau$) = length($\pi$) + 1 and $\pi < \tau \leq \delta(P)$ proves part 3 from part 1.

Now we prove part 1. Suppose $\pi' = \delta(P) \geq \pi$, and let $P' \subseteq P$ be the subword obtained by reading $P$ in order, omitting any reflections along the way that do not increase length. Then $P'$ represents $\pi'$ by definition, and contains $\pi$ because any reduced expression for $\pi'$ contains a reduced expression for $\pi$.

On the other hand, suppose $T \subseteq P$ represents $\pi$, with $\tau \in \Sigma$ being the last element of $T$. Use induction on $|P|$ as follows. If $\tau$ is also the last element of $P$, then $\delta(P \setminus \tau) = \delta(P)\tau \leq \delta(P)\pi$ and thus $\pi \leq \delta(P)$. If $\tau$ isn’t the last element in $P$, then $\pi \leq \delta(P \setminus \tau) = \delta(P)$ already.

Lemma 4.6.4 Let $T$ be a word in $\Pi$ and $\pi \in \Pi$ such that $|T| = \text{length}(\pi) + 1$.

1. There are at most two elements $\sigma \in T$ such that $T \setminus \sigma$ represents $\pi$.
2. If $\delta(T) = \pi$, then there are two distinct $\sigma \in T$ such that $T \setminus \sigma$ represents $\pi$.
3. If $T$ represents $\tau > \pi$, then $T \setminus \sigma$ represents $\pi$ for exactly one $\sigma \in T$.

Proof. Part 1 is obvious if $|T| \leq 2$, so choose elements $\sigma_1, \sigma_2, \sigma_3 \in T$ in order of appearance, and let $T_1\sigma_1T_2\sigma_2T_3\sigma_3T_4 = T$ be the resulting partition of $T$ into connected segments. If two of the words $T \setminus \sigma_1$ represent $\pi$, then assume $T \setminus \sigma_3$ represents $\pi$ (by reversing $T$ and replacing $\pi$ with $\pi^{-1}$ if necessary). Letting $\tau_1$ be the ordered product of $T_4$, then $P := T \setminus (\sigma_3T_4) = T_1\sigma_1T_2\sigma_2T_3$ represents $\pi\tau_4^{-1}$. That $P \setminus \sigma_1$ and $P \setminus \sigma_2$ can’t both represent $\pi\tau_4^{-1}$ is a consequence of the exchange condition [Hum90, Theorem 5.8], so multiplying on the right by $\sigma_3\tau_4$ yields the result.

In part 2, $\delta(T) = \pi$ means there is some $\sigma \in T$ such that (i) $T = T_1\sigma T_2$; (ii) $T_1T_2$ represents $\pi$, and (iii) $\tau_1 > \tau_1\sigma$, where $T_1$ represents $\tau_1$. Omitting some $\sigma'$ from $T_1\sigma$ leaves a reduced expression for $\tau_1\sigma$. Thus $T \setminus \sigma'$ represents $\pi$, and $\sigma'$ can’t be the original $\sigma$, by (iii). Part 3 is the exchange condition.

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Remark 4.6.5 When $\Delta(Q, \pi)$ is $L_w$ (Proposition 4.5.3), there is a “pipe dream-theoretic” reason for part 1 of Lemma 4.6.4. Let $D$ be an rc-graph for $w$ with no ‘$+$’ at $(i, j)$, and let $D_+$ be the pipe dream resulting by adding the extra ‘$+$’ at $(i, j)$. This new ‘$+$’ in $D_+$ switches the destinations of the pipes in $D$ squiggling through $(i, j)$. In order to get an rc-graph for the same $w$ out of $D_+$, we must reset the destinations of these pipes by removing a ‘$+$’. Since the two pipes in question crossed at most once in $D$, they can cross at most twice in $D_+$. \hfill $\Box$

Lemma 4.6.6 Suppose every codimension 1 face of a shellable simplicial complex $\Delta$ is contained in at most two facets. Then $\Delta$ is a topological manifold-with-boundary that is homeomorphic to either a ball or a sphere. The facets of the topological boundary of $\Delta$ are the codimension 1 faces of $\Delta$ contained in exactly one facet of $\Delta$.

Proof. Let $B_d$ and $B'_d$ be homeomorphic to $d$-dimensional balls. If $h : B_{d-1} \rightarrow B'_{d-1}$ is a homeomorphism between $(d-1)$-balls in their boundaries, then $B_d \cup h B'_d$ is again homeomorphic to a $d$-ball. On the other hand, if $h : \partial B_d \rightarrow \partial B'_d$ is a homeomorphism between their entire boundaries, then $B_d \cup h B'_d$ is homeomorphic to a $d$-sphere. Now use induction on the number of facets of $\Delta$; if the $d$-sphere case is reached, the assumption on codimension 1 faces of $\Delta$ implies that the shelling must be complete. \hfill $\Box$

Of course, the shellable complexes that interest us are the subword complexes.

Theorem 4.6.7 The subword complex $\Delta(Q, \pi)$ is a either a ball or a sphere. A face $Q \setminus P$ is in the boundary of $\Delta(Q, \pi)$ if and only if $P$ has Demazure product $\delta(P) \neq \pi$.

Proof. That every codimension 1 face of $\Delta(Q, \pi)$ is contained in at most two facets is the content of part 1 in Lemma 4.6.4, while shellability is Theorem 4.5.6. This verifies the hypotheses of Lemma 4.6.6 for the first sentence of the Theorem.

If $P$ has Demazure product $\neq \pi$, then $\delta(P) > \pi$ by part 1 of Lemma 4.6.3. Choosing $T$ as in part 3 of Lemma 4.6.3, we find by part 3 of Lemma 4.6.4 that $Q \setminus T$ is a codimension 1 face contained in exactly one facet of $\Delta(Q, \pi)$. Thus, using Lemma 4.6.6, we conclude that $Q \setminus P \subseteq Q \setminus T$ is in the boundary of $\Delta(Q, \pi)$.

If $\delta(P) = \pi$, on the other hand, part 2 of Lemmas 4.6.3 and 4.6.4 say that every codimension 1 face $Q \setminus T \in \Delta(Q, \pi)$ containing $Q \setminus P$ is contained in two facets of $\Delta(Q, \pi)$. Lemma 4.6.6 says each such $Q \setminus T$ is in the interior of $\Delta(Q, \pi)$, whence $Q \setminus P$ must itself be an interior face. \hfill $\Box$

Corollary 4.6.8 The initial complex $L_w$ is the join of a simplex with either a ball or a sphere.

Proof. Theorem 4.6.7 and Proposition 4.5.3. \hfill $\Box$
4.7 Combinatorics of Grothendieck polynomials

For many applications, substituted Hilbert numerators \( \mathcal{K}(1 - \mathbf{x}) \) are more natural than the usual versions, being in some sense more geometric. This is the essence behind the definition of multidegree from the Hilbert numerator. When \( \mathcal{K} = \mathcal{G}_w \) is a Grothendieck polynomial, for instance, the Schubert polynomial \( \mathcal{S}_w(\mathbf{x}) \) is the sum of the lowest degree terms in \( \mathcal{G}_w(1 - \mathbf{x}) \) by Lemma 1.5.6, and we have seen how these count re-graphs. The goal of this section is the analogue for \( \mathcal{G}_w(1 - \mathbf{x}) \) of Corollary 4.2.12 for \( \mathcal{S}_w(\mathbf{x}) \) due to Fomin and Kirillov [FK94]. However, as in Sections 4.5 and 4.6 (whose notation we retain), the result for general subword complexes specializes to pipe dreams. Our arguments are based on standard tools from combinatorial commutative algebra, and we assume the notation of Section 1.2.

If \( J \subseteq \mathbf{k}[z] \) is a squarefree monomial ideal with zero set \( \mathcal{L} \), then recall that the Alexander dual ideal is defined as \( J^* = \langle z_{D_L} \mid L \subseteq \mathcal{L} \text{ is a coordinate subspace} \rangle \). Viewing the collection of coordinate subspaces in \( \mathcal{L} \) as a simplicial complex, the minimal generators of \( J^* \) are therefore \( \{ z_{D_L} \mid L \in \mathcal{L} \text{ is a facet} \} \); smaller faces \( L \) have larger sets \( D_L \), so they yield monomials \( z_{D_L} \) that aren’t minimal generators of \( J^* \).

Each subword complex \( \Delta(Q, \pi) \) determines a squarefree monomial ideal \( J \subseteq \mathbf{k}[z] \), where the variables \( z_1, \ldots, z_m \) correspond to the elements in \( Q = (\sigma_1, \ldots, \sigma_m) \). The components of the zero set of \( J \) correspond both to the facets of \( \Delta(Q, \pi) \) as well as to the generators of the Alexander dual ideal \( J^* \). Since \( J^* \) is \( \mathbf{Z}^m \)-graded, it has a minimal \( \mathbf{Z}^m \)-graded free resolution

\[
0 \leftarrow J^* \leftarrow E_0 \leftarrow E_1 \leftarrow \cdots \leftarrow E_m \leftarrow 0, \quad E_i = \bigoplus_{P \subseteq Q} \mathbf{k}[z](- \deg z_P)_{\beta_{i,p}}, \tag{22}
\]

where \( \beta_{i,p} \) is the \( i \)th Betti number of \( J^* \) in \( \mathbf{Z}^m \)-graded degree \( \deg z_P \).

Recall Hochster’s formula [MP01, p. 45] for the \( \mathbf{Z}^m \)-graded Betti numbers of \( J^* \) in terms of reduced homology of \( \Delta(Q, \pi) \):

\[
\beta_{i,p} = \dim \mathbf{k} \tilde{H}_{i-1}(\text{link}(Q \setminus P, \Delta(Q, \pi)); \mathbf{k}). \tag{23}
\]

We employ this formula to calculate the Hilbert series of \( J^* \). For notation, the monomial \( z_P \) is short for \( \prod_{\sigma_j \in P} z_j \) when \( P \) is a subword of \( Q \). Note that the sum of the lowest degree terms in Lemma 4.7.1 is \( \sum z_P \) taken over \( P \) representing \( \pi \).

**Lemma 4.7.1** If \( J \) is the Stanley-Reisner ideal of \( \Delta(Q, \pi) \) and \( \ell = \text{length}(\pi) \), then

\[
\mathcal{K}(J^*; z) = \sum_{\delta(P) = \pi} (-1)^{|P| - \ell} z_P
\]

is the numerator Hilbert series of the Alexander dual ideal, where the denominator is \( \prod_{j=1}^m (1 - z_j) \), as usual.

**Proof.** Let \( Q \setminus P \in \Delta(Q, \pi) \), so \( P \subseteq Q \) contains \( \pi \). By Theorem 4.6.7, either \( \text{link}(Q \setminus P, \Delta(Q, \pi)) \) is contractible (if \( \delta(P) \neq \pi \)), or it is a sphere of dimension

\[
\dim \Delta(Q, \pi) - |Q \setminus P| = (|Q| - \ell - 1) - |Q \setminus P| = |P| - \ell - 1
\]

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(if $\delta(P) = \pi$), where a sphere of dimension $-1$ is taken to mean the empty complex $\{\emptyset\}$ having nonzero reduced homology in dimension $-1$ (Remark 4.5.5). Therefore $\widetilde{H}_{i-1}\text{link}(Q \setminus P, \Delta(Q, \pi))$ is zero unless $\delta(P) = \pi$ and $i = |P| - \ell$. Now apply (23) to (22), and use $H(J^*; z) = \sum_j (-1)^j H(E_j; z)$.

