GRADED GREENLEES-MAY DUALITY AND THE ČECH HULL

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Abstract. The duality theorem of Greenlees and May relating local cohomology with support on an ideal I and the left derived functors of I-adic completion [GM92] holds for rather general ideals in commutative rings. Here, simple formulas are provided for both local cohomology and derived functors of $\mathbb{Z}^n$-graded completion, when I is a monomial ideal in the $\mathbb{Z}^n$-graded polynomial ring $k[x_1, \ldots, x_n]$. Greenlees-May duality for this case is a consequence. A key construction is the combinatorially defined Čech hull operation on $\mathbb{Z}^n$-graded modules [Mil98, Mil00, Yan00]. A simple self-contained proof of GM duality in the derived category is presented for arbitrarily graded noetherian rings, using methods motivated by the Čech hull.

1. Introduction

Let $S = k[x_1, \ldots, x_n]$ be a polynomial ring over a field k, and $I = \langle m_1, \ldots, m_r \rangle \subset S$ a *Stanley-Reisner ideal* generated by squarefree monomials. Local cohomology $H^{\bullet}_{\mathfrak{m}}(S/I)$ with support on the maximal ideal $\mathfrak{m} = \langle x_1, \ldots, x_n \rangle$ of the *Stanley-Reisner ring* S/I has been familiar to combinatorialists and algebraists ever since the fundamental work of Hochster and Stanley (see [Hoc77, Sta96]) as well as Gräbe [Grä84] relating these objects to simplicial complexes, and Reisner's discovery of a simplicial criterion for S/I to be Cohen-Macaulay [Rei76]. Beginning with Lyubeznik [Lyu84] and continuing with a series of recent papers [Ter99, Mus00a, Yan00, Mil00], increasing attention has been paid to properties of the local cohomology $H_I^{\bullet}(M)$ of modules M with support on a Stanley-Reisner ideal I.

One consistent feature of the recent investigations into $H_I^{\bullet}$ is the presence of some sort of duality. The duality ranges in character, from the topological duality between homology and cohomology, to a more combinatorial duality in posets (Alexander duality, if the poset is boolean), to algebraic dualities such as Matlis duality and local duality. Elsewhere in this volume is a full treatment of the rather general *Greenlees-May duality* [GM92, AJL97], an adjointness between $H_I^{\bullet}$ and the left derived functors $L_{\bullet}^I$ of I-adic completion in commutative rings (or better yet, for sheaves of ideals on schemes), that as yet has not appeared explicitly in combinatorial studies.

The purpose of this paper is to compute simple formulas for both sides of the $\mathbb{Z}^n$-graded Greenlees-May isomorphism over S, and to show how the computation can be rephrased to give an easy proof of GM duality for arbitrary noetherian rings graded by commutative semigroups. The isomorphism between graded local homology and the left derived functors of graded completion is an easy consequence. The $\mathbb{Z}^n$-graded local duality theorem with monomial support and the Alexander duality between $H_{\mathfrak{m}}^{\bullet}(S/I)$ and $H_I^{n-\bullet}(S)$, whose original proofs were independent of GM duality, illustrate the theory as combinatorial examples.

The key construction for the $\mathbb{Z}^n$-graded case is the Čech hull, a combinatorial operation on $\mathbb{Z}^n$-graded modules. Although it was first defined in the context of Alexander duality quite independently of local cohomological considerations [Mil98], the Čech hull turns out to be a natural tool for computing left derived functors $\underline{L}_{\bullet}^I$ of $\mathbb{Z}^n$-graded I-adic completion. More precisely, Theorem 4.4 carries out the computation of $\underline{L}_{\bullet}^I$ in terms of the Čech hull of the $\mathbb{Z}^n$-graded Hom module $\underline{\mathrm{Hom}}_S(\mathcal{F}_{\bullet}, \omega_S)$ for a free resolution $\mathcal{F}_{\bullet}$ of I. When altered slightly to avoid using the $\mathbb{Z}^n$-grading, this method provides a similar approach to the isomorphism between $L_{\bullet}^I$ and local homology $H_{\bullet}^I$ in graded noetherian rings (the new proof of this known isomorphism is the interesting part; the arbitrary grading comes for free).

