Subword complexes in Coxeter groups and applications to Schubert varieties

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based on joint work with Allen Knutson

Contents

Throughout the following notes, check the overhead transparency for further examples.

1 Subword complexes

Let (W, S) be a *Coxeter system* consisting of a Coxeter group W and a set S of generators corresponding to simple roots. See [Hum90] for background and definitions; the applications to rc-graphs concern only the case where $W = \Sigma_n$ and S consists of simple reflections.

Definition 1.1 A *word* of size m is an ordered list $Q = (s_1, \ldots, s_m)$ of elements of S. An ordered sublist $P \subseteq Q$ is called a *subword* of Q.

1. Q *represents* $w \in W$ if the ordered product of the simple reflections in Q is a reduced decomposition for w.
2. Q *contains* $w \in W$ if some sublist of Q represents w.

The *subword complex* $\Delta(Q, w)$ is the set of subwords $P \subseteq Q$ whose complements $Q \smallsetminus P$ contain w.

Sometimes we abuse notation and say that Q is a word in W. Note that Q need not itself be a reduced expression. The following lemma is immediate from the definitions and the fact that all reduced expressions for $w \in W$ have the same length. A simplicial complex is *pure* if all of its facets have the same cardinality = dimension + 1.

Lemma 1.2 *$\Delta(Q, w)$ is a pure simplicial complex whose facets are the subwords $Q \smallsetminus P$ such that $P \subseteq Q$ represents w.*

2 Pipe dreams and rc-graphs

Let Q_n be the following word in $W = \Sigma_{2n}$, where $S = \{s_1, \ldots, s_{2n-1}\}$ and s_i switches i and $i+1$:

$$Q_n = s_n s_{n-1} \cdots s_2 s_1 \;\; s_{n+1} s_n \cdots s_3 s_2 \;\; s_{n+2} s_{n+1} \cdots s_4 s_3 \;\; \cdots \;\; s_{2n-1} s_{2n-2} \cdots s_{n+1} s_n.$$

Thus, for instance, when $n = 4$, we get

$$Q_4 = s_4 s_3 s_2 s_1 \;\; s_5 s_4 s_3 s_2 \;\; s_6 s_5 s_4 s_3 \;\; s_7 s_6 s_5 s_4$$

It's easier to visualize this word Q_4 by reading the grid

s_1	s_2	s_3	s_4
s_2	s_3	s_4	s_5
s_3	s_4	s_5	s_6
s_4	s_5	s_6	s_7

right to left, starting at the upper-right corner and snaking your way toward the lower-left corner. In general, of course, the grid is $n \times n$ rather than 4×4.

Definition 2.1 (Bergeron–Billey, Fomin–Kirillov) An *rc-graph for $w \in \Sigma_n$* is a subword $P \subseteq Q_n$ representing w. The set of rc-graphs for w is denoted by $\mathcal{RC}(w)$.

Thus an rc-graph $P \in \mathcal{RC}(w)$ is the complement $Q_n \smallsetminus P$ of a facet of $\Delta(Q_n, w)$. The name 'rc-graph' was coined by Bergeron and Billey [BB93], and refers to a more fun pictorial way of visualizing subwords of Q_n (this is the way both groups of authors above thought about rc-graphs).

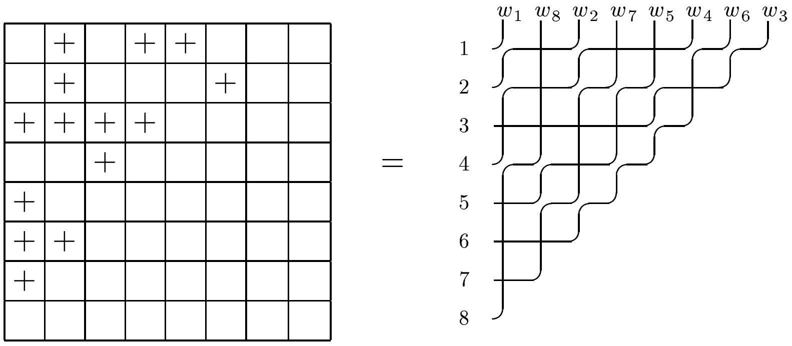
Consider a square $n \times n$ grid extending south and east, with the box in row i and column j labeled (i, j). If each box in the grid is covered with a square tile containing either ┼ or ┘┌, then one can think of the tiled grid as a network of pipes.

Definition 2.2 A *pipe dream*[1] is a subset of $[n] \times [n]$, identified as the set of crosses in a tiling by *crosses* and *elbow joints*.

[1]In the game Pipe Dream, the player is supposed to guide water flowing out of a spigot at one edge of the game board to its destination at another edge by laying down given square tiles with pipes going through them.

Reading off the antidiagonals on which the crosses sit in a pipe dream D, as we did with Q_4, a pipe dream is nothing more than a subword $P(D) \subseteq Q_n$. Pipe dreams are always drawn with crossing tiles denoted by '+' . However, the elbow tiles are often left blank, or denoted by dots, to make things easier to look at.

Example 2.3 A rather arbitrary example is $D =$



in which the first diagram represents D as a subset of $[8]^2$, whereas the second demonstrates how the tiles fit together. Since no cross in D occurs on or below the 8^{th} antidiagonal, the pipe entering row i exits column $w_i = w(i)$ for some permutation $w \in S_8$. In this case, $w = 13865742$. The elbows on and below the main diagonal are omitted from the right pipe dream, for clarity. $\square$

As in the previous example, all of the crosses in interesting pipe dreams sit in the region strictly above the main antidiagonal; we'll see why in a moment. When this is the case, there's no reason to draw the square tile boundaries and the "sea" of elbows below the main antidiagonal.

The permutation in the previous example suggests a more general statement.

Lemma 2.4 *If D is a pipe dream whose crosses lie above the main antidiagonal, then multiplying the reflections in $Q(D)$ yields the permutation w such that the pipe entering row i exits column $w(i)$. Furthermore, $|D| \geq \text{length}(w)$, with equality if and only if $D \in \mathcal{RC}(w)$.*

Proof. For the first statement, use induction on the number of crosses: adding a '+' in the i^{th} antidiagonal at the end of the list switches the destinations of the pipes beginning in rows i and $i+1$. This step implicitly uses the "above antidiagonal" condition, since we want i and $i+1$ to be $\leq n$. Each inversion in w contributes at least one crossing in D, whence $|D| \geq \text{length}(w)$. The expression $P(D)$ is reduced when D is an rc-graph because each inversion in w contributes ≤ 1 crossings in D. $\square$

Thus $P(D)$ gives a *reduced* expression for w if and only if $D \in \mathcal{RC}(w)$. Reiterating:

Proposition 2.5 *An rc-graph for $w \in \Sigma_n$ is a pipe dream such that*

1. *its crosses all lie strictly above the main antidiagonal;*

2. *the pipe entering row i exits from column $w(i)$; and*
3. *each pair of pipes crosses at most once.*

The ordered list of row indices for the crosses in D (taken in the same order as before) is what Bergeron and Billey call a *compatible sequence* for the expression $P(D)$. Given a reduced expression for w and a compatible sequence, it is possible (easy) to reconstruct the pipe dream from which they came. Although compatible sequences won't come up explicitly below, they do (finally) justify the prefix 'rc': it stands for the words 'reduced' and 'compatible' in the previous sentence. The interesting new aspect of rc-graphs here is their natural bijection with the facets of a simplicial complex.

3 Vertex-decomposability

Definition 3.1 Let Δ be a simplicial complex and $F \in \Delta$ a face.

1. The *deletion* of F from Δ is $\text{del}(F, \Delta) = \{G \in \Delta \mid G \cap F = \varnothing\}$.
2. The *link* of F in Δ is $\text{link}(F, \Delta) = \{G \in \Delta \mid G \cap F = \varnothing \text{ and } G \cup F \in \Delta\}$.