Of course, our goal is to get information about Grothendieck polynomials, which come from the Hilbert numerator of $J$ rather than $J^*$. However, these two are intimately related, as the next result demonstrates. It holds more generally for the “squarefree modules” of Yanagawa [Yan00], as shown in [Mil00b, Theorem 4.36]. A $\mathbb{Z}$-graded version of Proposition 4.7.2 was proved by Terao for squarefree ideals using some calculations involving $f$-vectors of simplicial complexes [Ter99, Lemma 2.3]. The $\mathbb{Z}^m$-grading here simplifies the proof.

**Proposition 4.7.2 (Alexander inversion formula)** For any squarefree monomial ideal $J \subseteq k[z]$, we have $\mathcal{K}(k[z]/J; z) = \mathcal{K}(J^*; 1 - z)$.

**Proof.** See (1) in Section 1.2 for the Hilbert series of $k[z]/J$. On the other hand, the Hilbert series of $J^*$ is the sum of all monomials $z^b$ divisible by $z^{D_L}$ for some $L \in \mathcal{L}$, that is for $b \in \mathbb{Z}^m$ having support $D_L$ for some $L \in \mathcal{L}$:

$$H(J^*; z) = \sum_{L \in \mathcal{L}} \prod_{i \in D_L} \frac{z_i}{1 - z_i} = \sum_{L \in \mathcal{L}} \prod_{i \in D_L} (z_i) \prod_{i \notin D_L} (1 - z_i) \prod_{i=1}^m (1 - z_i). \quad (24)$$

Now compare the last expressions of (1) and (24). \qed

**Theorem 4.7.3** If $J$ is the Stanley-Reisner ideal of $\Delta(Q, \pi)$ and $\ell = \text{length}(\pi)$, then

$$\mathcal{K}(k[z]/J; z) = \sum_{\delta(P) = \pi} (-1)^{|P| - \ell}(1 - z)^P,$$

where $(1 - z)^P = \prod_{i \in P}(1 - z_i)$.

**Proof.** Proposition 4.7.2 and Lemma 4.7.1. \qed

Special cases of Theorem 4.7.3 and [FK94, Theorem 2.3 and p. 190] agree. Their notation differs from ours: substituting $y_p \leftarrow 1 - y_p^{-1}$ and $x_q \leftarrow 1 - x_q$ in their polynomial $\mathcal{L}_w^{(-1)}(y, x)$ yields what we have called $\mathcal{G}_w(x, y)$. Therefore, in the following corollary, we write $\mathcal{G}_w(1 - x, 1 - y^{-1})$ for the polynomial obtained by substituting $x_q \leftarrow 1 - x_q$ and $y_p \leftarrow 1 - y_p^{-1}$ into $\mathcal{G}_w(x, y)$.

**Corollary 4.7.4 ([FK94])** The double Grothendieck polynomial $\mathcal{G}_w(x, y)$ satisfies

$$\mathcal{G}_w(1 - x, 1 - y^{-1}) = \sum_{\delta(D) = w} \prod_{(q, p) \in D} (-1)^{|D| - \ell}(x_q + y_p - x_q y_p),$$

where $\text{length}(w) = \ell$. The version for single Grothendieck polynomials reads

$$\mathcal{G}_w(1 - x) = \sum_{\delta(D) = w} (-1)^{|D| - \ell}x^D.$$
Proof. Any pipe dream in \([n]^2\) having Demazure product \(w \in S_n\) must be contained in \(D_0\) (from Example 4.1.2) by part 1 of Lemma 4.6.3. Therefore, Proposition 4.5.3 says \(\mathcal{L}_w = \Delta(Q, w)\), where \(Q\) is the word whose pipe dream fills \([n]^2\) with crosses. Apply Theorem 4.7.3 to this subword complex, so that \(J = J_w\) by definition. Specializing \(z_{ap}\) to \(x_q y_p\) yields the double version after calculating \(1 - (1 - x_q)(1 - y_p) = x_q + y_p - x_q y_p\), while the single version follows trivially.

Note that the sum of lowest degree terms in \(G_w(1 - x, 1 - y^{-1})\) equals \(G_w(x, -y)\), which has \((x_q + y_p)\) in place of the difference \((x_q - y_p)\) appearing in Corollary 4.2.13. The single version in Corollary 4.2.12 is derived unchanged from Corollary 4.7.4.

Remark 4.7.5 Corollary 4.7.4 implies that the coefficients of \(G_w(1 - x)\) alternate. To be precise, there are polynomials \(G_w^{(d)}(x)\) with nonnegative coefficients such that

\[
G_w(1 - x) = \sum_{d \geq \ell} (-1)^{d-\ell}G_w^{(d)}(x),
\]

where \(\ell = \text{length}(w)\). Since the results in [FK94] already imply this fact, it was one of our principal reasons for conjecturing the Cohen–Macaulayness of \(\mathcal{L}_w\) (Corollary 4.5.7) in the first place. The connection is through the Eagon-Reiner theorem [ER98]:

A simplicial complex \(\Delta\) is Cohen–Macaulay if and only if the Alexander dual \(J^*_\Delta\) of its Stanley–Reisner ideal has **linear free resolution**, meaning that the differential in its minimal \(\mathbb{Z}\)-graded free resolution over \(k[z]\) can be expressed using matrices filled with linear forms.

The numerator of the Hilbert series of any module with linear resolution alternates as in (25), so the Alexander inversion formula (Proposition 4.7.2) and the Eagon-Reiner theorem together say that (25) holds if \(\mathcal{L}_w\) is Cohen–Macaulay. It would take suspiciously fortuitous cancellation to have a squarefree monomial ideal whose Hilbert numerator behaves like (25) without the ideal actually having linear resolution.

Here is a weird consequence of the Demazure product characterization of \(K(J^*_w; z)\).

Porism 4.7.6 Each squarefree monomial \(z^D\) in \(z = (z_{ap})\) appears with nonzero coefficient in the Hilbert numerator of exactly one ideal \(J^*_w\), and its coefficient is \(\pm 1\).

Proof. The permutation \(w\) in question is \(\delta(D)\), by Lemma 4.7.1.

A Appendix

A.1 Gradings, Hilbert series, and \(K\)-theory

We review here some notions concerning \(\mathbb{Z}^n\)-gradings and Hilbert series, and point out the connection to equivariant \(K\)-theory of vector spaces with sufficiently nice torus actions. The algebraic side of this material pervades the paper, while the more
geometric aspects find applications in Sections 1.2 and 3.2. As in the rest of this monograph, the field $k$ can have arbitrary characteristic, and for convenience we assume that $k$ is algebraically closed, although this hypothesis can be dispensed with by resorting to sufficiently abstruse language.

Suppose a torus $T \cong (k^*)^n$ with weight lattice (character group) $W \cong \mathbb{Z}^n$ acts linearly on the vector space $M \cong k^m$. Then $M = k \cdot e_1 \oplus \cdots \oplus k \cdot e_m$ is a direct sum of characters of $T$, say with weights $a_1, \ldots, a_m \in W$. If $z = \{z_1, \ldots, z_m\} \in M^*$ is the basis dual to $e_1, \ldots, e_m$, then the coordinate ring of $M$ is $k[z]$, which has a grading induced by the action of $T$:

$$z^b = z_1^{b_1} \cdots z_m^{b_m} \implies \deg(z^b) = \sum_{p=1}^m b_p a_p.$$ 

Henceforth, we shall always assume that our torus actions have been given in this form, so that $M$ is identified with $k^m$, and we choose an identification $W = \mathbb{Z}^n$.

**Example A.1.1** Our primary concern is the case $m = n^2$ and $M = M_n$, but in fact the actions of four different tori play important roles. The largest, $(k^*)^{n^2}$, scales all of the entries in a matrix independently; the invertible $n \times n$ diagonal matrices $T$ act by multiplication on the left; $T^{-1}$ is the same diagonal matrices, but acting by inverse multiplication on the right; and $k^*$ acts by scaling all of the entries in a matrix simultaneously. The variables in $k[z]$ come in an array $\{z_{ij}\}_{i,j=1}^n$, and their **exponential weights** are

<table>
<thead>
<tr>
<th>torus action</th>
<th>$k^*$</th>
<th>$T$</th>
<th>$T \times T^{-1}$</th>
<th>$(k^*)^{n^2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>exponential weight</td>
<td>$t$</td>
<td>$x_i y_j$</td>
<td>$z_{ij}$</td>
<td></td>
</tr>
</tbody>
</table>

as Laurent monomials in the group algebra $\mathbb{Z}[\text{weight lattice}]$ over the integers. □

The set of weights $a \in \mathbb{Z}^n$ that are degrees of monomials in $k[z]$ is an **affine semigroup** (a finitely generated submonoid of $\mathbb{Z}^n$), which we call the **support semigroup**, with generators $a_1, \ldots, a_m$. A $k[z]$-module $\Gamma$ is $\mathbb{Z}^n$-**graded** when $\Gamma = \bigoplus_{a \in \mathbb{Z}^n} \Gamma_a$ and $z^b \cdot \Gamma_a \subseteq \Gamma_{a + \deg(b)}$, where we set $\deg(b) = \deg(z^b)$. We denote by $\Gamma(a)$ the $\mathbb{Z}^n$-**graded shift** of $\Gamma$ “down by $a$”, which is defined by $\Gamma(a)_{a'} = \Gamma_{a+a'}$. All of the homomorphisms $\Gamma \to \Gamma'$ of $\mathbb{Z}^n$-graded modules we use are **homogeneous**, meaning that the image of $\Gamma_a$ is contained in $\Gamma'_a$ for all $a \in \mathbb{Z}^n$. For instance, multiplication by $z^b$ induces an isomorphism between $k[z](-\deg(b))$ and the principal ideal $(z^b)$.

It is a standard fact that the kernel and cokernel of a homomorphism of $\mathbb{Z}^n$-graded modules is $\mathbb{Z}^n$-graded. In particular, every $\mathbb{Z}^n$-graded module $\Gamma$ is a quotient of a $\mathbb{Z}^n$-graded free module by a $\mathbb{Z}^n$-graded submodule. Furthermore, every $\mathbb{Z}^n$-graded free module is isomorphic to a direct sum of rank one graded free modules of the form $k[z](-a)$, whose generator is in degree $a \in \mathbb{Z}^n$.

The “sufficiently nice” torus actions mentioned at the beginning of this subsection are those satisfying the conditions in the next definition. Such is the case with Example A.1.1, for instance, where the support semigroup is $\mathbb{N}^n$.  

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**Definition A.1.2** The action of $T$ on $k^m$ is **positive** if

1. the degrees $a_1, \ldots, a_m$ of $z_1, \ldots, z_m$ are nonzero; and
2. the support semigroup has no nontrivial units (i.e. the real cone spanned by $a_1, \ldots, a_m$ does not contain a linear subspace of positive dimension).

The reason for defining positivity is that we want to work with Hilbert series, as the following lemma allows.

**Lemma A.1.3** The torus $T$ acts positively on $k^m$ if and only if $\dim_k(\Gamma_a)$ is finite for all finitely generated $k[z]$-modules $\Gamma$ and all degrees $a \in \mathbb{Z}^n$.

**Proof.** If some variable $z_p$ has degree $0 \in \mathbb{Z}^n$, then the subring $k[z_p] \subseteq k[z]_0$ has infinite dimension over $k$. If the support semigroup has a unit $a = \deg(z^b)$, so that $-a = \deg(z^{b'})$, then the subring $k[z^{b+b'}] \subseteq k[z]_0$ has infinite dimension over $k$. Therefore the finiteness condition implies positivity.