The organization is as follows. The Čech hull and $\mathbb{Z}^n$-graded construction of Matlis duality are reviewed in Section 2. The relation between the Čech hull and local cohomology is demonstrated in Section 3. The left derived functors $\underline{L}_{\bullet}^I$ of graded I-adic completion are introduced in Section 4, and computed over the polynomial ring via the Čech hull. Section 5 treats $\mathbb{Z}^n$-graded Greenlees-May duality (and its consequences) over S. Finally, Section 6 covers the reformulation of the methods to give a conceptually easy proof for noetherian graded rings.

The reader interested only in the (short) proof of Greenlees-May duality for graded noetherian rings is advised to read Sections 4.1–4.2 and Definition 4.7 before continuing on to Section 6, which requires no other results.

This paper is intended to be accessible to those unfamiliar with Greenlees-May duality as well as to those unfamiliar with the combinatorial side of local cohomology and $\mathbb{Z}^n$-gradings. Therefore many parts are longer than strictly necessary, and have a decidedly expository feel. This seems to be in the spirit of the "first ever" conference on local cohomology, as well as this workshop proceedings.

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2. Basic constructions

2.1. **Matlis duality.** Let $S = k[x_1, \ldots, x_n]$ be a polynomial ring over a field k, and set $\mathbf{x}^\mathbf{a} = x_1^{a_1} \cdots x_n^{a_n}$ for $\mathbf{a} \in \mathbb{N}^n$. An S-module J is $\mathbb{Z}^n$*-graded* if $J = \bigoplus_{\mathbf{b} \in \mathbb{Z}^n} J_\mathbf{b}$ and $\mathbf{x}^\mathbf{a} J_\mathbf{b} \subseteq J_{\mathbf{a}+\mathbf{b}}$ for all monomials $\mathbf{x}^\mathbf{a} \in S$. Thus the polynomial ring S is $\mathbb{Z}^n$-graded, as is the localization $S[x_i^{-1} \mid i \in F]$ for any subset $F \subseteq \{1, \ldots, n\}$.

Definition 2.1. Define the *Matlis dual* $J^\vee$ of J by

$$(J^\vee)_{-\mathbf{b}} = \mathrm{Hom}_k(J_\mathbf{b}, k),$$

with S-module structure determined by letting $\mathbf{x}^\mathbf{a} : (J^\vee)_\mathbf{b} \to (J^\vee)_{\mathbf{a}+\mathbf{b}}$ be the transpose of $\mathbf{x}^\mathbf{a} : J_{-\mathbf{a}-\mathbf{b}} \to J_{-\mathbf{b}}$.

It is obvious from this definition that Matlis duality is an exact contravariant functor on $\mathbb{Z}^n$-graded modules, and that $(J^\vee)^\vee = J$ if $\dim_k J_\mathbf{b} < \infty$ for all $\mathbf{b} \in \mathbb{Z}^n$ (such a module J is said to be $\mathbb{Z}^n$*-finite*). To orient the reader, this Matlis duality restricts to the usual one between finitely generated and artinian $\mathbb{Z}^n$-graded modules (see Lemma 2.3), although this fact won't arise here.

Example 2.2. The Matlis dual of an S-module that is expressed as

$$J = k[x_i \mid i \in F][x_j^{-1} \mid j \in G] \quad \text{is} \quad J^\vee = k[x_i^{-1} \mid i \in F][x_j \mid j \in G].$$

It is not necessary that F and G be disjoint, or that $F \supseteq G$, or that their union be $\{1, \ldots, n\}$; when $J_\mathbf{b} \neq 0$ but $J_{\mathbf{a}+\mathbf{b}} = 0$, it is understood that $\mathbf{x}^\mathbf{a} J_\mathbf{b} = 0$. The easiest and most important example along these lines is when $J = S$, so that $J^\vee = S^\vee = k[x_1^{-1}, \ldots, x_n^{-1}]$ is the *injective hull* of k. □

Let R be any $\mathbb{Z}^n$-graded ring (we will use only S and its $\mathbb{Z}^n$-graded subring k concentrated in degree $\mathbf{0}$). A map $\phi : M \to N$ of $\mathbb{Z}^n$-graded R-modules is called *homogeneous of degree* $\mathbf{b} \in \mathbb{Z}^n$ (or just *homogeneous* when $\mathbf{b} = \mathbf{0}$)

if $\phi(M_{\mathbf{c}}) \subseteq N_{\mathbf{b}+\mathbf{c}}$. When $R = S$ and $\mathbf{b}$ is fixed, the set of such maps is a k-vector space denoted

$$\underline{\mathrm{Hom}}_S(M,N)_{\mathbf{b}} = \text{ degree } \mathbf{b} \text{ homogeneous maps } M \to N.$$

