Potential applications of commutative algebra to combinatorial game theory

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Finite combinatorial games involving two players taking turns on the same game board are much more complex when the last player to move loses (misère play, as in Dawson's chess) instead of winning (normal play, as in Nim). The goal of Combinatorial Game Theory, in this setting, is to describe—abstractly and algorithmically—the set of winning positions for any given game, or any given class of games.

Recent developments by Plambeck [5], and later also with Siegel [6], have introduced certain commutative monoids, called *misère quotients*, as contexts in which to classify winning positions in misère play; see [7] for a gentle introduction. Our aim is to develop *lattice games*, played on affine semigroups, to place arbitrary impartial combinatorial games—but particularly the historically popular notion of octal game—in a general context where commutative algebra might be brought to bear on periodicity questions. In this note, we state a precise conjecture to the effect that sets of winning positions in lattice games are finite unions of translates of affine semigroups, drawing analogies and connections to the combinatorics of monomial local cohomology and binomial primary decomposition.

In what follows, C is a fixed *affine semigroup* in $\mathbb{Z}^d$; thus C is a finitely generated submonoid of $\mathbb{Z}^d$. We additionally require that C be *pointed*: its identity is its only unit (i.e., invertible element). The games are basically played on C, with allowed lattice moves taken from a fixed set Γ.

Definition 1. A finite subset $\Gamma \subset \mathbb{Z}^d \smallsetminus \{0\}$ is a *rule set* if

1. $\mathbb{N}\Gamma$ is pointed, and
2. $\mathbb{N}\Gamma \supseteq C$.

A *game board* G is the complement in C of a finite Γ-order ideal in C called the set of *defeated positions*. Elements in G are called *positions*. A *move* proceeds from a position $p \in G$ to a point $p - \gamma$ for some $\gamma \in \Gamma$; the move is *legal* if $p - \gamma \in G$.

Normal play corresponds to the choice of $D = \varnothing$; in that case, the goal is to be the player whose move lands at the origin. *Misère play* corresponds to the choice of $D = \{0\}$; in that case, the goal is *not* to be the player whose move lands at the origin. In this sense, misère play is "normal play in which one tries to lose".

It is implicit in the definition that Γ induces a partial order on C; the elementary proof is omitted. Conjecture 3 will make sense with the above notion of rule set Γ, but one or more of the following stronger conditions on Γ might be required.

3. Γ is the minimal generating set for $\mathbb{N}\Gamma$.
4. For each ray ρ of C, there exists $\gamma_i \in \Gamma$ lying in the *negative tangent cone* $-T_\rho C = -\bigcap_{H \supset \rho} H_+$ of C along ρ, where $H_+ \supset C$ is the positive closed half space defined by a supporting hyperplane H for C.

5. Every $p \in C$ has a Γ-path to 0 contained in C; that is, given p, there exists a sequence $0 = p_0, p_1, \ldots, p_r = p$ in C with $p_k - p_{k-1} \in \Gamma$ for all $k > 0$.

Condition 4 is precisely what is necessary to guarantee that from every position there is a Γ-path ending in a neighborhood of the origin. Thus condition 5 implies condition 4, though we omit the proof.

Definition 2. Fix a game board G with rule set Γ. Then $W \subseteq G$ is the set of *winning positions*, and $L \subseteq G$ is the set of *losing positions*, if

1. W and L partition G,
2. $(W + \Gamma) \cap G = L$, and
3. $(W - \Gamma) \cap W = \varnothing$.

The last player to make a legal move wins; this holds both for normal play and misère play, as well as the generalizations for larger D. A position is winning if the player who just moved there can force a win. Condition 1 says that every position is either winning or losing. Condition 2 says that the losing positions are precisely those positions possessing legal moves to W: if your opponent lands on a losing position, then you can always move to W to force a win. Condition 3 says that it is impossible to move directly from one winning position to another.

Conjecture 3. *If W is the set of winning positions for a lattice game, then W is a finite union of translates of affine semigroups.*

The conjecture, if true, would furnish a finite data structure in which to encode the set of winning positions. This would be the first step toward effectively computing the winning positions. Note that the misère octal game *Dawson's chess* [1] remains open; that game initiated and still motivates much of the research on misère games, and one hope would be to use lattice games, along with whatever computational commutative algebra might arise, to crack it.

According to the theory invented by Sprague and Grundy in the 1930s [3, 8], all normal play impartial games are equivalent to the particularly simple game Nim under a certain equivalence relation. For misère games, the analogous equivalence relation is too weak (i.e., too many equivalence classes: not enough games are equivalent to one another). That is why Plambeck invented misère quotients [5]. In the setting of lattice games, the equivalence relation is as follows.

Definition 4. Fix a game board $G = C \smallsetminus D$ with winning positions W. Two lattice points p and $q \in C$ are *congruent* if $(p + C) \cap W = p - q + (q + C) \cap W$. The *misère quotient* of C is the set Q of congruence classes.

Thus p and q are congruent if the winning positions in the "cones" above them are translates of one another. It is elementary to verify that the quotient map $C \to Q$ is a morphism of monoids. Plambeck and Siegel have studied misère quotients in quite a bit of algebraic detail [6]; as this is the proceedings for a conference on commutative algebra, it is strongly recommended that the reader have a look at their work, as it is filled with commutative algebra of finitely generated monoids.

Theorem 5. *Conjecture 3 holds when the misère quotient is finite.*

Proof. Follows by properly interpreting the combinatorial description of primary decomposition [2] of the binomial presentation ideal of the semigroup ring for Q inside of the semigroup ring for C. □

From discussions with Plambeck, Siegel, and others, as well as from examples, it seems likely that binomial primary decomposition has a further role to play in open questions about misère quotients. Such questions include when finiteness occurs, and more complex "algebraic periodicity" questions, which have yet to be formulated precisely [7].

There is another analogy with commutative algebra that is worth bearing in mind. When I is a monomial ideal in an affine semigroup ring, and M is a finitely generated finely graded (i.e., $\mathbb{Z}^d$-graded) module, then the local cohomology $H^i_I(M)$ is supported on a finite union of translates of affine semigroups [4]. If Conjecture 3 is true, then perhaps one could develop a homological theory for winning positions in combinatorial games that explains why.

References

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