Δ is *vertex-decomposable* if Δ is pure and either (1) $\Delta = \{\varnothing\}$, or (2) for some vertex $v \in \Delta$, both $\text{del}(v, \Delta)$ and $\text{link}(v, \Delta)$ are vertex-decomposable. A *shelling* of Δ is an ordered list $F_1, F_2, \ldots, F_t$ of its facets such that $\bigcup_{j<i} F_j \cap F_i$ is a union of codimension 1 faces of F_i for each $i \leq t$. We say Δ is *shellable* if it is pure and has a shelling.

Remark 3.2 The empty set $\varnothing$ is a perfectly good face of Δ, representing the empty set of vertices. Thus the *empty complex* $\Delta = \{\varnothing\}$, whose unique face has dimension -1, is to be distinguished from the *void complex* $\Delta = \{\}$ consisting of no subsets of the vertex set. Writing out their reduced chain complexes, the empty complex has integral reduced homology $\mathbb{Z}$ in dimension -1, while the void complex has entirely zero homology. This distinction reveals itself most prominently when taking the link of a facet (maximal face) of Δ: the result is the empty complex, not the void complex. The link of $\varnothing$ in Δ is just Δ itself, including $\varnothing \in \Delta$. □

Provan and Billera [BP79] introduced the notion of vertex-decomposability and proved that it implies shellability (proof: use induction on the number of vertices by first shelling $\text{del}(v, \Delta)$ and then shelling the cone from v over $\text{link}(v, \Delta)$ to get a shelling of Δ). It is well-known that shellability implies the desirable topological condition of *Cohen-Macaulayness* [BH93, Theorem 5.1.13]. Here is the central observation concerning subword complexes for words and elements in arbitrary Coxeter groups.

Theorem 3.3 *$\Delta(Q, w)$ is vertex-decomposable. Therefore subword complexes are Cohen-Macaulay and even shellable.*

Proof. Induct on the size of Q, with $|Q| \leq 1$ being trivial. Supposing that $Q = (s, s_2, s_3, \ldots, s_m)$, there are two cases. If sw is longer than w, then s can't be first in a subword representing w, and $\Delta(Q, w)$ is therefore a cone over $\Delta(Q', w)$, where $Q' = (s_2, ..., s_m)$. On the other hand, if sw is shorter than w, then the link of s in $\Delta(Q, w)$ is $\Delta(Q', w)$ while the deletion of s from $\Delta(Q, w)$ is $\Delta(Q', sw)$. □

It seems quite difficult to prove directly that $\Delta(Q_n, w)$ is shellable, without resorting to a larger class of subword complexes (those appearing in the proof above have facets that usually don't correspond to rc-graphs). This remains true even in the face of conjectured shelling orders (see Section 9, below).

Example 3.4 Besides subword complexes, the only other natural families of vertex decomposable simplicial complexes that I know of are the *shifted complexes* [Duv96, HT99] and the *dual greedoid complexes* [BKL85], the latter containing the class of *matroid complexes.* Although subword complexes strongly resemble dual greedoid complexes, the exchange axiom defining greedoids seem to be slightly stronger than the exchange axioms for facets of subword complexes imposed by Coxeter relations.

To be precise, a collection M of *feasible subsets* of a finite vertex set Q is a *greedoid* when (1) $\varnothing \in M$, and (2) if X and Y are in M with $|X| > |Y|$, then there is some element $x \in X \smallsetminus Y$ such that $Y \cup x$ is in M. The facets of the dual greedoid complex are then the complements in Q of the maximal elements (*bases*) in M.

There is a natural attempt at defining a greedoid whose dual complex is $\Delta(Q, w)$: namely, let $M(Q, w)$ be the collection of subwords of Q that are themselves *reduced* subwords of some $P \subseteq Q$ representing w. Thus an element $Y \in M(Q, w)$ is a sublist of Q such that (1) the ordered product of elements in Y has length $|Y|$, and (2) there is some sublist $Z \subseteq Q$ such that $Y \cup Z$ is a reduced expression for w.

This $M(Q, w)$ is not necessarily a greedoid, even for the rc-graph case. An easy non-greedoid example is when $Q = s_4s_3s_2s_1s_4s_3s_2s_4s_3s_4$ is the unique rc-graph for the long word $w_0 \in S_5$ and $w = 12543 = s_3s_4s_3 = s_4s_3s_4$:

$$\begin{array}{rcl} Q & = & s_4s_3s_2s_1s_4s_3s_2s_4s_3s_4 \\ X & = & s_3s_4s_3 \\ Y & = & s_4s_3 \\ Z & = & s_4 \end{array}$$

Moving any of the elements from X down to Y creates a non-reduced expression.

You are invited to construct greedoids so that subword complexes are dual greedoid complexes. I personally don't think any systematic construction exists. Note that any successful attempt to prove me wrong will exclude the subword Y above from the feasible set. $\square$

4 Antidiagonals

Let $M_n\mathbb{C}$ be the $n \times n$ matrices of complex numbers, with coordinate ring $\mathbb{C}[\mathbf{z}]$ in indeterminates $\{z_{ij}\}_{i,j=1}^n$. In what follows, the integers r and s will always satisfy $1 \le r, s \le n$, while Z will stand for an $n \times n$ matrix. Most often, Z will be the *generic matrix* of variables (z_{ij}), although occasionally Z will be an element of $M_n\mathbb{C}$. Denote by $Z_{[r,s]}$ the northwest $r \times s$ submatrix of Z.

For example, identify a permutation $w \in \Sigma_n$ with the permutation matrix w^T having a 1 in the i^{th} row and $w(i)^{\text{th}}$ column. Then

$$\operatorname{rank}(w^T_{[r,s]}) = \#\{(i,j) \le (r,s) \mid w(i) = j\}$$

is the number of 1's in the submatrix $w^T_{[r,s]}$.

An *antidiagonal of length* p in the generic matrix (z_{ij}) is the product of p distinct variables whose indices $(i_1, j_1), \ldots, (i_p, j_p)$ are incomparable, say

$$i_1 < i_2 < \cdots < i_p \quad \text{and} \quad j_1 > j_2 > \cdots > j_p.$$

Definition 4.1 Given $w \in \Sigma_n$, let p_{rs} be the number of 1's in the northwest $r \times s$ submatrix $w^T_{[r,s]}$ of the permutation matrix w^T. Let $J_w \subseteq \mathbb{C}[\mathbf{z}]$ be the ideal generated by the union over all r, s of the length $(p_{rs} + 1)$ antidiagonals in $Z_{[r,s]}$.

Since the ideal J_w is generated by squarefree monomials, it is the *Stanley–Reisner ideal* for some simplicial complex Δ_w having n^2 vertices. Now there are two simplicial complexes Δ_w and $\Delta(Q_n, w)$ floating around that depend on the same information: a permutation $w \in \Sigma_n$. It is natrual to guess (the presentation above having been ordered the way it has!) that these two complexes are equal. There is hope: the vertex set of $\Delta(Q_n, w)$ is the same as the set $\{z_{rs} \mid 1 \leq r, s \leq n\}$ of variables, so the facets of Δ_w correspond to (complements of) pipe dreams in $[n]^2$.

On the other hand, the subword complex $\Delta(Q_n, w)$ is presented in terms of its facets, which correspond to the irreducible components of its Stanley–Reisner ideal, while the antidiagonal complex Δ_w is presented in terms of the generators of its Stanley–Reisner ideal J_w. By definition of Δ_w, these generators are $\langle \mathbf{z}^D \mid D \notin \Delta_w \rangle$, where $\mathbf{z}^D = \prod_{r,s \in D} z_{rs}$ is the squarefree monomial corresponding to the crosses in the pipe dream D.

The next theorem can be thought of either as producing explicitly the generators of the Stanley–Reisner ideal of $\Delta(Q_n, w)$, or as producing the irreducible components of J_w (i.e. the facets of Δ_w).