As the notation suggests,

$$\underline{\mathrm{Hom}}_S(M,N) = \bigoplus_{\mathbf{b}\in\mathbb{Z}^n} \underline{\mathrm{Hom}}_S(M,N)_{\mathbf{b}}$$

is a $\mathbb{Z}^n$-graded S-module, with $\mathbf{x}^{\mathbf{a}}\phi$ defined by $(\mathbf{x}^{\mathbf{a}}\phi)(m) = \mathbf{x}^{\mathbf{a}}(\phi m) = \phi(\mathbf{x}^{\mathbf{a}} m)$. Matlis duality can now be expressed without resorting to degree-by-degree vector space duals.

Lemma 2.3. *For any $\mathbb{Z}^n$-graded modules J and M,*

$$\underline{\mathrm{Hom}}_S(J, M^\vee) = (M \otimes_S J)^\vee = \underline{\mathrm{Hom}}_S(M, J^\vee).$$

In particular, $J^\vee = \underline{\mathrm{Hom}}_S(J, S^\vee)$.

Proof. $J^\vee$ can be expressed as the $\mathbb{Z}^n$-graded module $\underline{\mathrm{Hom}}_k(J,k)$, because a k-vector space homomorphism $J \to k$ that is homogeneous of degree $\mathbf{b}$ is the same thing as a vector space map $J_{-\mathbf{b}} \to k$, the k being concentrated in degree $\mathbf{0}$. The result is a consequence of the adjointness between $\underline{\mathrm{Hom}}$ and $\otimes$ that holds for arbitrary $\mathbb{Z}^n$-graded k-algebras S and S-modules J, M:

$$\underline{\mathrm{Hom}}_k(M \otimes J, k) = \underline{\mathrm{Hom}}_S(M, \underline{\mathrm{Hom}}_k(J,k)) = \underline{\mathrm{Hom}}_S(M, J^\vee).$$

That M and J can be switched is by the symmetry of $\otimes$. □

Matlis duality switches flat and injective modules in the category of $\mathbb{Z}^n$-graded modules.

Lemma 2.4. *$J \in \mathcal{M}$ is flat if and only if $J^\vee$ is injective.*

Proof. The functor $\underline{\mathrm{Hom}}_S(-, J^\vee)$ on the right in

$$\underline{\mathrm{Hom}}_k(- \otimes_S J, k) = \underline{\mathrm{Hom}}_S(-, \underline{\mathrm{Hom}}_k(J,k)) .$$

is exact $\Leftrightarrow$ the functor on the left is exact $\Leftrightarrow$ $- \otimes_S J$ is, because k is a field. □

Example 2.5. The $\mathbb{Z}^n$-graded dualizing complex for S is the Matlis dual of the Čech complex on $x_1, \ldots, x_n$, appropriately shifted homologically. □

See [GW78, Sta96, Mil00, MP01] for more on the category of $\mathbb{Z}^n$-graded modules and its relation to combinatorial commutative algebra.

2.2. **The Čech hull.**

Definition 2.6 ([Mil00]). *The* Čech hull *of a $\mathbb{Z}^n$-graded module M is the $\mathbb{Z}^n$-graded module $\check{C}M$ whose degree $\mathbf{b}$ piece is*

$$(\check{C}M)_{\mathbf{b}} = M_{\mathbf{b}_+} \quad \text{where} \quad \mathbf{b}_+ = \sum_{b_i \geq 0} b_i \mathbf{e}_i$$

is the positive part of $\mathbf{b}$*. Equivalently,*

$$\check{C}M = \bigoplus_{\mathbf{b}\in\mathbb{N}^n} M_{\mathbf{b}} \otimes_k k[x_i^{-1} \mid b_i = 0].$$

If $\mathbf{e}_i \in \mathbb{Z}^n$ *is the* i^{th} *standard basis vector, the action of multiplication by* x_i *is*

$$\cdot x_i : (\check{C}M)_{\mathbf{b}} \to (\check{C}M)_{\mathbf{e}_i+\mathbf{b}} = \begin{cases} \text{identity} & \text{if } b_i < 0 \\ \cdot x_i : M_{\mathbf{b}_+} \to M_{\mathbf{e}_i+\mathbf{b}_+} & \text{if } b_i \geq 0 \end{cases},$$