On the other hand, suppose the action is positive. By condition 2 of positivity, there is a linear functional $\xi$ on $\mathbb{Z}^n$ which is minimized on the support semigroup precisely at $0$. By condition 1 of positivity, $\xi(a_p) > 0$ for $p = 1, \ldots, m$. It follows that for each nonnegative $a \in \mathbb{Z}^n$, there are only finitely many monomials $z^b \in k[z]$ such that $\xi(\deg(z^b)) \leq a$. More generally, in any $\mathbb{Z}^n$-graded shift $k[z](-b)$ there are finitely many monomials with $\xi$-degree at most $a$. Since the finiteness condition is stable under finite direct sums and then quotients, it holds for all finitely generated free modules and then all finitely generated modules. \hfill \Box

Now we can define the **Hilbert series**

$$H(\Gamma; x) := \sum_{a \in \mathbb{Z}^n} \dim_k(\Gamma_a)x^a.$$

of a finitely generated $\mathbb{Z}^n$-graded module $\Gamma$ over the polynomial ring $k[z]$, in the presence of a positive $T$-action on $k^m$. As it stands, this Hilbert series makes sense only as an element in a direct product of copies of $\mathbb{Z}$, one for each $a \in \mathbb{Z}^n$. To remedy this, let $A \subset \mathbb{Z}^n$ be the support semigroup for the positive action of $T$ on $k^m$. The set of monomials $x^a \neq 1$ in the semigroup ring $\mathbb{Z}[A] = \mathbb{Z}[x^{a_1}, \ldots, x^{a_m}]$ generate a proper ideal. Completion with respect to this ideal yields a ring $\mathbb{Z}[[A]]$ of power series supported on $A$. Let $\mathbb{Z}[x^{\pm 1}]$ be the ring of Laurent polynomials in $x$, and define the **conical ring** for the positive action to be

$$\mathbb{Z}[x^{\pm 1}][[A]] := \mathbb{Z}[x^{\pm 1}] \otimes_{\mathbb{Z}[A]} \mathbb{Z}[[A]].$$

Its elements are those formal sums in which the Laurent monomials appearing with nonzero (integer) coefficient are contained in a finite union of translates of $A$ inside $\mathbb{Z}^n$. Note that $A$ need not generate $\mathbb{Z}^n$ as a group—$A$ may even have dimension less than $n$—unless the action of $T$ is faithful.
Example A.1.4 Consider what happens when \( n = 2 \) and \( z_1, \ldots, z_m \) all have weight \((1, 1)\). The grading on \( k[z] \) is the usual \( Z \)-grading, but \((a, a^{-1}) \in T = k^* \times k^* \) acts trivially on \( k^m \), and \( A = \mathbb{N} \cdot (1, 1) \leq \mathbb{Z}^2 \) only spans a subgroup of rank 1. The Hilbert series of finitely generated \( \mathbb{Z}^2 \)-graded modules can only have infinitely many terms along finitely many rays pointing up diagonally with slope 1.

An example in a different direction is when \( n = m = 2 \), and the weights of \( z_1 \) and \( z_2 \) are \( a_1 = (2, 0) \) and \( a_2 = (0, 2) \). In this case, the kernel of the action of \( T = (k^*)^2 \) on \( k^2 \) is finite, of order 4. There is a \( \mathbb{Z}^2 \)-graded free module \( k[z_1, z_2][(-1, -1)] \) generated in degree \((1, 1)\), say, even though the support of its Hilbert series is disjoint from \( A \). \( \square \)

For positive actions, the completeness of \( \mathbb{Z}[[A]] \) implies that \((1 - x^{ap})^{-1} \) is, for each \( p \), represented by a well-defined power series (i.e., convergent in the appropriate topology) in \( \mathbb{Z}[[A]] \subset \mathbb{Z}[x^{\pm 1}][[A]] \). This allows us to write

\[
H(k[z](-a); x) = \frac{x^a}{\prod_{p=1}^m (1 - x^{ap})} \in \mathbb{Z}[x^{\pm 1}][[A]]
\]

for the Hilbert series of a rank one free module, and (given the Hilbert syzygy theorem) proves the following proposition.

Proposition A.1.5 Let \( T \) act positively on \( k^m \). The Hilbert series of each finitely generated \( \mathbb{Z}^n \)-graded \( k[z] \)-module \( \Gamma \) is an element of the conical ring \( \mathbb{Z}[x^{\pm 1}][[A]] \). More precisely, there is a unique Hilbert numerator Laurent polynomial \( K(\Gamma; x) \) with

\[
H(\Gamma; x) = \frac{K(\Gamma; x)}{\prod_{p=1}^m (1 - x^{ap})}.
\]

Note that only certain very special elements of the conical ring are actually Hilbert series of \( \mathbb{Z}^n \)-graded modules, and that Hilbert series of ideals and quotients by them are always in \( \mathbb{Z}[[A]] \) itself.

Our reason for caring about \( \mathbb{Z}^n \)-graded modules is that they are global sections of \( T \)-equivariant sheaves on \( k^m \), as we now show. Recall that an equivariant sheaf \( \mathcal{F} \) on \( k^m \) is a sheaf along with isomorphisms \( \mathcal{F} \to \alpha_* \mathcal{F} \), for each \( \alpha \in T \), that are compatible with the group action. For instance, the isomorphism \( \mathcal{F} \to (\alpha \beta)_* \mathcal{F} \) is obtained by composing \( \mathcal{F} \to \beta_* \mathcal{F} \) with \( \beta_* \mathcal{F} \to \alpha_* (\beta_* \mathcal{F}) \).

Lemma A.1.6 Let \( \Gamma \) be a module over \( k[z] \), and \( \mathcal{F} \) the corresponding sheaf. Then \( \mathbb{Z}^n \)-gradings on \( \Gamma \) correspond bijectively to \( T \)-equivariant structures on \( \mathcal{F} \).

Proof. The direct sum decomposition of \( \Gamma \) indexed by \( \mathbb{Z}^n \) is the splitting into isotypic components for the \( T \)-action. It is trivial to verify that \( z^b \Gamma_a \subseteq \Gamma_{a + \deg(b)} \). On the other hand, the sheaf \( \mathcal{F} \) associated to a \( \mathbb{Z}^n \)-graded module \( \Gamma \) comes with the torus action in which \( \alpha \in T \) acts on the global section \( \sigma \in \Gamma_a \) via \( \alpha \cdot \sigma = \chi_\alpha(\sigma) \sigma \), where \( \chi_\alpha : T \to k^* \) is the character corresponding to \( a \). Note that this action is compatible with multiplication by \( z^b \) because \( \alpha \cdot z^b = \chi_{\deg(b)} z^b \). \( \square \)
Recall that the **equivariant K-cohomology ring** $K_G^*(X)$ of a variety $X$ with an action of a group $G$ is generated as a group by isomorphism classes of $G$-equivariant vector bundles on $X$ modulo the relations $[\mathcal{E}]_G = [\mathcal{E}'_G] + [\mathcal{E}^*_G]$ determined by short exact sequences $0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}^* \to 0$, and has ring operations induced by direct sum and tensor product. Allowing all equivariant sheaves instead of just vector bundles yields the **equivariant K-homology group** $K_G^T(X)$, which is a module over $K_G^0(X)$ via direct sum and tensor product.

**Example A.1.7** When $X$ is a single point, an equivariant sheaf is the same thing as an equivariant vector bundle: a representation of $G$. In this case $K_G^0(\text{point})$ is the **representation ring** of $G$. When $G = T$ is a torus, $K_T^0(\text{point}) \cong \mathbb{Z}[x^{\pm 1}]$ is generated as an abelian group by the characters of $T$.

When $G = \{1\}$ is the trivial group, an equivariant sheaf is just a sheaf. In this case, $K_{\{1\}}^0(X) = K^0(X)$ and $K_{\{1\}}^1(X) = K_1(X)$ are called the **ordinary K-groups** of $X$. \hfill \Box

If $X = k^m$ (or any smooth variety $M$), the Poincaré homomorphism $K_G^0(X) \to K_G^T(X)$ sending an equivariant vector bundle to its (automatically equivariant) sheaf of sections is an isomorphism. More explicitly, an equivariant coherent sheaf $\mathcal{F}$ has a finite resolution by equivariant vector bundles,

\begin{equation}
0 \leftarrow \mathcal{F} \leftarrow \mathcal{E}_0 \leftarrow \cdots \leftarrow \mathcal{E}_{m-1} \leftarrow \mathcal{E}_m \leftarrow 0,
\end{equation}

and we have $[\mathcal{F}]_G = \sum_{p=0}^m (-1)^p [\mathcal{E}_p]_G$ as a class in $K_G^*(X)$.

**Proposition A.1.8** $K_T^0(k^m) \cong \mathbb{Z}[x^{\pm 1}]$ for positive actions of $T$, and the $K_T^*$-class of a $T$-equivariant coherent sheaf $\mathcal{F}$ is the Hilbert numerator $K(\Gamma(\mathcal{F}); x)$.

**Proof.** Every $\mathbb{Z}^n$-graded projective module is isomorphic to a direct sum of free modules of the form $k[z](-a)$, by Nakayama’s lemma (this requires the positive grading if we are to avoid using the Quillen-Suslin theorem [Qui76, Sus76]). In other words, every equivariant vector bundle on $k^m$ is a direct sum of equivariant line bundles of the form $\mathcal{O}_{k[z]} \otimes_k k_a$, where $k_a$ is the 1-dimensional $T$-representation with weight $a$. Since $k^m$ is affine, the global section functor is exact, and it follows that every short exact sequence of equivariant vector bundles is split. Thus $K_T^0(k^m) \cong \mathbb{Z}[x^{\pm 1}]$.

Under this isomorphism, the $K_T^*$ class of $\mathcal{O}_{k[z]} \otimes_k k_a$ is just $x^a$, so the result holds for line bundles by (26). The result holds for equivariant vector bundles on $k^m$ since these are all direct sums of equivariant line bundles. Finally, the result holds for arbitrary equivariant coherent sheaves by Lemma A.1.6 and the line after (26). \hfill \Box

**Example A.1.9** Everything in this subsection is functorial with respect to the choice of torus $T$. In the situation of Example A.1.1, for example, the inclusions of tori induce maps

\[
\begin{align*}
k^* &\hookrightarrow T & T &\hookrightarrow T \times T^{-1} &\to (k^*)^{n^2} \\
\mathbb{Z}[x^{\pm 1}] &\hookrightarrow \mathbb{Z}[x^{\pm 1}] & \mathbb{Z}[x^{\pm 1}, y^{\pm 1}] &\hookrightarrow \mathbb{Z}[z^{\pm 1}] \\
t &\hookrightarrow x_i & x_i/y_j &\hookrightarrow z_{ij}
\end{align*}
\]
on equivariant $K$-theory of $M_n$. These maps amount to **coarsening the grading:** every $\mathbb{Z}^{n^2}$-graded module is also $\mathbb{Z}^{2n}$-graded; every $\mathbb{Z}^{2n}$-graded module is also $\mathbb{Z}^n$-graded, and so on, via the maps between weight lattices. □

More generally, an arbitrary action of $T$ on $k^m$ is induced by a map from $T$ to the torus $(k^*)^m$ scaling each of the variables $z_1, \ldots, z_m$ separately. Taking the standard basis $e_1, \ldots, e_m$ for $\mathbb{Z}^m$, positivity can be rephrased in terms of the resulting map $\mathbb{Z}^m \to W$ of weight lattices by saying that the kernel intersects $\mathbb{N}^m$ trivially.

This perspective on $k[z]$ with its grading by $W$ leads naturally to toric geometry in two forms. The extra information of a certain radical monomial ideal makes $k[z]$ into a “homogeneous coordinate ring” [Cox95]. On the other hand, the extra information of the surjection $k[z]$ onto the semigroup ring $k[A]$, where $A$ is the image in $W$ of the standard basis in $\mathbb{Z}^m$, has kernel $I_A$, which is a “lattice ideal” [PS98].

### A.2 Term orders and Gröbner bases

This appendix collects some standard definitions and facts surrounding the theory of Gröbner bases. At the end, we point out where in this monograph the various aspects of “Gröbnerness” turn up.