Theorem 4.2 $\Delta_w = \Delta(Q_n, w)$. *In other words, J_w is the Stanley–Reisner ideal of $\Delta(Q_n, w)$.*

No simple proof of this statement is known. (Find one, and tell me about it!) The known proof is more or less a corollary of methods used to prove Theorem 5.3, given the facts about rc-graphs determined by Bergeron–Billey and Fomin–Kirillov. The methods rely on algebraic geometry, algebraic topology, and commutative algebra.

5 Schubert varieties and Gröbner bases

It is well-known that Schubert varieties in flag manifolds are Cohen-Macaulay. The first proof was given by Ramanathan [Ram85], in the contex of general algebraic groups and quotients by parabolic subgroups. One of the motivations for finding Theorem 3.3 was to give a combinatorial justification for the Cohen-Macaulayness of Schubert varieties in the ordinary (type A) flag variety.

The transition from Schubert varieties to subword complexes begins with:

Definition 5.1 Let $w \in \Sigma_n$.

1. The *matrix Schubert variety* $\overline{X}_w \subseteq M_n\mathbb{C}$ consists of the matrices $Z \in M_n\mathbb{C}$ such that $\operatorname{rank}(Z_{[r,s]}) \leq \operatorname{rank}(w^T_{[r,s]})$ for all r, s.
2. The *determinantal ideal* $I_w \subset \mathbb{C}[\mathbf{z}]$ is generated by all minors in $Z_{[r,s]}$ of size $1 + \operatorname{rank}(w^T_{[r,s]})$ for all r, s, where $Z = (z_{ij})$ is the generic matrix of variables.

The matrix Schubert variety $\overline{X}_w$ is the zero set of I_w in $M_n\mathbb{C}$, because the rank of a matrix is $\leq p$ if and only if the minors of size $p+1$ all vanish. (It turns out that I_w is a radical ideal; this actually follows from Theorem 5.3, because J_w is a radical ideal.) The varieties $\overline{X}_w$ are related to the usual Schubert varieties in a straightforward way.

First, recall that the *variety of flags in* $\mathbb{C}^n$ is the quotient $B\backslash GL_n$ of the general linear group $GL_n = GL_n(\mathbb{C})$ by the *Borel subgroup* B consisting of lower-triangular matrices. It turns out that the matrix Schubert varieties $\overline{X}_w$ are closed under that the left action of B on $M_n\mathbb{C}$ (check this: it comes down to the fact that adding a multiple of some row to another row below it preserves the rank inequality in part 1 of Definition 5.1). Of course, GL_n is itself stable under this action by B, so $\widetilde{X}_w = \overline{X}_w \cap GL_n$ is, as well. The ordinary Schubert variety $X_w \subseteq B\backslash GL_n$ is then the quotient $B\backslash\widetilde{X}_w$. The general setup looks like the diagram below:

$$\begin{array}{ccccc} X_w & & \widetilde{X}_w & & \overline{X}_w \\ \cap & & \cap & & \cap \\ B\backslash GL_n & \twoheadleftarrow & GL_n & \hookrightarrow & M_n\mathbb{C} \end{array} \tag{1}$$

Suppose $Z' \subseteq (z_{rs})$ is a square submatrix of the generic matrix of variables. The *antidiagonal* of Z' or of $\det(Z')$ is the product of the variables on its main antidiagonal. Any term order on $\mathbb{C}[\mathbf{z}]$ such that the leading term of $\det(Z')$ is (for every each Z') the antidiagonal is called an *antidiagonal term order*.

Theorem 4.2 is the bridge between Schubert varieties and subword complexes, via the following almost content-free lemma.

Lemma 5.2 *J_w is generated by the antidiagonals of the minors generating I_w.*

Just because J_w is generated by initial terms of generators for I_w doesn't mean that J_w equals the ideal generated by the initial terms of *all* elements in I_w. When the initial terms of a generating set for I_w do generate the entire initial ideal of I_w, the generating set is called a *Gröbner basis*.

Theorem 5.3 *The minors generating I_w constitute a Gröbner basis for any antidiagonal term order, and the initial ideal is J_w.*

The proof draws on tools ranging from the combinatorics of antidiagonals to the algebra of Grothendieck polynomials (Section 8), and applies equivariant cohomology to commutative algebra. A hint of the methods involved appears in the proof sketches for Theorem 8.2 and Theorem 9.6.

Now Theorem 5.3 and Theorem 3.3 combine nicely.

Corollary 5.4 *Every matrix Schubert variety $\overline{X}_w$, and hence every Schubert variety $X_w \subseteq B\backslash GL_n$, is Cohen-Macaulay.*

Proof. Theorem 3.3 implies that $\mathbb{C}[\mathbf{z}]/J_w$ is Cohen-Macaulay. Whenever $I \subseteq \mathbb{C}[\mathbf{z}]$ is an ideal having an initial ideal J such that $\mathbb{C}[\mathbf{z}]/J$ is Cohen-Macaulay, the quotient $\mathbb{C}[\mathbf{z}]/I$ is itself also Cohen-Macaulay, thanks to the upper-semicontinuity of Betti numbers in flat families. Thus $\overline{X}_w$ is also Cohen-Macaulay by Theorem 5.3. Now $\overline{X}_w \cap GL_n = \widetilde{X}_w$ is an open subset of $\overline{X}_w$ and hence also Cohen-Macaulay (which is a local property). Finally, $\widetilde{X}_w$ is a smooth bundle over X_w with smooth fiber B, so X_w is Cohen-Macaulay. □

One final comment: there are lots of antidiagonal term orders. These include

(i) the reverse lexicographic term order that snakes its way from the northwest corner to the southeast corner, $z_{11} > z_{12} > \cdots > z_{1n} > z_{21} > \cdots > z_{nn}$; and

(ii) the lexicographic term order which snakes its way from the northeast to the southwest $z_{1\,n} > z_{1\,n-1} > \cdots > z_{1\,1} > z_{2\,n} > \cdots > z_{n\,1}$.

6 Demazure products

Let's return to the combinatorial setting, where (W, S) is a Coxeter system, Q is a word, and $w \in W$. The simplicial complex $\Delta(Q, w)$ being Cohen-Macaulay is perhaps not the most appealing, concrete, or precise statement. While vertex-decomposability and shellability are both quite a bit better, further analysis of Coxeter groups in fact produces even stronger results.

Definition 6.1 Let R be a commutative ring, and $\mathcal{D}$ a free R-module with basis $\{e_w \mid w \in w\}$. Defining a multiplication on $\mathcal{D}$ by

$$e_w e_s = \begin{cases} e_{ws} & \text{if length}(ws) > \text{length}(w) \\ e_w & \text{if length}(ws) < \text{length}(w) \end{cases} \tag{2}$$

yields the *Demazure algebra* of (W, S) over R. Define the *Demazure product* $\delta(Q)$ of the word $Q = (s_1, \ldots, s_m)$ by $e_{s_1} \cdots e_{s_m} = e_{\delta(Q)}$.

Example 6.2 When $R = \mathbb{Z}$ is the integers, $W = \Sigma_n$ and S is the set of simple reflections $s_1, \ldots, s_{n-1}$, the algebra $\mathcal{D}$ is isomorphic to the algebra generated over $\mathbb{Z}$ by the *Demazure operators* $\overline{\partial}_i = e_{s_i}$. By definition, the i^{th} Demazure operator $\overline{\partial}_i : \mathbb{Z}[[\mathbf{x}]] \to \mathbb{Z}[[\mathbf{x}]]$ (where $\mathbf{x} = x_1, \ldots, x_n$) sends a power series $f(\mathbf{x})$ to

$$\frac{x_{i+1} f(x_1, \ldots, x_n) - x_i f(x_1, \ldots, x_{i-1}, x_{i+1}, x_i, x_{i+2}, \ldots, x_n)}{x_{i+1} - x_i}. \tag{3}$$

Note that $\overline{\partial}_i$ does indeed take a power series to another power series, because $\mathbb{Z}[[\mathbf{x}]]$ has unique factorization: the numerator in Eq. (3) is zero when $x_i = x_{i+1}$, so it is divisible by $x_{i+1} - x_i$.