Resume the general notation $k[z] = k[z_1, \ldots, z_m]$ of Appendix A.1. A total order ‘$>$’ on the monomials in $k[z]$ is a **term order** if

$$z^{a+b} > z^{a+c} \quad \text{whenever} \quad z^b > z^c \quad (a, b, c \in \mathbb{N}^m), \quad \text{and} \quad (27)$$

$$z^b > 1 \quad \text{whenever} \quad 0 \neq b \in \mathbb{N}^m. \quad (28)$$

The first of these two conditions is read “the total order ‘$>$’ is **multiplicative.**” Together, (27) and (28) imply that

$$z^{a+b} > z^a \quad \text{for all} \quad a, b \in \mathbb{N}^m \text{ with } b \neq 0, \quad (29)$$

and the following key property is a consequence.

**Lemma A.2.1** Term orders are **artinian**, meaning that every set of monomials in $k[z]$ possesses a (necessarily unique) minimal element. In particular, every descending sequence of monomials stabilizes.

**Proof.** If $A$ is a set of monomials, then a finite subset $A' \subseteq A$ generates the ideal $\langle A \rangle$, because $k[z]$ is noetherian. The smallest element in $A'$ is minimal in $A$ by (29). □

One of the best known term orders is the **lexicographic** term order, in which $z^a >_{\text{lex}} z^b$ if the earliest nonzero coordinate of $a - b$ is positive. As B. Sturmfels likes to say, $z_1$ is so expensive, that any monomial containing more $z_1$ than another must automatically be larger in the total order. And if two monomials have the same amount of $z_1$, then $z_2$ is the expensive variable, and so on. Technically, there is a lexicographic term order for each total ordering of the variables themselves; the lex term order in this paragraph has $z_1 > z_2 > \cdots > z_m$.  

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The other standard term order with $z_1 > z_2 > \cdots > z_m$ is the reverse lexicographic term order, in which $z^a >_{\text{revlex}} z^b$ whenever $\sum a_i = \sum b_i$ and the latest nonzero coordinate of $a - b$ is negative. Again, as Sturmfels explains it, $z_m$ is so cheap, that any monomial containing more $z_m$ than another monomial of the same degree must automatically be smaller in the total order.

Given a term order ‘$>$’ (or indeed, any total order on monomials in $k[z]$), every polynomial $f \in k[z]$ has an initial term in$_\succ(f)$, which we abbreviate to in$(f)$ when the term order ‘$>$’ is clear from context. The term in$(f)$ is defined to be $\lambda z^b$, where $z^b$ is the largest monomial in the total order that has nonzero coefficient in $f$, and the nonzero coefficient on $z^b$ in $f$ is $\lambda \in k^*$.

**Example A.2.2** Consider two term orders on $k[z]$, where $z = (z_{ij})_{i,j=1}^n$:

1. the reverse lexicographic term order that makes its way from the northwest corner to the southeast corner, $z_{11} > z_{12} > \cdots > z_{1n} > z_{21} > \cdots > z_{mn}$; and

2. the lexicographic term order that makes its way from the northeast to the southwest, $z_{1,n} > z_{2,n} > \cdots > z_{n,n} > z_{1,n-1} > \cdots > z_{n-1,n} > z_{n,1}$.

Let us verify the statement made after Definition 2.1.1 to the effect that for either term order, the initial term of any minor in the generic matrix $Z = (z_{ij})$ is the antidiagonal term. For simplicity, let’s just check this for det$(Z)$ itself.

Start with the revlex order. Every monomial in det$(Z)$ contains a variable from the last row. If this variable isn’t $z_{n1}$, then the monomial is so cheap that it can’t be the initial term. Therefore $z_{n1}$ divides in$_{\text{revlex}}(\det(Z))$. Repeat the argument on the submatrix of $Z$ obtained by crossing out its left column and bottom row, and so on.

Now do the lex order. Every monomial in det$(Z)$ contains a variable from the last column. If this variable is $z_{1n}$, then the monomial is so expensive that it must be larger than any monomial without $z_{1n}$. Therefore $z_{1n}$ divides in$_{\text{lex}}(\det(Z))$. Repeat the argument on the submatrix of $Z$ obtained by crossing out its top row and rightmost column, and so on. Observe that snaking the other way from northeast to southwest (that is, across the top row first instead of down the rightmost column first) produces the same result, by the symmetry reflecting across the antidiagonal. \[\square\]

Algebraically, the distinction between term orders and arbitrary total orders on monomials becomes clear when considering the initial ideal of an ideal $I \subseteq k[z]$, defined as in$_\succ(I) := \{\text{in}_\succ(f) \mid f \in I\}$. Again, we write in$(I)$ when the context is clear. That in$(I)$ is indeed an ideal follows easily from (27), (28), and (29).

**Proposition A.2.3** The monomials outside in$(I)$ form a $k$-vector space basis for $k[z]/I$.

**Proof.** The initial term of any nonzero $k$-linear combination of monomials outside in$(I)$ lies outside in$(I)$. Therefore, no such linear combination can lie in $I$, so the monomials outside in$(I)$ remain linearly independent modulo $I$.
Given a polynomial \( f \in k[z] \), let \( \text{max}(f) \) be the largest monomial of \( \text{in}(I) \) with nonzero coefficient in \( f \). If no such monomial exists, then \( f \) lies in the span \( S \) of monomials outside \( \text{in}(I) \). Otherwise, pick \( g \in I \) with \( \text{in}(g) = \text{max}(f) \), and subtract the appropriate scalar multiple of \( g \) from \( f \). This produces a polynomial \( f' \) that either lies in \( S \) or has \( \text{max}(f) > \text{max}(f') \). This process must terminate by Lemma A.2.1. □

A set \( \{f_1, \ldots, f_r\} \) of elements in \( I \) is called a **Gröbner basis** for \( I \) if \( \text{in}(I) = \langle \text{in}(f_1), \ldots, \text{in}(f_r) \rangle \) is generated by the initial terms of the polynomials \( f_1, \ldots, f_r \). Such finite sets always exist because \( k[z] \) is noetherian. Our main examples of Gröbner bases are, of course, the minors in Theorem 2.1.2. The property of Gröbner bases that we exploit most is the following.

**Corollary A.2.4** Assume \( k[z] \) is graded via a positive torus action. Then \( k[z]/I \) and \( k[z]/\text{in}(I) \) have the same \( \mathbb{Z}^n \)-graded Hilbert series.

**Proof.** The two Hilbert series in question count the dimensions of vector spaces having the same set of monomials for bases, namely the monomials outside \( \text{in}(I) \). □

Various aspects of Gröbner bases and the passage to initial ideals arise in this monograph. Roughly, they can be classified as algebraic, combinatorial, and geometric, the same way as in the Introduction.

In the next appendix, Gröbner degeneration plays an algebraic role: the preservation of Hilbert series (and therefore Hilbert numerators) under taking initial ideals helps us define the notion of multidegree. This occurs particularly in Proposition A.3.7. Multidegrees, which are significantly coarser invariants than Hilbert series, are one of the two avenues by which Corollary A.2.4 enters into the proof of Theorem 2.1.2, via Lemma 2.1.5 in this case. For the second (and more direct) application, we calculate the Hilbert series of matrix Schubert varieties by finding the Hilbert series of their initial “antidiagonal” ideals.

The calculation of the initial ideals’ Hilbert series relies on actually knowing the initial terms that generate them. In other words, we take advantage of a specific Gröbner basis rather than some abstract Gröbner degeneration. The Gröbner basis consists of geometrically meaningful polynomials (minors), whose initial terms (antidiagonals) have crucial combinatorial properties. The analysis of these antidiagonal terms occupies most of Section 2; in addition to the Hilbert series of Theorem 2.1.2, it results also in the purely combinatorial theorems forming the first half of Section 4.

Geometrically, Corollary A.2.4 says that Gröbner degeneration is flat, or in other words that the Gröbner degenerated monomial subscheme defines the same class in equivariant \( K \)-theory. This geometric perspective on Gröbner degeneration as an algebraic homotopy is the motor behind our intuition and proof of the positivity of coefficients in Schubert polynomials, thought of as representing the cohomology classes of Schubert varieties in flag manifolds (see the Introduction and the beginning Appendix A.3). It also provides the backdrop for Section 3.2, which identifies the Grothendieck polynomials as \( K \)-classes of Schubert varieties in flag manifolds.
As a final remark, it is possible to make sense of initial ideals with respect to “partially defined” term orders, which fail to distinguish between some monomials, but which admit refinements to honest term orders. Such is the case in Remark 4.5.9, for instance.

A.3 Multidegrees of graded modules

Resume the notation from Appendix A.1, so \( k[z] \) is a \( \mathbb{Z}^n \)-graded polynomial ring in \( m \) variables. Let \( \Gamma \) be a \( \mathbb{Z}^n \)-graded module. We shall define in this Appendix the multigraded degree of \( \Gamma \), in a way that generalizes the usual notion of degree for \( \mathbb{Z} \)-graded ideals and subschemes of projective space. We assume in this appendix that the action of the torus \( T = (\mathbb{C}^*)^n \) on \( k^m \) is positive, but see Remark A.3.14.

The source of our intuition is that the multidegree of the quotient \( k[z]/I \) for a homogeneous radical ideal \( I \) ought to be nothing more than the \( T \)-equivariant cohomology class of the subvariety of \( I \), at least when \( k = \mathbb{C} \). This can be made completely precise, although we would need better machinery for producing equivariant cohomology classes from subvarieties than the Vassiliev–Kazarian method in the next appendix. The more powerful machinery would be forced by the fact that \( T \) usually fails to have finitely many orbits on \( \mathbb{C}^m \), so that \( T \)-stable subvarieties will rarely be \( (\mathbb{C}^*)^m \)-stable as would be required by Corollary A.4.6.

In algebraic terms, this means that \( I \) need not be a monomial ideal, so we can’t just specialize the natural \( \mathbb{Z}^m \)-graded degree of \( I \) (we shall determine what that is for nonreduced monomial ideals) to the \( \mathbb{Z}^n \)-graded degree of \( I \). On the other hand, Gröbner deformation always respects the \( \mathbb{Z}^n \)-grading: homogenizing a \( \mathbb{Z}^n \)-graded polynomial with respect to some weight vector and then setting the homogenizing variable to some constant always results in another \( \mathbb{Z}^n \)-graded polynomial. Therefore, we can simply define the multidegree of \( I \) in terms of its initial ideal, having faith that because Gröbner deformation preserves equivariant cohomology classes, our definition of multidegree remains independent of the term order (this independence is proved in Proposition A.3.7, below).

To begin with, recall the standard notion of multiplicity \( \text{mult}_X(\Gamma) \) of a module \( \Gamma \) along an irreducible subvariety \( X \) in \( k^m \): it is the length of the largest submodule of finite length in the localization of \( \Gamma \) at the prime ideal of \( X \). When \( X \) has Krull dimension \( \text{dim} \Gamma \), this localization already has finite length \( \text{mult}_X(\Gamma) \).

Our initial use of multiplicity will be for modules \( \Gamma \) that are direct sums of quotients by monomial ideals, in which case we can safely assume that \( X \) is a coordinate subspace \( L \subseteq k^m \). We mention this because the reader may know other more combinatorial characterizations of multiplicity at monomial primes, for instance in terms of ‘standard pairs’ [STV95]. Let \( D_L \) be the subset of \( \{1, \ldots, m\} \) such that \( \langle z_i \mid i \in D_L \rangle \) is the ideal of functions vanishing on \( L \).

Our main new ingredient for multidegrees looks a little strange at first sight. Set

\[
\log_k(z^b) = \prod_{b_i \neq 0} \left( b_i \sum_{j=1}^{n} a_{ij} x_j \right) \quad \text{for any monomial } z^b \in k[z].
\]
The numbers $a_{ij}$ are, for each $i = 1, \ldots, m$, the components of the degree $a_i \in \mathbb{Z}^n$ of $z_i$. The right hand side of (30) should be interpreted as an element in the ring $H_T^*(k^m) = \text{Sym}^*_Z((\mathbb{Z}[z])^n)$, and could more “correctly” be written as the element $\prod_{i=1}^m b_i \deg(z_i) \in \text{Sym}^*_Z((\mathbb{Z}[z])^n)$, where $c$ is the number of nonzero coordinates of $b$.

**Definition A.3.1** Let $J \subseteq k[z]$ be a monomial ideal with zero scheme $\mathcal{L} \subseteq k^m$. The $(\mathbb{Z}^n)$-graded geometric multidegree of $\mathcal{L}$ (or of $k[z]/J$) is the sum

$$[\mathcal{L}]_{\mathbb{Z}^n} = \sum \text{mult}_L(k[z]/J) \cdot \log_\mathbb{C}(z^D_L)$$

over all reduced subspaces $L \subseteq \mathcal{L}$ of maximal dimension. Suppose that $J_1, \ldots, J_r$ are monomial ideals with zero schemes $\mathcal{L}_1, \ldots, \mathcal{L}_r$, and that $a_1, \ldots, a_r \in \mathbb{Z}^n$. The geometric multidegree of the module $\Gamma = \bigoplus_{r=1}^r (k[z]/J_r)(a_r)$ is the sum $\sum [\mathcal{L}_\ell]_{\mathbb{Z}^n}$ over all $\ell$ such that $\dim(\mathcal{L}_\ell) = \dim(\Gamma)$.

Note that geometric multidegrees ignore $\mathbb{Z}^n$-graded shifts. Before going ahead and defining multidegrees of arbitrary $\mathbb{Z}^n$-graded quotients by taking initial ideals, we have to know that the multidegree is independent of the term order. This will be evident once we can read the multidegree of a direct sum of monomial quotients of $k[z]$ off of its Hilbert series (Proposition A.3.3). In order to prove the desired result, we need to make a short foray into combinatorial commutative algebra.

A $\mathbb{Z}^m$-graded $k[z]$-module $\Gamma$ is called $\mathbb{N}^m$-graded if the graded pieces of $\Gamma$ are nonzero only in degrees from $\mathbb{N}^m$. Such is the case, for instance, for all monomial ideals and quotients by them. An irreducible monomial ideal is an ideal generated by powers of the variables $z_i$. For convenience, we denote the ideal $\langle z_i^{b_i} \mid b_i \neq 0 \rangle$ by $m^b$ whenever $b \in \mathbb{N}^m$. An irreducible resolution of $\Gamma$ is an exact sequence

$$0 \rightarrow \Gamma \rightarrow W^0 \rightarrow W^1 \rightarrow \cdots$$

in which $W^k = \bigoplus_{\ell} k[z]/m^{b_{k\ell}}$ is a direct sum of quotients by irreducible monomial ideals.

**Lemma A.3.2** Every $\mathbb{N}^m$-graded $k[z]$-module $\Gamma$ has a finite irreducible resolution in which $\dim W^k \leq \dim(\Gamma)$ for all $k \geq 0$.

**Proof.** There are a number of ways to prove the existence of finite irreducible resolutions. One could apply the Alexander duality functor with respect to some degree $a \in \mathbb{Z}^m$ to a free resolution of the Alexander dual module $\Gamma^a$ [Mil00a]. Alternatively, one can take the $\mathbb{N}^m$-graded part of a $\mathbb{Z}^m$-graded injective resolution of $\Gamma$; this is the way the proof goes in [Mil01], where irreducible resolutions were first defined (for monomial ideals in semigroup rings). The dimension inequality actually follows from the injective resolution argument. \qed

Given a Laurent monomial $x^a$ for some $a \in \mathbb{Z}^n$, the rational function $\prod_{j=1}^m (1-x_j)^{a_j}$ can be expanded as a well-defined (i.e. convergent in the $x$-adic topology) formal power series $\prod_{j=1}^m (1-a_j x_j + \cdots)$ in $x$. Doing the same for an arbitrary Laurent polynomial $K(x)$ results in a power series $K(1-x)$ called a substituted Laurent polynomial.
\textbf{Proposition A.3.3} If \( J \) is a monomial ideal with zero scheme \( \mathcal{L} \), then the sum of all lowest degree terms in \( \mathcal{K}(k[z]/J; 1 - \mathbf{x}) \) equals the geometric multidegree \( |\mathcal{L}|_{\mathbb{Z}^n} \). This sum is a homogeneous form (in the usual sense) of degree \( m - \text{dim}(\mathcal{L}) \) in \( \mathbf{x} \).

\textit{Proof.} First we prove the result when \( J = m^b \). Since the lowest free resolution of this \( J \) is a Koszul complex, the \( \mathbb{Z}^m \)-graded Hilbert numerator of \( k[z]/J \) is \( \prod_{b_i \neq 0} (1 - z_i^{b_i}) \).

The Hilbert numerator for the \( \mathbb{Z}^n \)-grading is therefore equal to \( \prod_{b_i \neq 0} (1 - x_i^{b_i \alpha_i}) \). Substituting \( 1 - x_j \) for each occurrence of \( x_j \) (\( j = 1, \ldots, n \)) yields

\[
\prod_{b_i \neq 0} \left( 1 - \prod_{j=1}^n (1 - x_j)^{b_ia_{ij}} \right) = \prod_{b_i \neq 0} \left( 1 - \prod_{j=1}^n (1 - b_ia_{ij}x_j + \text{h.o.t.}) \right)
\]

\[
= \prod_{b_i \neq 0} \left( \sum_{j=1}^n (b_ia_{ij}x_j) + \text{h.o.t.} \right)
\]

\[
= \log_x(z^b) + \text{h.o.t.}
\]

\[
= \left( \prod_{b_i \neq 0} b_i \right) \cdot \left( \prod_{b_i \neq 0} \log_x(z_i) \right) + \text{h.o.t.,}
\] (31)

as desired (\text{h.o.t.} denotes higher order terms in the \( \mathbf{x} \) variables). We have used the fact that the multiplicity of \( k[z]/J \) at its unique associated prime is \( \prod_{b_i \neq 0} b_i \).

Now let \( J \) be an arbitrary monomial ideal, and consider an irreducible resolution \( W^* \) of \( k[z]/J \) satisfying the conditions of Lemma A.3.2. Substituting \( 1 - x \) for \( x \) in the Hilbert numerator of \( k[z]/J \) yields the finite alternating sum

\[
\mathcal{K}(k[z]/J; 1 - \mathbf{x}) = \sum_{k \geq 0} (-1)^k \mathcal{K}(W^k; 1 - \mathbf{x}).
\] (32)

The terms of lowest degree on the right side of this equation result only from summands \( k[z]/m^{b_k} \) for which the support of \( b_k \) is \textit{minimal}, by (31). In other words, only the summands of \( W^* \) whose associated primes \( m^{D_L} \) have \textit{maximal} dimension \( \text{dim}(\mathcal{L}) \) contribute to the sum of lowest degree terms, of degree \( |D_L| = m - \text{dim}(\mathcal{L}) \).

Let \( L \subseteq \mathcal{L} \) be a reduced coordinate subspace of maximal dimension, so \( m^{D_L} \) is an associated prime of maximal dimension. Collecting the coefficients on the terms expressed as \( \log_x(z^{D_L}) \) on the right hand side of (32) returns the alternating sum \( \sum_k (-1)^k \text{mult}_L(W^k) \), by (31) again. Fortunately, this alternating sum of multiplicities also equals \( \text{mult}_L(k[z]/J) \); indeed, this is a consequence of localizing \( W^* \) at \( m^{D_L} \), because length is additive on short exact sequences of finite-length modules. We conclude that the reduced coordinate subspace \( L \) contributes \( \text{mult}_L(k[z]/J) \) to the coefficient on \( \log_x(z^{D_L}) \) in (32), as required by Definition A.3.1.

The wording in the last paragraph, including the phrases ‘terms expressed as’ and ‘contributes \( \text{mult}_L(k[z]/J) \) to the coefficient on’ may seem a bit odd. We had no choice, really: two distinct subspaces \( L \) might have equal homogeneous logarithms \( \log_x(z^{D_L}) \). This becomes especially obvious for the usual \( \mathbb{Z} \)-grading, in which every
subspace $L$ of codimension $c$ has the same homogeneous logarithm. Nonetheless, irreducible resolutions separate out the contributions from distinct coordinate subspaces.

Proposition A.3.3 holds for all of the modules $\Gamma$ in Definition A.3.1. The main reason is an easy lemma, which also finds an application in Theorem A.3.8.

**Lemma A.3.4** For any Laurent polynomial $\mathcal{K}(\mathbf{x})$ and $\mathbf{a} \in \mathbb{Z}^n$, the substituted Laurent polynomials obtained from $\mathcal{K}(\mathbf{x})$ and $\mathbf{x}^\mathbf{a}\mathcal{K}(\mathbf{x})$ have equal sums of lowest degree terms.

*Proof.* The power series $(1 - \mathbf{x})^\mathbf{a}$ has lowest degree term $1$. 

**Corollary A.3.5** With notation as in Definition A.3.1, the geometric multidegree of $\Gamma = \bigoplus_{t=1}^r (k[z]/J_t)(\mathbf{a}_t)$ equals the sum of lowest degree terms in $\mathcal{K}(\Gamma; 1 - \mathbf{x})$. This sum is a homogeneous form (in the usual sense) of degree $m - \operatorname{dim}(\Gamma)$ in $\mathbf{x}$.

*Proof.* When $r = 1$, so there is only one summand, the Hilbert numerator satisfies $\mathcal{K}(\Gamma; \mathbf{x}) = \mathbf{x}^\mathbf{a}\mathcal{K}(k[z]/J_1; \mathbf{x})$. Using Lemma A.3.4, the corollary for $r = 1$ follows from Proposition A.3.3 and the definitions. The case $r \geq 1$ follows from the additivity of Hilbert numerators for direct sums. 

The previous result motivates the algebraic definition of multidegree.

**Definition A.3.6** The multidegree of a $\mathbb{Z}^n$-graded $k[z]$-module $\Gamma$ is the sum $\mathcal{C}(\Gamma; \mathbf{x})$ of the lowest degree terms in the substituted Hilbert numerator $\mathcal{K}(\Gamma; 1 - \mathbf{x})$. If $\Gamma = k[z]/I$ is the coordinate ring of a subscheme $X \subseteq k^m$, then write $[X]_{\mathbb{Z}^n} = \mathcal{C}(\Gamma; \mathbf{x})$.

Now we can finally show that the multidegree of $k[z]/\operatorname{in}(I)$ is independent of the term order used to calculate the initial ideal $\operatorname{in}(I)$, for any $\mathbb{Z}^n$-graded ideal $I$. More generally, we work with initial modules for submodules of free modules.

**Proposition A.3.7** Fix a $\mathbb{Z}^n$-graded module $\Gamma$, and let $\Gamma \cong F/K$ be an expression of $\Gamma$ as the quotient of a free module $F$ with kernel $K$. The multidegree $\mathcal{C}(\Gamma; \mathbf{x})$ equals the geometric multidegree of $F/\operatorname{in}(K)$ for any initial submodule $\operatorname{in}(K)$ of $K$. In particular, $\mathcal{C}(\Gamma; \mathbf{x})$ is a homogeneous polynomial of degree $m - \operatorname{dim}(\Gamma)$.

*Proof.* This follows from Corollary A.3.5, because Hilbert series—and therefore multidegrees—are preserved under taking initial submodules. Note that $\operatorname{in}(K)$ does indeed have the form of the direct sum in Corollary A.3.5. 