If $f(\mathbf{x})$ is symmetric in x_i and x_{i+1} and $g(\mathbf{x})$ is arbitrary, then $\overline{\partial}_i(fg) = f\overline{\partial}_i(g)$. It is less obvious (but no less true) that the operators $\overline{\partial}_i$ statisfy the *Coxeter relations* $\overline{\partial}_i\overline{\partial}_{i+1}\overline{\partial}_i = \overline{\partial}_{i+1}\overline{\partial}_i\overline{\partial}_{i+1}$ and $\overline{\partial}_i\overline{\partial}_{i'} = \overline{\partial}_{i'}\overline{\partial}_i$ when $|i-i'| \geq 2$. Given the Coxeter relations, the fact that $\mathcal{D}$ is an associative algebra with given free $\mathbb{Z}$-basis is immediate. □

In general, the fact that the equations in (2) define an associative algebra is the special case of [Hum90, Theorem 7.1] where all of the 'a' variables equal 1 and all of the 'b' variables are zero. Observe that the ordered product of a word equals the Demazure product if the word is reduced. Here are some basic properties of Demazure products, in which the '$\geq$' and '$>$' signs denote the *Bruhat partial order* on W [Hum90, Section 5.9].

Lemma 6.3 *Let P be a word in W and let $w \in W$.*

1. *The Demazure product $\delta(P)$ is $\geq w$ if and only if P contains w.*
2. *If $\delta(P) = w$, then every subword of P containing w has Demazure product w.*
3. *If $\delta(P) > w$, then P contains a word T representing an element $t > w$ satisfying $|T| = \text{length}(t) = \text{length}(w) + 1$.*

Proof. If $P' \subseteq P$ and P' contains w, then P' contains $\delta(P')$ and $w = \delta(P) \geq \delta(P') \geq w$, proving part 2 from part 1. Choosing any $t \in W$ such that $\text{length}(t) = \text{length}(w)+1$ and $w < t \leq \delta(P)$ proves part 3 from part 1.

Now we prove part 1. Suppose $w' = \delta(P) \geq w$, and let $P' \subseteq P$ be the subword obtained by reading P in order, omitting any reflections along the way that do not increase length. Then P' represents w' by definition, and contains w because any reduced expression for w' contains a reduced expression for w.

On the other hand, suppose $T \subseteq P$ represents w, with $s \in S$ being the last element of T. Observe that $\delta(P)s < \delta(P)$, and that $\delta(P \smallsetminus s)$ equals either $\delta(P)s$ or $\delta(P)$. Since also $ws < w$, we know a priori that $ws \leq \delta(P)$ if and only if $w \leq \delta(P)$. Now use induction on $|P|$: if s isn't the last element in P, then $w \leq \delta(P \smallsetminus s) \leq \delta(P)$; on the other hand, if s is the last element of P, then $ws \leq \delta(P \smallsetminus s) \leq \delta(P)$, whence $w \leq \delta(P)$. □

Lemma 6.4 *Let T be a word in W and $w \in W$ such that $|T| = \text{length}(w) + 1$.*

1. *There are at most two elements $s \in T$ such that $T \smallsetminus s$ represents w.*
2. *If $\delta(T) = w$, then there are two distinct $s \in T$ such that $T \smallsetminus s$ represents w.*
3. *If T represents $t > w$, then $T \smallsetminus s$ represents w for exactly one $s \in T$.*

Proof. Part 1 is obvious if $|T| \leq 2$, so choose elements $s_1, s_2, s_3 \in T$ in order of appearance, and let $T_1s_1T_2s_2T_3s_3T_4 = T$ be the resulting partition of T into connected segments. If two of the words $T \smallsetminus s_i$ represent w, then assume $T \smallsetminus s_3$ represents w (by reversing T and replacing w with w^{-1} if necessary). Letting t_4 be the ordered product of T_4, then $P := T \smallsetminus (s_3T_4) = T_1s_1T_2s_2T_3$ represents wt_4^{-1}. That $P \smallsetminus s_1$ and $P \smallsetminus s_2$ can't both represent wt_4^{-1} is a consequence of the *exchange condition* [Hum90, Theorem 5.8], so mutliplying on the right by s_3t_4 yields the result.

In part 2, $\delta(T) = w$ means there is some $s \in T$ such that (i) $T = T_1 s T_2$; (ii) $T_1 T_2$ represents w, and (iii) $t_1 > t_1 s$, where T_1 represents t_1. Omitting some s' from $T_1 s$ leaves a reduced expression for $t_1 s$. Thus $T \smallsetminus s'$ represents w, and s' can't be the original s, by (iii). Part 3 is the exchange condition. $\square$

Remark 6.5 For $\Delta(Q_n, w)$, there is a "pipe dream-theoretic" reason for part 1 of Lemma 6.4. Let D be an rc-graph for w with no '+' at (i,j), and let D_+ be the pipe dream resulting by adding the extra '+' at (i,j). This new '+' in D_+ switches the destinations of the pipes in D squiggling through (i,j). In order to get an rc-graph *for the same* w out of D_+, we must reset the destinations of these pipes by removing a '+'. Since the two pipes in question crossed at most once in D, they can cross at most twice in D_+. $\square$

Lemma 6.6 *Suppose every codimension 1 face of a shellable simplicial complex Δ is contained in at most two facets. Then Δ is a topological manifold-with-boundary that is homeomorphic to either a ball or a sphere. The facets of the topological boundary of Δ are the codimension 1 faces of Δ contained in exactly one facet of Δ.*

Proof. Let B_d and B'_d be homoemorphic to d-dimensional balls. If $h : B_{d-1} \to B'_{d-1}$ is a homeomorphism between $(d-1)$-balls in their boundaries, then $B_d \cup_h B'_d$ is again homoemorphic to a d-ball. On the other hand, if $h : \partial B_d \to \partial B'_d$ is a homeomorphism between their entire boundaries, then $B_d \cup_h B'_d$ is homeomorphic to a d-sphere. Now use induction on the number of facets of Δ; if the d-sphere case is reached, the assumption on codimension 1 faces of Δ implies that the shelling must be complete. $\square$

Of course, the shellable complexes that interest us are the subword complexes.

Theorem 6.7 *The subword complex $\Delta(Q, w)$ is a either a ball or a sphere. A face $Q \smallsetminus P$ is in the boundary of $\Delta(Q, w)$ if and only if P has Demazure product $\delta(P) \neq w$.*

Proof. That every codimension 1 face of $\Delta(Q, w)$ is contained in at most two facets is the content of part 1 in Lemma 6.4, while shellability is Theorem 3.3. This verifies the hypotheses of Lemma 6.6 for the first sentence of the Theorem.

If P has Demazure product $\neq w$, then $\delta(P) > w$ by part 1 of Lemma 6.3. Choosing T as in part 3 of Lemma 6.3, we find by part 3 of Lemma 6.4 that $Q \smallsetminus T$ is a codimension 1 face contained in exactly one facet of $\Delta(Q, w)$. Thus, using Lemma 6.6, we conclude that $Q \smallsetminus P \subseteq Q \smallsetminus T$ is in the boundary of $\Delta(Q, w)$.