Multidegrees are truly geometric invariants, depending only on the support of a module (i.e., the union of all subvarieties along which the module has nonzero multiplicity) and the module’s multiplicities along the irreducible components.

**Theorem A.3.8** Fix a $\mathbb{Z}^n$-graded module $\Gamma$, and let $X_1, \ldots, X_r$ be the maximal-dimensional irreducible components of the support variety of $\Gamma$. Then

$$\mathcal{C}(\Gamma; \mathbf{x}) = \sum_{t=1}^r \mathrm{mult}_{X_t}(\Gamma) \cdot [X_t]_{\mathbb{Z}^n}.$$ 

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Proof. Choose a filtration $\Gamma = \Gamma_0 \supset \Gamma_{s-1} \supset \cdots \supset \Gamma_0 = 0$ in which the successive quotients $\Gamma_e/\Gamma_{e-1}$ are $\mathbb{Z}^n$-graded shifts of quotients of $k[z]$ by $\mathbb{Z}^n$-graded primes. The multiplicity along $X_\ell$ is the number of times a shift of the quotient $k[z]/I(X_\ell)$ appears, because $X_\ell$ is maximal-dimensional. The Hilbert numerator of $\Gamma$ is the sum of the Hilbert numerators of the successive quotients: $\mathcal{K}(\Gamma; x) = \sum_{e=1}^s \mathcal{K}(\Gamma_e/\Gamma_{e-1}; x)$. However, the terms $\mathcal{K}(\Gamma_e/\Gamma_{e-1}; 1 - x)$ for which $\dim(\Gamma_e/\Gamma_{e-1}) < \dim(\Gamma)$ contribute nothing to the multidegree of $\Gamma$, by Proposition A.3.7. Finally, all of the terms $\mathcal{K}(\Gamma_e/\Gamma_{e-1}; 1 - x)$ such that $\Gamma_e/\Gamma_{e-1}$ is a shift of $k[z]/I(X_\ell)$ contribute $[X_\ell]_{\mathbb{Z}^n}$ to $\mathcal{C}(\Gamma; x)$, by Lemma A.3.4.

We record the most easily applied special case of Theorem A.3.8. It is the version of Definition A.3.1 for arbitrary $\mathbb{Z}^n$-graded ideals.

**Corollary A.3.9** If the subscheme $X \subseteq k^m$ has maximal-dimensional irreducible components $X_1, \ldots, X_r$, then

$$[X]_{\mathbb{Z}^n} = \sum_{i=1}^r \text{mult}_{X_i}(X) \cdot [X_i]_{\mathbb{Z}^n}.$$ 

Our final result on multidegrees concerns coarsening the grading.

**Proposition A.3.10** Let $T \to T'$ be a map of tori acting positively on $k[z]$, with the action of $T$ factoring through that of $T'$, and denote the induced map of weight lattices by $\mathbb{Z}^n' \to \mathbb{Z}^n$. Any $\mathbb{Z}^n$-graded module $\Gamma$ is also $\mathbb{Z}^n'$-graded, and the $\mathbb{Z}^n$-graded multidegree $\mathcal{C}(\Gamma; x)$ maps to the $\mathbb{Z}^n'$-graded multidegree $\mathcal{C}(\Gamma'; x')$ under the natural homomorphism $\text{Sym}^c_\mathbb{Z}(\mathbb{Z}^n) \to \text{Sym}^c_\mathbb{Z}(\mathbb{Z}^n')$, where $c = m - \dim(\Gamma)$.

Proof. This holds when $\Gamma = k[z]/J$ is a monomial quotient, because of Proposition A.3.3 and the definition of $\log_\mathbb{Z}(z^{D_{\chi}})$ appearing in the geometric multidegree. It therefore holds for all $\mathbb{Z}^n$-graded $\Gamma$ by Corollary A.3.5 and Proposition A.3.7.

At long last, here are some examples of multidegrees; see Section 1.2 for more.

**Example A.3.11** The usual degree of a $\mathbb{Z}$-graded ideal is a special case, as is the bigraded degree of a doubly homogeneous ideal in two sets of variables. These degrees are usually thought of as associated to subschemes of projective space or a product of projective spaces.

If the support semigroup $A = \mathbb{N} \cdot \{a_1, \ldots, a_m\}$ admits a homomorphism to $\mathbb{N}$ taking $a_i$ to $1 \in \mathbb{N}$ for all $i$, then the induced map $\mathbb{Z}[x] \to \mathbb{Z}[t]$ takes the $\mathbb{Z}^n$-graded multidegree of $k[z]/I$ to $\sum_{\text{dim } I}$ times the ordinary $\mathbb{Z}$-graded degree of $I$.

**Example A.3.12** The $\mathbb{Z}^{2n}$-graded multidegree of a vexillary matrix Schubert variety is a multi-Schur polynomial. This follows from Theorem 1.4.3 along with the fact that double Schubert polynomials for vexillary permutations are multi-Schur polynomials [Ful92, Mac91].

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Example A.3.13 In the situation of Example A.1.1, the ordinary weights of the variables are given as elements of $\text{Sym}^1_Z$ (weight lattice) in the bottom row of the diagram. The second row contains the natural maps on multidegree rings of $M_n$.

$$\begin{align*}
\mathbf{k}^* &\leftrightarrow T &\leftrightarrow T \times T^{-1} &\rightarrow (\mathbf{k}^*)^{n^2} \\
\mathbb{Z}[t] &\leftrightarrow \mathbb{Z}[x] &\leftrightarrow \mathbb{Z}[x,y] &\leftrightarrow \mathbb{Z}[z] \\
t &\leftrightarrow x_i &\leftrightarrow x_i - y_j &\leftrightarrow z_{ij}
\end{align*}$$

The similarity of this diagram to that in Example A.1.9 is a little bit misleading: Laurent monomials more complicated than simple variables for exponential weights will give rise to more complicated linear combinations for ordinary weights. Note that $\mathbb{Z}[x,y] \leftarrow \mathbb{Z}[z]$ is not surjective, since $T \times T^{-1} \rightarrow (\mathbf{k}^*)^{n^2}$ has nontrivial kernel $(\mathbf{k}^*)^{\pm 1}$ consisting of the elements $(\alpha, \ldots, \alpha) \times (\alpha, \ldots, \alpha)$. Nonetheless, the image of $\mathbb{Z}[z]$ inside $\mathbb{Z}[x,y]$ still surjects onto $\mathbb{Z}[x]$ because $T \cap (\mathbf{k}^*)^{\pm 1}$ is trivial. \hfill \Box

Remark A.3.14 Just as Hilbert numerators can be defined for nonpositive gradings simply by using the equivariant $K$-class, multidegrees can also be defined for nonpositive gradings, simply by taking the terms of lowest degree in the substituted equivariant $K$-class. Slightly odd phenomena begin to occur, such as subspaces with multidegree zero, but only when such subspaces have trivial equivariant $K$-class. Geometrically, it means that the subspace can be moved off to infinity torus-equivariantly. In any case, we shouldn’t be too surprised: every proper subspace in $\mathbf{k}^m$ with the trivial torus action has multidegree zero (and defines the zero element in cohomology). \hfill \Box

Remark A.3.15 It can be important to think of the multidegree as associated not to a subscheme of $\mathbf{k}^m$, but to a multigraded quotient of $\mathbf{k}[z]$. The distinction arises in toric geometry, where the same multigraded ideal can represent subschemes of various rather different toric varieties, via their Cox homogeneous coordinate rings. In each smooth such toric variety, the multigraded degree of $\mathbf{k}[z]/I$ maps to the appropriately equivariant cohomology class of the subscheme determined by $I$. \hfill \Box

A.4 Equivariant cohomology

This appendix contains the few facts about equivariant cohomology that we use in Section 3.6 and Appendix A.5. Most of this is very standard (see [Gui94, GZ99] for combinatorial approaches, or [Bri98, EG98] for ones closer to algebraic geometry). The only nonstandard point comes when we want to associate equivariant cohomology classes to closed subvarieties of a noncompact space (since the references above require compactness). We repeat\(^{10}\) the argument from [Kaz97] that works in the special case we need. The field in this appendix is $\mathbf{k} = \mathbb{C}$.

Given a locally compact group $G$, there exists a (left) $G$-space $EG$ that is contractible, and on which $G$ acts freely; the quotient is called $BG$. To an action of $G$ on another topological space $M$, we associate the mixing space of Borel,

\[ M_G = M \times_G EG = G/(M \times EG). \]

\(^{10}\)for the convenience of the reader, as this issue of this journal seems to be largely unavailable
This has a natural map to $BG$ coming from projection onto the second factor, with fibers $M$. The $G$-equivariant cohomology $H^*_G(M)$ is defined as the cohomology of the mixing space, and the projection $M_G \to BG$ makes $H^*_G(M)$ into a module over $H^*(BG) = H^*_G(pt)$.

**Example A.4.1** The group $\mathbb{C}^*$ acts on Hilbert space by rescaling. This is contractible, but the action is not free; once we remove the zero vector it becomes free (and the loss of an infinite-codimensional set does not spoil the contractibility). The quotient $BC^*$ is thus $\mathbb{C}P^\infty$, and its cohomology ring is a polynomial ring $\mathbb{Z}[x]$ in one generator of degree 2, “the universal first Chern class”.

More generally, if $T$ is a complex torus $(\mathbb{C}^*)^n$, the space $BT$ is a product of $\mathbb{C}P^\infty$s, and the cohomology ring is a polynomial ring in $n$ generators, most naturally identified with the symmetric algebra of the weight lattice $W$ of $T$. \qed

If $V \to X$ is a vector bundle, and $G$ acts on both such that the bundle map is $G$-equivariant, the mixing construction makes $V_G \to X_G$ a bundle with the same fiber. In particular, one can define the **equivariant characteristic classes** of $V$, which live in $H^*_G(X)$, as the usual characteristic classes of $V_G$.

These two lemmata are straightforward from the definitions:

**Lemma A.4.2** If $G \times G'$ acts on $M$ with $G = G \times \{1\}$ acting freely, then the natural map $H^*_G\!(G\!\setminus\!M) \to H^*_G\!(G\times G\setminus M)$ is an isomorphism.

**Lemma A.4.3** If $G' \subseteq G$ and $G'\setminus G$ is contractible, then the natural map $H^*_G(M) \to H^*_{G'}(M)$ is an isomorphism.

Recall the notation from the beginning of Section A.1 regarding torus actions. Define the **equivariant Chern classes** of a vector bundle $E$ by $c^G_i(E) = c_i(E_G)$.

**Lemma A.4.4** Suppose $M$ is equivariantly contractible.

1. The natural map $H^*(BG) \cong H^*_G(M)$ is an isomorphism.
2. If $G = T \cong (\mathbb{C}^*)^n$, then $H^*_T\!(M) = \mathbb{Z}[W]$ consists of polynomials in the equivariant first Chern classes of $T$-equivariant line bundles on $M$.
3. If $T' \to T$ is a homomorphism of tori with weight lattices $W'$ and $W$, then $H^*_T\!(M) \leftarrow H^*_T\!(M)$ is induced by the natural map $W' \leftarrow W$.

**Proof.** The mixing space of an equivariant contraction of $M$ to a point is an ordinary contraction of $M_G$ to $BG$, proving part 1. Because of the contractions, line bundles on $M_T$ and equivariant line bundles on $M$ are (up to isomorphism) all pulled back from $BT$ and a point, respectively. Since pullback commutes with formation of mixing spaces, each line bundle on $M_T$ can be expressed as $L_T$ for some equivariant line bundle $L$ on $M$. The equivariant Chern class $c^T_1(L)$ is the ordinary Chern class $c_1(L_T)$ by definition, so part 2 is a consequence of the standard isomorphism $H^*(BT) = \text{Sym}^*_\mathbb{Z}(W)$. Part 3 is functoriality. \qed

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Equivariant cohomology classes of subvarieties.