If $\delta(P) = w$, on the other hand, part 2 of Lemmas 6.3 and 6.4 say that every codimension 1 face $Q \smallsetminus T \in \Delta(Q, w)$ containing $Q \smallsetminus P$ is contained in two facets of $\Delta(Q, w)$. Lemma 6.6 says each such $Q \smallsetminus T$ is in the interior of $\Delta(Q, w)$, whence $Q \smallsetminus P$ must itself be an interior face. $\square$

7 Hilbert series

Suppose $Q = (s_1, \ldots, s_m)$ and that $J \subseteq \mathbb{C}[z_1, \ldots, z_m] =: \mathbb{C}[\mathbf{z}]$ is the Stanley–Reisner ideal of $\Delta(Q, w)$. Write monomials in $\mathbb{C}[\mathbf{z}]$ as $\mathbf{z}^{\mathbf{b}} = z_1^{b_1} \cdots z_m^{b_m}$, and call the set $\{i \mid b_i \neq 0\}$ the *support* of $\mathbf{b}$ or of $\mathbf{z}^{\mathbf{b}}$, and identify it with the obvious subword of Q. Then J is the squarefree monomial ideal such that $\mathbf{z}^{\mathbf{b}}$ remains nonzero in $\mathbb{C}[\mathbf{z}]/J$ (i.e. $\mathbf{z}^{\mathbf{b}} \notin J$) if and only if the support of $\mathbf{b}$ is in $\Delta(Q, w)$.

Theorem 6.7 allows for easy computation of the *Hilbert series* of $\mathbb{C}[\mathbf{z}]/J$. Recall what this means for the Stanley–Reisner ideal J of any simplicial complex Δ. The set of monomials nonzero in $\mathbb{C}[\mathbf{z}]/J$ forms an *order ideal* in the poset $\mathbb{N}^m$, meaning that if $\mathbf{z}^{\mathbf{b}'}$ divides $\mathbf{z}^{\mathbf{b}}$ and the latter is nonzero in $\mathbb{C}[\mathbf{z}]/J$, then $\mathbf{z}^{\mathbf{b}'}$ is nonzero in $\mathbb{C}[\mathbf{z}]/J$. The Hilbert series of $\mathbb{C}[\mathbf{z}]/J$ is then simply the generating function for monomials not in J:

$$H(\mathbb{C}[\mathbf{z}]/J; \mathbf{z}) = \sum_{\mathbf{z}^{\mathbf{b}} \notin J} \mathbf{z}^{\mathbf{b}}.$$

The Hilbert series above can always be expressed as a rational function

$$H(\mathbb{C}[\mathbf{z}]/J; \mathbf{z}) = \frac{\mathcal{N}(\Delta; \mathbf{z})}{\prod_{j=1}^{m}(1 - z_j)}$$

in which $\mathcal{N}(\Delta; \mathbf{z})$ is a polynomial with integer coefficients (one proof is simply by inclusion-exclusion). There are a number of formulas for $\mathcal{N}(\Delta; \mathbf{z})$ in terms of numerical invariants the simplicial complex Δ. The invariants usually take the form of dimensions of (co)homology groups of various subcomplexes associated to Δ.

One of these formulas uses the i^{th} *Betti numbers*

$$\beta_{i,P} = \dim_{\mathbb{C}} \widetilde{H}_{i-1}(\operatorname{link}(\overline{P}, \Delta); \mathbb{C}), \tag{4}$$

where $\overline{P}\{1, \ldots, m\} \smallsetminus P$ is the face complementary to P, and $\widetilde{H}$ denotes reduced homology. Links were defined in Section 3.

Proposition 7.1 *For any simplicial complex Δ, the numerator of the Hilbert series of its Stanley–Reisner ideal satisfies*

$$\mathcal{N}(\Delta; \mathbf{1} - \mathbf{z}) = \sum_{\overline{P} \in \Delta} (-1)^i \beta_{i,P}\, \mathbf{z}^P,$$

where $\mathbf{z}^P = \prod_{j \in P} z_j$ and $\mathcal{N}(\Delta; \mathbf{1} - \mathbf{z})$ is obtained by substituting $1 - z_j$ for every occurrence of $1 - z_j$ in $\mathcal{N}(\Delta; \mathbf{z})$.

This formula probably doesn't appear verbatim in the literature: its proof (which isn't hard) uses a known formula of Hochster along with the *Alexander inversion formula* for Hilbert series of Alexander dual monomial ideals.

The Cohen-Macaulay property of $\Delta(Q, w)$ makes the formula in Proposition 7.1 particularly nice, because the Betti numbers $\beta_{i,P}$ will be zero unless i is as big as possible. Moreover, the ball/sphere property makes simplifies things even further: essentially, the links will all be balls or spheres, and the link will be a sphere precisely when $Q \smallsetminus P$ is an *interior* face of $\Delta(Q, w)$—i.e. when the Demazure product of P equals w.

Theorem 7.2 *If J is the Stanley-Reisner ideal of $\Delta(Q, w)$ and $\ell = \text{length}(w)$, then the numerator Hilbert series of $\mathbb{C}[\mathbf{z}]/J$ satisfies*

$$\mathcal{N}(\Delta(Q, w); \mathbf{1} - \mathbf{z}) = \sum_{\delta(P)=w} (-1)^{|P|-\ell} \mathbf{z}^P,$$

where the denominator is $\prod_{j=1}^{m}(1 - z_j)$, as usual.

Proof. Let $Q \smallsetminus P \in \Delta(Q, \pi)$, so $P \subseteq Q$ contains w. By Theorem 6.7, either $\text{link}(Q \smallsetminus P, \Delta(Q, w))$ is contractible (if $\delta(P) \neq w$), or it is a sphere of dimension

$$\dim \Delta(Q, w) - |Q \smallsetminus P| = (|Q| - \ell - 1) - |Q \smallsetminus P| = |P| - \ell - 1$$

(if $\delta(P) = w$), where a sphere of dimension -1 is taken to mean the empty complex $\{\varnothing\}$ having nonzero reduced homology in dimension -1 (Remark 3.2). Therefore $\widetilde{H}_{i-1}\text{link}(Q \smallsetminus P, \Delta(Q, w))$ is zero unless $\delta(P) = w$ and $i = |P| - \ell$. Now apply (4) to Proposition 7.1. □

The rc-graph case $\Delta(Q_n, w)$ of Theorem 7.2 deserves a special name.

Definition 7.3 Suppose $w \in \Sigma_n$ has length ℓ. The *Demazure polynomial* is $\mathfrak{D}_w(\mathbf{z}) = \mathcal{N}(\Delta(Q_n, w); \mathbf{1} - \mathbf{z}) = \sum_{\delta(D)=w} (-1)^{|D|-\ell} \mathbf{z}^D$.

Here is a weird consequence of the characterization of $\mathcal{N}(\Delta(Q_n, w); \mathbf{z})$ via $\mathfrak{D}_w(\mathbf{z})$.

Porism 7.4 *Viewing $w \in \Sigma_n$ as being an element of Σ_N for all $N \geq n$, the Hilbert numerators $\mathcal{N}(\Delta(Q_N, w); \mathbf{1} - \mathbf{z})$ are the same for all $N \geq n$. Each squarefree monomial $\mathbf{z}^D$ in $(z_{rs})_{r,s=1}^{\infty}$ appears with nonzero coefficient in a polynomial $\mathcal{N}(\Delta(Q_n, w); \mathbf{1} - \mathbf{z})$ for exactly one permutation $w \in \Sigma_\infty$, and its coefficient is ± 1.*

Proof. The permutation w in question is $\delta(D)$, by Theorem 7.2. □

8 Grothendieck polynomials

The Stanley–Reisner ideal of $\Delta(Q_n, w)$ turned out to be pretty important from both geometric and algebraic points of view. In fact, there is more useful information to be reaped from it, this time coming from K-theory (and later, cohomology).

Thanks to Example 6.2, the polynomial $\mathcal{G}_w(\mathbf{x})$ in the next definition depends only on w and not on the reduced expression for $w_0 w$.

Definition 8.1 The *Grothendieck polynomial* $\mathcal{G}_w(\mathbf{x})$ is defined recursively by setting $\mathcal{G}_{w_0} = \prod_{i=1}^n (1-x_i)^{n-i}$ and declaring that $\mathcal{G}_{ws_i} = \overline{\partial}_i \mathcal{G}_w$ if $\text{length}(ws_i) < \text{length}(w)$. In other words, $\mathcal{G}_w = \overline{\partial}_{i_k} \cdots \overline{\partial}_{i_1} \mathcal{G}_{w_0}$, where $s_{i_1} \cdots s_{i_k}$ is a *reduced expression* for $w_0 w$, meaning that $k = \text{length}(w_0 w)$.

Briefly, the closed subvariety $X_w \subseteq B\backslash GL_n$ determines a class in the ordinary K-theory of $B\backslash GL_n$. Such classes can be represented (nonuniquely) by polynomials in n variables $x_1, \ldots, x_n$. Lascoux and Schützenberger [LS82] showed that the Grothendieck polynomial $\mathcal{G}_w(\mathbf{x})$ is a particularly well-behaved representative.

Its connection to the present discussion is via the $\mathbb{Z}^n$-graded Hilbert series. Recall: the ring $\mathbb{C}[\mathbf{z}]$ is graded by $\mathbb{Z}^n$, the degree of each variable z_{ij} being the i^{th} standard basis vector in $\mathbb{Z}^n$. The ideal I_w is generated by homogeneous elements for this $\mathbb{Z}^n$-grading (determinants are sums of monomials with one variable in each column and row). For any $\mathbb{Z}^n$-graded module $\Gamma = \bigoplus_{\mathbf{b}\in\mathbb{Z}^n} \Gamma_{\mathbf{b}}$, the $\mathbb{Z}^n$-graded Hilbert series is

$$H(\Gamma; \mathbf{x}) = \sum_{\mathbf{b}\in\mathbb{Z}^n} \dim_{\mathbb{C}}(\Gamma_{\mathbf{b}}) \mathbf{x}^{\mathbf{b}}.$$

Like monomial ideals in the "finer" grading by $\mathbb{Z}^m$ (here $m = n^2$, of course), Hilbert series of finitely generated $\mathbb{Z}^n$-graded modules can be expressed as rational functions.

Theorem 8.2 *$\mathbb{Z}^n$-graded Hilbert series of $\mathbb{C}[\mathbf{z}]/I_w$ is*

$$H(\mathbb{C}[\mathbf{z}]/I_w; \mathbf{x}) = \frac{\mathcal{G}_w(\mathbf{x})}{\prod_{i=1}^n (1-x_i)^n}.$$

Proof sketch. Under the various maps in Eq. (1) on page 7, the ordinary K-theory class $[\mathcal{O}_{X_w}] \in K^\circ(B\backslash GL_n)$ pulls back to the equivariant K-theory class $[\mathcal{O}_{\tilde{X}_w}]_B \in K_B^\circ(GL_n)$. This class is mapped to by $[\mathcal{O}_{\overline{X}_w}]_B \in K_B^\circ(M_n\mathbb{C}) = K_T^\circ(M_n\mathbb{C})$, and the "well-behaved" properties of Grothendieck polynomials imply that $[\mathcal{O}_{\overline{X}_w}]_T = \mathcal{G}_w(\mathbf{x})$. On the other hand, it follows almost directly from the definitions that the numerator of the Hilbert series of $\mathbb{C}[\mathbf{z}]/I_w$ represents the class $[\mathcal{O}_{\overline{X}_w}]_T$ (take the alternating sum of the summands in a graded free resolution of $\mathbb{C}[\mathbf{z}]/I_w$). $\square$

Part of the next corollary is due to Fomin and Stanley [FS94]. It should be viewed as a pipe dream interpretation for the coefficients of canonical representatives for K-theory classes in flag manifolds. Observe that no cancellation occurs when $\mathbf{z}$ is specialized to $\mathbf{x}$, because the terms in $\mathfrak{D}_w(\mathbf{z})$ having the same total degree all have the same sign.

Corollary 8.3 *If $w \in \Sigma_n$ has length ℓ, then specializing $z_{rs} \mapsto x_r$ in the Demazure polynomial $\mathfrak{D}_w(\mathbf{z})$ yields the Grothendieck polynomial $\mathcal{G}_w(\mathbf{1}-\mathbf{x}) = \mathcal{G}_w(1-x_1, \ldots, 1-x_n)$:*

$$\mathcal{G}_w(\mathbf{1}-\mathbf{x}) = \mathfrak{D}(\mathbf{z})|_{z_{rs}=x_r} = \sum_{\delta(D)=w} (-1)^{|D|-\ell} \mathbf{x}^D.$$

Proof. Since J_w is an initial ideal of I_w, the numerator of the $\mathbb{Z}^n$-graded Hilbert series of $\mathbb{C}[\mathbf{z}]/I_w$ equals that of $\mathbb{C}[\mathbf{z}]/J_w$. After substituting $\mathbf{1}-\mathbf{x}$ for $\mathbf{x}$, the latter is obtained from $\mathfrak{D}_w(\mathbf{z})$ by setting $z_{rs} = x_r$. □

For many applications, the polynomial $\mathcal{G}_w(\mathbf{1}-\mathbf{x})$ is more natural than $\mathcal{G}_w(\mathbf{x})$, being in some sense more geometric. For instance, the cohomology class of a Schubert variety is represented by a *Schubert polynomial* $\mathfrak{S}_w(\mathbf{x})$, which is the sums of the lowest degree terms in $\mathcal{G}_w(\mathbf{1}-\mathbf{x})$, as we will see shortly.

Like the Grothendieck polynomials, the usual definition of Schubert polynomials is inductive. Instead of using Demazure operators $\overline{\partial}_i$, Schubert polynomials require the *divided difference operators* ∂_i, that take a polynomial $f(x_1, x_2, \ldots)$ to

$$\partial_i f(x_1, x_2, \ldots) = \frac{f(x_1, x_2, \ldots,) - f(x_1, \ldots, x_{i-1}, x_{i+1}, x_i, x_{i+2}, \ldots)}{x_i - x_{i+1}}.$$

The divided differences also satisfy the Coxeter relations, so the next defintion makes sense.

Definition 8.4 Set $\mathfrak{S}_{w_0} = \prod_{i=1}^n x_i^{n-i}$. The *Schubert polynomial for w* is defined by $\mathfrak{S}_w = \partial_{i_k} \cdots \partial_{i_1} \mathfrak{S}_{w_0}$, where $w_0 w = s_{i_1} \cdots s_{i_k}$ and $\operatorname{length}(w_0 w) = k$.

Lemma 8.5 *The Schubert polynomial $\mathfrak{S}_w(\mathbf{x})$ is the sum of lowest-degree terms in $\mathcal{G}_w(\mathbf{1}-\mathbf{x})$.*

Proof. Assuming $f(\mathbf{1}-\mathbf{x})$ is homogeneous, plugging $\mathbf{1}-\mathbf{x}$ for $\mathbf{x}$ into Eq. (3) and taking the lowest degree terms yields $\partial_i f(\mathbf{1}-\mathbf{x})$. Since $\mathfrak{S}_{w_0}$ is homogeneous, the result follows by induction on $\operatorname{length}(w_0 w)$. □

The combinatorial interpretation of coefficients in the K-theoretic Grothendieck polynomials generalizes (and implies) the following cohomological version.