**Proposition A.4.5** Let $M$ be a smooth complex variety and $G$ an algebraic group acting algebraically on $M$. Assume $G$ has finitely many orbits on $M$. Every $G$-orbit class $[\mathcal{O}]_G \in H^*_G(M)$.

The word “naturally” in the Proposition means that given a $G$-equivariant map $\phi : M' \to M$ of varieties satisfying the hypotheses, the class $[\mathcal{O}]_G \in H^*_G(M)$ of an orbit on $M$ pulls back under $\phi$ to the class $[\phi^{-1}(\mathcal{O})]_G \in H^*_G(M')$ of its preimage (if the preimage has the same codimension). The only examples of $M$ and $G$ that we actually see in this text are

$$M = GL_n \text{ or } M_n \text{ with } G = B \times B_+,$$

and

$$M = M_n \text{ with } G = (\mathbb{C}^*)^n.$$  

We present here the appropriate special case of the Vassiliev–Kazarian method for constructing the equivariant cohomology classes of the $G$-orbit closures on $M$ (see [Kaz97]). We owe a debt of gratitude to Richárd Rimányi for showing us the construction that comprises the proof below.

**Proof.** Let $M^p$ be the union of all $G$-orbits in $M$ of complex codimension exactly $p$. The mixing space $M^p_G$ of each locally closed subvariety $M^p$ is a topological subspace of the total space $M_G$. There is a resulting filtration $\mathcal{O} = X_{-1} \subset X_0 \subset X_1 \subset \cdots$ of $M_G$ in which the $p^{th}$ piece

$$X_p = \bigcup_{j \leq p} M^j_G$$

is the union of all orbit mixing spaces having codimension at most $p$.

The filtration $\{X_p\}$ of $M_G$ induces a filtration

$$C^*(M_G) \supset C^*(M_G, X_0) \supset \cdots \supset C^*(M_G, X_{p-1}) \supset C^*(M_G, X_p) \supset \cdots$$

of the singular cochain complex of $M_G$. Here, $C^*(M_G, X_p)$ is the relative cochain complex of the pair $(M_G, X_p)$ with coefficients in $\mathbb{Z}$. The cohomology first quadrant spectral sequence

$$\delta_r : E^{p,q}_r \to E^{p+r,q-r+1}_r$$

associated to this filtration of $C^*(M_G)$ has $E^{p,q}_1 = H^{p+q}(X_p, X_{p-1})$, because of the canonical isomorphism $C^*(M_G, X_{p-1})/C^*(M_G, X_p) = C^*(X_p, X_{p-1})$.

Even better, we can identify the relative cohomology in the $E_1$ term as the absolute cohomology group

$$E^{p,q}_1 = H^q(M^p_G)$$

of the codimension $p$ orbit mixing space. Indeed, each of the orbits of $G$ on $M$ is a complex submanifold, with a resulting canonically (co)oriented normal bundle, so
the isomorphism \( H^p(X_p, X_{p-1}) \cong H^q(M^p) \) is an immediate consequence of excision followed by the Thom isomorphism. In the case \( q = 0 \), the orientations on the codimension \( p \) orbits \( \mathcal{O} \subseteq M \) identify generators of the zeroth cohomology groups \( H^0(\mathcal{O}_G) = \mathbb{Z} \), and the \( E_1 \) terms on the bottom row are direct sums

\[
E_1^{p,0} = \bigoplus_{\text{codim}(\mathcal{O}) = p} H^0(\mathcal{O}_G) = \bigoplus_{\text{codim}(\mathcal{O}) = p} \mathbb{Z}.
\]

All of the odd-numbered columns in the Kazarian spectral sequence (33) are zero, because the \( G \)-orbits on \( M \) are complex manifolds and hence even dimensional. It follows that \( E_1^{p,0} = E_2^{p,0} \) for all \( p \). Therefore, the edge homomorphisms \( E_2^{p,0} \to H^p(M_G) \) automatically induce maps \( H^0(\mathcal{O}_G) \to H^p(M_G) \) for codimension \( p \) orbits \( \mathcal{O} \), thereby producing each cohomology class \( [\mathcal{O}_G] \in H^p(M_G) \) as the image of \( 1 \in \mathbb{Z} = H^0(\mathcal{O}_G) \). Of course, \( H^p(M_G) = H^p_G(M) \) by definition, so the class \( [\mathcal{O}]_G = [\mathcal{O}_G] \) is the desired equivariant cohomology class determined by the orbit \( \mathcal{O} \).

All of the constructions used in defining the \( G \)-equivariant cohomology class \( [\mathcal{O}]_G \in H^*_G(M) \) of an orbit \( \mathcal{O} \) are functorial in \( M \) and \( \mathcal{O} \), proving the second claim of the Proposition. \( \square \)

The above proof demonstrates how exceedingly useful it is to be dealing with matrix Schubert varieties and zero sets of monomial ideals, which are orbit closures for Borel group and torus actions with finitely many orbits. Nonetheless, our main applications, in Section 3.1 and Appendix A.5, require matrix Schubert varieties and coordinate subspace arrangements to define equivariant cohomology classes for proper subgroups of these groups. This is not a problem:

**Corollary A.4.6** Assume the hypotheses of Proposition A.4.5, and that \( G' \subseteq G \) is an algebraic subgroup.

1. Every \( G \)-orbit \( \mathcal{O} \) of codimension \( p \) in \( M \) naturally determines a \( G' \)-equivariant cohomology class \( [\mathcal{O}]_{G'} \in H^*_G(M) \).
2. Given a \( G \)-equivariant map \( \phi : M' \to M \), the class \( [\mathcal{O}]_{G'} \in H^*_G(M) \) of an orbit on \( M \) pulls back under \( \phi \) to the class \( [\phi^{-1}(\mathcal{O})]_{G'} \in H^*_G(M') \) of its preimage.

**Proof.** Once we've defined the class \( [\mathcal{O}]_G \in H^*_G(M) \), the rest comes for free, under the natural map \( H^*_G(M) \to H^*_G(M') \). \( \square \)

### A.5 Flag manifolds

Having supplanted the topology of the flag manifold with multigraded commutative algebra in the main body of the exposition, we would like now to connect back to the topological language. The material in this appendix actually formed the basis for our original proof of Theorem 2.1.2 over \( k = \mathbb{C} \), and therefore of Theorem 3.1.1, before we discovered the technology of multidegrees. The first half of this appendix also contains background material for Sections 3.2 and 3.5.
Let $B \subseteq GL_n$ denote the group of lower triangular matrices, and $B \setminus GL_n \mathbb{C}$ the manifold of flags in $V = \mathbb{C}^n$. Thus we think of $V$ as consisting of row vectors. The group of upper triangular matrices $B_+$ has a left action on the flag manifold $B \setminus GL_n \mathbb{C}$ in which a matrix $b \in B_+$ acts via right multiplication by $b^{-1}$. Identifying a permutation $w \in S_n$ with the permutation matrix $w^T$ having a 1 in the $i$th row and $w(i)$th column, we denote by $X_w$ the Schubert variety inside $B \setminus GL_n \mathbb{C}$ that is the closure of the orbit $B w B_+$. For instance, the smallest Schubert variety is the point $X_{w_0}$, where $w_0$ is the long permutation $n \cdots 21$.

We recall the identification of the class $[\mathcal{O}_{X_{w_0}}] \in K^*(B \setminus GL_n \mathbb{C})$ in the $K$-theory of algebraic vector bundles on $B \setminus GL_n \mathbb{C}$ (Appendix A.1) as a mode for introducing some more standard objects. The ring $K^*(B \setminus GL_n \mathbb{C})$ is generated over $\mathbb{Z}$ by the classes $x_1, \ldots, x_n$ of the tautological line bundles $L_1, \ldots, L_n$, which are defined as follows. To begin with, $B \setminus GL_n \mathbb{C}$ has tautological subbundles $S_0, \ldots, S_n$ of the trivial bundle $\widetilde{V} = V \otimes_{\mathbb{C}} \mathcal{O}_{B \setminus GL_n \mathbb{C}}$, the fiber of $S_i$ at a flag $F_i : 0 = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_n = V$ being the $i$-plane $F_i$. We set $L_i = S_i/S_{i-1}$.

It follows from the inclusion $S_i \to \widetilde{V}$ that the dual bundle $S_i^* = Q_i$ is a quotient of the trivial bundle $\widetilde{V}^* = V^* \otimes_{\mathbb{C}} \mathcal{O}_{B \setminus GL_n \mathbb{C}}$, where $V^* = \text{Hom}_C(V, \mathbb{C})$. In particular, if $e_1, \ldots, e_n$ are the standard basis row vectors in $V$, then every dual basis vector $e_j^*$ gives rise to a section of $Q_i$, generating a trivial line subbundle $\langle e_j^* \rangle_i$. We therefore have maps $S_i \otimes \langle e_j^* \rangle_i \to \mathcal{O}_{B \setminus GL_n \mathbb{C}}$ for all $i$ and $j$. The image $e_j^*(S_i)$ is the ideal sheaf of the locus where $F_i \subseteq \ker e_j^*$ as subspaces of $V$. Therefore, the image of the map

$$S := \bigoplus_{i=1}^n S_i \otimes \langle e_{n-i}^* \rangle_i \to \mathcal{O}_{B \setminus GL_n \mathbb{C}}$$

is the ideal sheaf of the locus of flags $F$ where $F_i$ is spanned by $\{e_n, \ldots, e_{n+1-i}\}$ for all $i$. For example, any vector $v$ spanning $F_1$ must have $(n-1)^{st}$ coordinate zero because $v \in F_1 \subseteq \ker e_{n-1}^*$, and $(n-2)^{nd}$ coordinate zero because $v \in F_2 \subseteq \ker e_{n-2}^*$, and so on. The image of $\phi$ is hence the ideal of the point $X_{w_0}$.

Now $\text{rank}(S) = \dim(B \setminus GL_n \mathbb{C})$, so the Koszul complex on $(S, \phi)$ is a locally free resolution of the skyscraper sheaf at $X_{w_0}$. Thus $[\mathcal{O}_{X_{w_0}}] = \sum_{d \geq 0} (-1)^d [\Lambda^d S]$. On the other hand, the filtration of each $S_i$ by the subbundles $S_h$ for $h \leq i$ has associated graded vector bundle $\text{gr}(S_i) = \bigoplus_{h \leq i} L_h$. Therefore, we have $[S_i] = \sum_{h \leq i} x_i$ in $K^*(B \setminus GL_n \mathbb{C})$, where again $x_i = [L_i]$. Finally, we conclude that $[S] = \sum_{i+j \leq n} x_i = \sum_{i+j \leq n} x_i$, so

$$[\mathcal{O}_{X_{w_0}}] = \sum_{d \geq 0} (-1)^d [\Lambda^d (\bigoplus_{i+j \leq n} L_i)] = \prod_{i=1}^n (1 - x_i)^{n-i}.$$

The polynomial $G_{w_0}(x) = \prod_{i=1}^n (1 - x_i)^{n-i}$ is called the Grothendieck polynomial for $w_0$. It is the “top” member of a family of polynomials constructed from it in Definition 1.5.5.