Corollary 8.6 (Billey-Jockusch-Stanley [BJS93], Fomin-Stanley [FS94])

$$\mathfrak{S}_w(\mathbf{x}) = \sum_{D \in \mathcal{RC}(w)} \mathbf{x}^D.$$

Proof. Apply Lemma 8.5 to the sum in Corollary 8.3. □

9 Mitosis algorithm

The formulas in Section 8 for Grothendieck and Schubert polynomials work by taking a collection of pipe dreams D (subsets of $[n]^2$), associating a monomial $\pm\mathbf{z}^D$ to each one, and then specializing the sum by setting $z_{rs} = x_r$. In a similar vein, A. Kohnert conjectured an algorithmic procedure for producing, inductively on the weak Bruhat order, another (definitely distinct) set of diagrams whose associated sum is $\mathfrak{S}_w(\mathbf{x})$

[Koh91] (see also [Mac91, Appendix to Chapter IV, by N. Bergeron]). The *mitosis*[2] algorithm introduced here sidesteps Kohnert's conjecture by producing rc-graphs inductively on the weak Bruhat order.

Definition 9.1 Given a pipe dream $D \subseteq [n]^2$ and a row index i, construct a set $\text{mitosis}_i(D)$ of pipe dreams contained in $[n]^2$ as follows. Define the column index

$$\text{start}_i(D) = \min(\{j \mid (i,j) \notin D\} \cup \{n+1\})$$

of the westernmost elbow joint in row i of D, and set

$$\begin{aligned} \mathcal{J}(D) &= \{j < \text{start}_i(D) \mid (i+1,j) \notin D\}; \\ \mathcal{J}_{i,\leq s} &= \{(i,j) \mid j \in \mathcal{J}(D) \text{ and } j \leq s\} \text{ for } s \in \mathcal{J}(D); \\ \mathcal{J}_{i+1,<s} &= \{(i+1,j) \mid j \in \mathcal{J}(D) \text{ and } j < s\} \text{ for } s \in \mathcal{J}(D). \end{aligned}$$

Define the *offspring* $D_s = D \cup \mathcal{J}_{i+1,<s} \smallsetminus \mathcal{J}_{i,\leq s}$ and $\text{mitosis}_i(D) = \{D_s \mid s \in \mathcal{J}(D)\}$. Given a set $\mathcal{P}$ of pipe dreams, write $\text{mitosis}_i(\mathcal{P}) = \bigcup_{D \in \mathcal{P}} \text{mitosis}_i(\mathcal{P})$.

For a pipe dream D, the offspring D_s is obtained by first deleting the cross at (i,s) in D, and then moving the crosses west of it from row i south to empty boxes in row $i+1$ (mitosis only cares about the columns $\mathcal{J}(D)$, which are empty in row $i+1$). In the region of D that is west of (i,s), row i is filled solidly with crosses. Observe that $\text{mitosis}_i(D)$ is an empty set whenever $\mathcal{J}(D)$ is empty.

Proposition 9.4 presents another verbal description of mitosis—this one a little more algorithmic—using a certain local transformation on pipe dreams.

Definition 9.2 (Bergeron-Billey [BB93]) A *chutable rectangle* is a connected $2 \times k$ rectangle C inside a pipe dream D such that $k \geq 2$ and all but the following 3 locations in C are crosses: the northwest, southwest, and southeast corners. Applying a *chute move*[3] to D is accomplished by placing a '+' in the southwest corner of a chutable rectangle C and removing the '+' from the northeast corner of the same C.

Heuristically, a chute move therefore looks like:

$$\begin{array}{|c|c|c|c|c|c|c|}\hline \cdot & + & + & & + & + & + \\ \hline \cdot & + & + & \cdots & + & + & \cdot \\ \hline \end{array} \quad \overset{\text{chute}}{\rightsquigarrow} \quad \begin{array}{|c|c|c|c|c|c|c|}\hline \cdot & + & + & & + & + & \cdot \\ \hline + & + & + & \cdots & + & + & \cdot \\ \hline \end{array}$$

Remark 9.3 In their main result [BB93, Theorem 3.7(d)], Bergeron and Billey give an algorithm for generating $\mathcal{RC}(w)$ by applying chute moves to a single "top" rc-graph for w (some rc-graphs are produced twice). In particular, $\mathcal{RC}(w)$ is closed under chute operations. Though motivated by and related to Kohnert's conjecture, their algorithm is not inductive on the weak Bruhat order, instead remaining inside $\mathcal{RC}(w)$. $\square$

[2]The term *mitosis* is biological lingo for cell division in multicellular organisms.
[3]The transpose of a chute move is called a *ladder move* in [BB93]

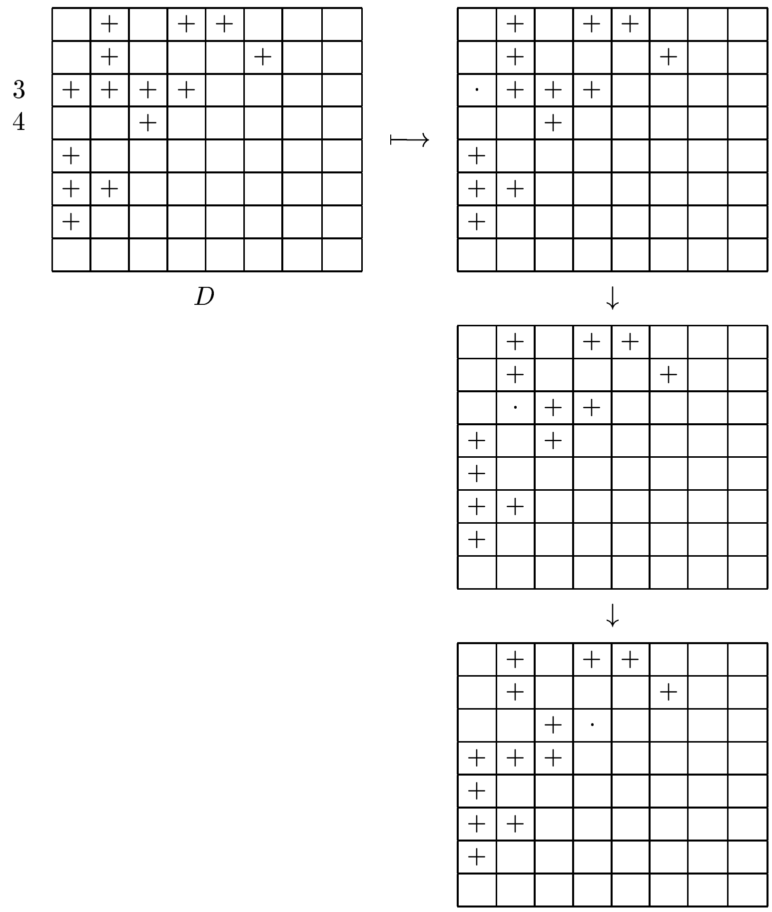


Figure 1: Mitosis

Proposition 9.4 *Let D be a pipe dream, and suppose j is the smallest column index such that $(i+1, j) \notin D$ and $(i, s) \in D$ for all $s \leq j$. Then $D_s \in \mathrm{mitosis}_i(D)$ is obtained from D by*

1. *removing (i, j), and then*
2. *performing chute moves from row i to row $i+1$, each one as far west as possible, so that (i, s) is the last '+' removed.*

Proof. Immediate from Definitions 9.2 and 9.1. $\square$

Example 9.5 Fig. 1 depicts the algorithmic 'chute' form of mitosis in Proposition 9.4, with $i = 3$. The dot in each pipe dream represents the last '+' removed. $\square$

Mitosis is how $\mathcal{RC}(w)$ reproduces to form $\mathcal{RC}(ws_i)$.

Theorem 9.6 *$\mathcal{RC}(ws_i)$ is the disjoint union $\dot{\bigcup}_{D \in \mathcal{RC}(w)} \mathrm{mitosis}_i(D)$ whenever $w < ws_i$. If $s_{i_1} \cdots s_{i_k}$ is a reduced expression for $w_0 w$, then*

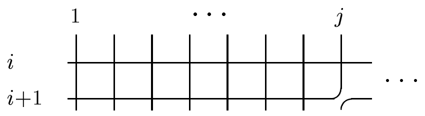
$$\mathcal{RC}(w) = \mathrm{mitosis}_{i_k} \cdots \mathrm{mitosis}_{i_1}(D_0), \tag{5}$$

where D_0 is the unique rc-graph for w_0.

Proof sketch. The proof is somewhat involved. One approach gives a complete argument based entirely on Corollary 8.6 and the characterization of Schubert polynomials by divided differences, along with elementary combinatorial properties of rc-graphs. Another possibility exploits the combinatorics of antidiagonals, translating back to rc-graph language at the end. The latter approach has the advantage of drawing on methods developed anyway for the proofs of Theorems 4.2 and 5.3.

In either case, one shows that $\mathrm{mitosis}_i(\mathcal{RC}(w)) \subseteq \mathcal{RC}(ws_i)$, and that the polynomials in $\mathbf{x}$ determined by these two sets (see the beginning of this section) are equal. (Actually, it's enough to show that the two sets have equal size, which one gets by plugging in $x_j = 1$ for all j.) Remark 9.3 combines with Lemma 9.7 to prove containment. A certain involution on $\mathcal{RC}(w)$ shows why the polynomial formed from $\mathrm{mitosis}_i(\mathcal{RC}(w))$ is $\partial_i \mathfrak{S}_w(\mathbf{x}) = \mathfrak{S}_{ws_i}(\mathbf{x})$. Corollary 8.6 completes the proof. $\square$

To give some flavor of the combinatorics involved in checking Theorem 9.6, here is a criterion for when removing a '+' from a pipe dream $D \in \mathcal{RC}(w)$ yields a pipe dream in $\mathcal{RC}(ws_i)$. Specifically, it concerns the removal of a cross at (i, j) from configurations that look like



$$= \quad \begin{array}{c|cccccccc|c} & 1 & & & \cdots & & & & j & \\ i & + & + & + & + & + & + & + & + & \cdots \\ i{+}1 & + & + & + & + & + & + & + & \cdot & \end{array}$$

at the west end of rows i and $i{+}1$ in D. It implies that the beginning of the algorithmic form of mitosis in Proposition 9.4 outputs an rc-graph for ws_i when the input is an rc-graph for w.

Lemma 9.7 *Let $D \in \mathcal{RC}(w)$ and j be a fixed column index with $(i+1, j) \notin D$, but $(i, s) \in D$ for all $s \leq j$, and $(i{+}1, s) \in D$ for all $s < j$. Then* $\mathrm{length}(ws_i) < \mathrm{length}(w)$, *and if $D' = D \smallsetminus (i, j)$ then $D' \in \mathcal{RC}(ws_i)$.*

Proof. Removing (i, j) only switches the exit points of the two pipes starting in rows i and $i + 1$, so the pipe starting in row k of D' exits out of column $ws_i(k)$ for each k. The result follows from Lemma 2.4. $\square$

The mitosis algorithm for generating $\mathcal{RC}(w)$ is irredundant, in the sense that each rc-graph appears exactly once in the implicit union on the right hand side of (5). However, the efficiency of the algorithm can vary widely with the reduced expression $s_{i_1} \cdots s_{i_k}$ for $w_0 w$. For instance, the set $\mathcal{RC}(\mathrm{id}_n)$ for the identity permutation consists of one element, $\varnothing \subset [n]^2$, even though the repeated mitosis in (5) for $w = \mathrm{id}_n$ can pass through $\mathcal{RC}(w)$ for *any* permutation w. As a consequence, huge numbers of rc-graphs must be killed along the way, without producing any offspring.

Definition 9.8 A reduced expression for $w_0 w$ is called *poptotic*[4] if every rc-graph has at least one offspring along the way to $\mathcal{RC}(w)$ via Theorem 9.6.

[4] The word *apoptosis* is a biological term referring to "programmed cell death", in which some cell in a multicellular organism in effect commits suicide for the greater good of the organism. Thus *apoptotic* is an apt term for a total order in which some rc-graph dies along the way. In analogy with the way 'crepant' is formed from 'discrepancy', we take the opposite of 'apoptotic' to be 'poptotic'.

Example 9.9 The three pipe dreams from the second column of Fig. 1 are all rc-graphs for $v = 13685742$ (apply Theorem 9.6 to Examples 2.3 and 9.5). Setting $i = 4$ and either inspecting the inversions of v or applying Lemma 9.7 to the last rc-graph in Fig. 1, we find that $\text{length}(vs_4) < \text{length}(v)$. But mitosis_4 kills the first two of the three rc-graphs, whereas the last has two offspring. Thus any reduced expression for $w_0 v$ ending s_4 is necessarily apoptotic.

On the other hand, the *lexicographically first* reduced expression (with $s_1 > s_2 > \cdots > s_{n-1}$), which is $w_0 v = s_2 s_1 s_3 s_5 s_4 s_3 s_2 s_1 s_7 s_6 s_5 s_4 s_3 s_2 s_1$ in the present case, is always poptotic. In particular, the lex first reduced expression for $w_0 = w_0 \text{id}_n$ passes through permutations with exactly one rc-graph (each is a *dominant* permutation, whose unqiue rc-graph is shaped like a Young diagram). □

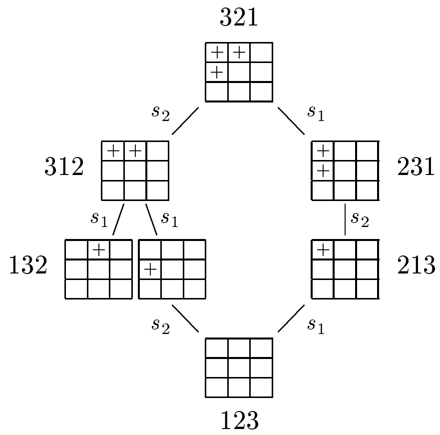
Theorem 9.6 makes the set $\mathcal{RC}_n = \bigcup_{w \in \Sigma_n} \mathcal{RC}(w)$ of rc-graphs for permutations of n elements into a poset determined by

$$D' \prec D \text{ if } D' \in \text{mitosis}_i(D) \text{ for some } i.$$

If $D' \prec D \in \mathcal{RC}(w)$, then automatically $D' \in \mathcal{RC}(ws_i)$, where $\text{length}(ws_i) < \text{length}(w)$ by Lemma 9.7 and the fact that $\mathcal{RC}(ws_i)$ is closed under chute moves. Therefore the poset $\mathcal{RC}_n$, which is ranked by length = cardinality, fibers over the weak Bruhat order on Σ_n, with the preimage of $w \in \Sigma_n$ being $\mathcal{RC}(w)$.

A reduced decomposition for $w_0 w$ can be thought of as a decreasing path in the weak Bruhat order on Σ_n, beginning at w_0 and ending at w. The preimage in $\mathcal{RC}_n$ of such a path is a tree having $\mathcal{RC}(w)$ among its leaves. The path is poptotic, as in Definition 9.8, when the leaves are precisely $\mathcal{RC}(w)$.

Example 9.10 Here is the Hasse diagram for $\mathcal{RC}_3$:



The right hand path from 321 to 123 is poptotic because there's only one rc-graph at each stage. The left hand path is apoptotic because the first rc-graph for 132 has no offspring under mitosis_2. □

Whether or not a path from w_0 to w is poptotic, breadth-first search on the preimage tree (ordering the mitosis offspring as in Proposition 9.4) yields a total order on $\mathcal{RC}(w)$. It can be shown that *poptotic* total orders by breadth-first search are linear extensions of the partial order on rc-graphs determined by chute operations. Through heuristic arguments, computer calculations in small symmetric groups, and the fact that $\Delta(Q_n, w)$ is a priori shellable (Theorem 3.3), it should be that:

Conjecture 9.11 *Poptotic total orders on $\mathcal{RC}(w)$ are shellings of $\Delta(Q_n, w)$.*

It is conceivable that all of the apoptotic total orders are shellings, too, but I hold out less hope for these.

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