**Remark A.5.1** The substitution $x \mapsto 1 - x$ is a change of basis accompanying the Poincaré isomorphism. In general geometric terms, $c_i(L_i)$ is the cohomology
class Poincaré dual to the divisor $D_i$ of the line bundle $L_i$, and the exact sequence

$0 \to L_i^\vee \to \mathcal{O} \to \mathcal{O}_{D_i} \to 0$ implies that the $K$-homology class $[\mathcal{O}_{D_i}]$ equals the $K$-cohomology class $1 - [L_i^\vee]$. Thus $G_w(\mathfrak{x})$ writes $[\mathcal{O}_{X_w}]$ as a polynomial in the Chern characters $x_i = e^{c_1(L_i)}$ of the line bundles $L_i$, whereas $G_w(1 - \mathfrak{x})$ writes $[\mathcal{O}_{X_w}]$ as polynomial (over $\mathbb{Q}$, perhaps) in the expressions

$$1 - e^{c_1(L_i)} = 1 - e^{-c_1(L_i^\vee)} = c_1(L_i^\vee) - \frac{c_1(L_i^\vee)^2}{2!} + \frac{c_1(L_i^\vee)^3}{3!} - \frac{c_1(L_i^\vee)^4}{4!} + \cdots,$$

whose lowest degree terms are the first Chern classes $c_1(L_i^\vee)$ of the dual bundles $L_i^\vee$. Forgetting the higher degree terms here, in Definition 1.2.2, and in Lemma 1.5.6 amounts to taking the image in the associated graded ring of $K_*(B\backslash GL_n \mathbb{C})$, which is $H^*(B\backslash GL_n \mathbb{C})$. See [Fu98, Chapter 15] for details.

It is an often annoying quirk of history that we end up using the same variable $x_i$ for both $c_1(L_i^\vee) \in H^*$ and $[L_i] \in K^*$. We tolerate (and sometimes even come to appreciate) this confusing abuse of notation because it can be helpful at times. In terms of algebra, it reinterpret the displayed equation as: the lowest degree term in $1 - e^{-x_i}$ is just $x_i$ again.}

In the rest of this appendix, we use known statements about the relation between Schubert polynomials and the cohomology of the flag manifold to derive a statement about the torus-equivariant cohomology classes of matrix Schubert varieties in $M_n$—that is, a statement about multidegrees.

According to the standard literature [Bor53], the cohomology $H^*(B\backslash GL_n \mathbb{C})$ with coefficients in $\mathbb{Z}$ is the quotient of the polynomial ring $\mathbb{Z}[\mathfrak{x}]$ in the Chern classes $x_i = c_1(L_i^\vee)$ by the ideal $K_n = \langle e_d(\mathfrak{x}) \mid 1 \leq d \leq n \rangle$ generated by the nonconstant elementary symmetric functions. (These relations hold in $H^*(B\backslash GL_n \mathbb{C})$ because the trivial rank $n$ bundle $\tilde{V}$ has Chern roots $x_1, \ldots, x_n$.) We require an equivariant cohomological justification for this presentation. As in Appendix A.1, denote by $\mathbb{C}_{-e_i}$ the 1-dimensional representation of $T = (\mathbb{C}^*)^n$ whose weight is the negative of the $i^{th}$ standard basis vector of the weight lattice.

**Lemma A.5.2** Let $x_i = c_1^T(\mathbb{C}_{-e_i} \otimes \mathcal{O}_{M_n})$ be the torus-equivariant Chern class. There is a natural map $\mathbb{Z}[\mathfrak{x}] = H^*_T(M_n) \to H^*(B\backslash GL_n \mathbb{C})$ sending $x_i \mapsto c_1(L_i^\vee)$.

**Proof.** We begin by justifying the following diagram.

\[
\begin{array}{c c c}
X_w & \cap & \overline{X}_w \\
\cap & B\backslash GL_n \mathbb{C} & \leftrightarrow GL_n \leftrightarrow M_n \\
\cap & H^*(B\backslash GL_n \mathbb{C}) & \cong H^*_T(GL_n) \leftrightarrow H^*_T(M_n)
\end{array}
\]  

(36)

The isomorphism $H^*(B\backslash GL_n \mathbb{C}) \cong H^*_T(GL_n)$ is by Lemma A.4.2 with $G = B$ and $G'$ trivial, while $H^*_B(GL_n) = H^*_T(GL_n)$ by Lemma A.4.3. Explicitly, the isomorphism $H^*(B\backslash GL_n \mathbb{C}) \cong H^*_T(GL_n)$ takes the first Chern class of $L_i^\vee$ to the equivariant Chern class $c_1^T(L_i^\vee)$ of its pullback to $GL_n$.  

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The bundle $\tilde{L}_i$ is isomorphic $T$-equivariantly (but not $B$-equivariantly) to the line bundle on $GL_n$ whose fiber over an invertible matrix is the subspace of $V = \mathbb{C}^n$ spanned by its $i^{th}$ row. Moreover, the inclusion $\tilde{L}_i \hookrightarrow V \times GL_n$ into the trivial bundle (with fiber $V$ having trivial $T$-action) is $T$-equivariant. The dual bundle $\tilde{L}_i^\vee$ is therefore generated by its sections $z_1, \ldots, z_m$ of weight zero. In other words, the canonical map $V \otimes \mathbb{C}[z] \to \mathbb{C}[z](e_i)$, whose image is the ideal $\langle z_1, \ldots, z_m \rangle(e_i)$, becomes a surjection $V \otimes \mathcal{O}_{GL_n} \to \tilde{L}_i^\vee$ when restricted to $GL_n$ (concretely, this is because the generic $n \times n$ determinant is in the ideal generated by any single row). In particular, $\tilde{L}_i^\vee$ is isomorphic to the restriction of $\mathbb{C}_{-e_i} \otimes \mathcal{O}_{M_n}$ to $GL_n$.

Applying the mixing construction for $T$ to these line bundles, we find that $x_i$ pulls back to $c_i^T(\tilde{L}_i^\vee)$ under the inclusion $GL_n \hookrightarrow M_n$. The second part of Lemma A.4.4 implies that $\mathbb{Z}\langle x \rangle = H^*_T(M_n)$, and completes the proof. 

Ehresmann [Ehr34] showed that the Schubert classes $[X_w] \in H^*(B\backslash GL_n \mathbb{C})$ are a $\mathbb{Z}$-basis. In the equivariant setting on $M_n$, where the classes are uniquely defined as polynomials, the Schubert classes arise from the matrix Schubert varieties.

**Lemma A.5.3** The polynomial $[\overline{X}_w]_T \in H^*_T(M_n)$ represents the Schubert class $[X_w] \in H^*(B\backslash GL_n \mathbb{C})$ under the map $\mathbb{Z}\langle x \rangle \to H^*(B\backslash GL_n \mathbb{C})$ of Lemma A.5.2.

**Proof.** Given that $T$-stable subvarieties of $M_n$ and $GL_n$ determine $T$-equivariant cohomology classes (Proposition A.4.5), it is clear from (36) that $[\overline{X}_w]_T \in H^*_T(M_n)$ maps to $[\overline{X}_w \cap GL_n]_T \in H^*_T(GL_n)$, which equals $[X_w] \in H^*(B\backslash GL_n \mathbb{C})$. 

The insight of Lascoux and Schützenberger [LS82a], based on work of Bernstein-Gelfand-Gelf'fand [BGG73] and Demazure [Dem74], was to represent the classes $[X_w] \in \mathbb{Z}\langle x \rangle / K_n$ of Schubert varieties independently of $n$, using Schubert polynomials. To make a precise statement, let $B\backslash GL_N \mathbb{C}$ be the manifold of flags in $\mathbb{C}^N$ for $N \geq n$, so $B$ is understood to consist of $N \times N$ lower triangular matrices. Let $X_{w,N} \subseteq B\backslash GL_N \mathbb{C}$ and $S_{w,N}$ be the Schubert variety and polynomial for the permutation $w \in S_n$ considered as an element of $S_N$ that fixes $n+1, \ldots, N$.

**Proposition A.5.4** ([LS82a]) $S_w(x)$ is the unique polynomial representing the class $[X_{w,N}] \in H^*(B\backslash GL_N \mathbb{C})$ for all $N \geq n$.

**Proof.** The Appendix to [Mac91] is a self-contained proof whose only prerequisite other than the elementary algebra of divided differences (especially [Mac91, (4.5)]) is a formula of Monk [Mon59]. Note that $X_{w_0w}$ in [Mac91] corresponds to $X_w$ here. 

Here is a geometric explanation for the naturality of Schubert polynomials.

**Theorem A.5.5** The class $[\overline{X}_w]_T \in H^*_T(M_n)$ is the Schubert polynomial $S_w(x)$.

**Proof.** We can deduce the result from Proposition A.5.4 and Lemma A.5.3 as soon as we show why $[\overline{X}_w]_T$ is independent of $n$. But $X_{w,N} \subseteq M_N$ is the preimage of $\overline{X}_w$ under the projection $M_N \to M_n$ forgetting the last $N-n$ rows and columns. Hence $[X_{w,N}]_T$ is the image of $[\overline{X}_w]_T$ under the pullback map $H^*_T(M_n) \to H^*_T(M_N)$. This pullback map is the inclusion $\mathbb{Z}[x_1, \ldots, x_n] \hookrightarrow \mathbb{Z}[x_1, \ldots, x_N]$. 

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Exactly the same proof gives a stronger statement: the class \( [\overline{X}_w]_T \in H^*_T(\mathcal{M}_n) \) is the double Schubert polynomial.

**Remark A.5.6** It is the projection \( M_N \to \mathcal{M}_n \) that is crucial here but unavailable when dealing with \( B/GL_n \mathbb{C} \) or even \( GL_n \): the result of removing the last row and column of an invertible matrix need not be invertible. □

**Corollary A.5.7** The \( \mathbb{Z} \)-graded degree of \( \overline{X}_w \) is \( \deg(I(\overline{X}_w)) = \mathcal{S}_w(1, \ldots, 1) \).

*Proof.* By Example A.3.13, substituting \( x_i = t \) in \( \mathcal{S}_w \) for all \( i \) yields the \( \mathbb{C}^* \)-equivariant cohomology class \( \mathcal{S}_w(t, \ldots, t) = \mathcal{S}_w(1, \ldots, 1)^{\text{length}(w)} \) for \( \overline{X}_w \). By an argument analogous to that of Lemma A.5.3, we have \( H^*_\mathbb{C}(\mathcal{M}_n) \to H^*_\mathbb{C}(\mathcal{M}_n \setminus \{ \emptyset \}) = H^*(\mathbb{P}^{n^2-1}) = \mathbb{Z}[t]/t^{n^2} \). In general, the degree of an \( \ell \)-dimensional variety in projective space is the coefficient on \( t^{\ell} \) in its cohomology class. Here, we take that variety to be the projective variety associated to the \( \mathbb{Z} \)-graded ideal \( I(\overline{X}_w) \). □

Corollary A.5.7 can enter into a proof of Theorem 2.1.2 as follows. Calculate the multidegree of \( \mathcal{L}_w = \text{zero set of } J_w \) as in Section 2.1, using the combinatorial machinery in Section 2. This yields \( [\mathcal{L}_w]_{\mathbb{Z}^n} = \mathcal{S}_w(\mathbf{x}) \). Coarsening to the \( \mathbb{Z} \)-grading by Proposition A.3.10 yields \( \deg(J_w) = \mathcal{S}_w(1, \ldots, 1) \). Therefore, we can apply the \( \mathbb{Z} \)-graded version of Lemma 2.1.5 and complete the proof of Theorem 2.1.2 as in Section 2.1.

Observe that this argument does not directly show the equality \( [\overline{X}_w]_{\mathbb{Z}^n} = [\overline{X}_w]_T = \mathcal{S}_w(\mathbf{x}) \) in Theorem 3.1.1 between the multidegree and the \( T \)-equivariant cohomology class. However, once Theorem 2.1.2 is known, we can still conclude that the \( \mathbb{Z}^{2n^2} \)-graded multidegrees \( [\overline{X}_w]_{\mathbb{Z}^{2n^2}} \) and \( [\mathcal{L}_w]_{\mathbb{Z}^{2n^2}} \) are equal, by Proposition A.3.7.

**References**


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