Note that $(\mathbf{e}_i+\mathbf{b})_+ = \mathbf{b}_+$ *whenever* $b_i < 0$*, and* $(\mathbf{e}_i+\mathbf{b})_+ = \mathbf{e}_i+\mathbf{b}_+$ *whenever* $b_i \geq 0$*.*

Heuristically, the first description of $\check{C}M$ in the definition says that if you want to know what $\check{C}M$ looks like in degree $\mathbf{b} \in \mathbb{Z}^n$, then check what M looks like in the nonnegative degree closest to $\mathbf{b}$; the second description says that the vector space $M_{\mathbf{a}}$ for $\mathbf{a} \in \mathbb{N}^n$ is copied into all degrees $\mathbf{b}$ such that $\mathbf{b}_+ = \mathbf{a}$. The Čech hull "forgets" everything about M that isn't in $\mathbb{N}^n$.

The Čech hull can just as well be applied to a homogeneous map of degree $\mathbf{0}$ between two modules, by copying the maps in the $\mathbb{N}^n$-graded degrees as prescribed. Some properties of $\check{C}$ are now immediate from checking things degree by degree.

Lemma 2.7. *The Čech hull is an exact covariant functor on* $\mathbb{Z}^n$*-graded modules.*

The utility of the Čech hull stems from its ability to localize free modules and take injective hulls of quotients by primes simultaneously. For instance, $\check{C}S = S[x_1^{-1}, \dots, x_n^{-1}]$, while $\check{C}(S/\mathfrak{m}) = S^\vee$; see [Mil00, Example 2.8] for more details. Here, the applications require only the localization property.

The notation henceforth is as follows. For $\mathbf{b} \in \mathbb{Z}^n$, the $\mathbb{Z}^n$-graded shift $M(\mathbf{b})$ is the module satisfying $M(\mathbf{b})_{\mathbf{c}} = M_{\mathbf{b}+\mathbf{c}}$. Thus, for $\mathbf{a} \in \mathbb{N}^n$, the free module of rank 1 generated in degree $\mathbf{a}$ is $S(-\mathbf{a})$. If $F \in \{0,1\}^n$, the localization $M[\mathbf{x}^{-F}]$ is $M \otimes_S S[x_i^{-1} \mid F_i = 1]$. Setting $\mathbf{1} = (1, \dots, 1)$, the next lemma is straightforward.

Lemma 2.8. *If* $F \in \{0,1\}^n$ *then* $\check{C}(S(F - \mathbf{1})) \cong S(F - \mathbf{1})[\mathbf{x}^{-F}]$*.*

Lemma 2.10 will clarify the pertinence of the next definition.

Definition 2.9. Define the t^{th} *Frobenius power* of a monomial ideal $I = \langle m_1, \dots, m_r\rangle$ to be $I^{[t]} = \langle m_1^t, \dots, m_r^t\rangle$ for $1 \leq t \in \mathbb{Z}$. Any direct sum $\mathcal{F} = \bigoplus_j S(-F_j)$ with each $F_j \in \mathbb{N}^n$ is isomorphic to a direct sum $\bigoplus_j \langle \mathbf{x}^{F_j}\rangle$ of ideals, so $\mathcal{F}^{[t]} = \bigoplus_j S(-tF_j)$ is similarly defined.

The advantage to Frobenius powers is that their free resolutions are all related (no assumption is required on the characteristic of S). Let $\mathcal{F}_{\bullet}$ be a free resolution of S/I. Then there is an induced free resolution $\mathcal{F}_{\bullet}^{[t]}$ of $S/I^{[t]}$: choose matrices for the differentials in $\mathcal{F}_{\bullet}$ and replace every occurrence of x_i with x_i^t, for all i. Equivalently, if $\varphi^{[t]}$ is the k-algebra isomorphism mapping

$S = k[x_1, \dots, x_n]$ to $S^{[t]} = k[x_1^t, \dots, x_n^t]$ via $x_i \mapsto x_i^t$ for all i, then $\mathcal{F}_\bullet$ can be considered as a complex $\varphi^{[t]}(\mathcal{F}_\bullet)$ of $S^{[t]}$-modules, and $\mathcal{F}_\bullet^{[t]} = S \otimes_{S^{[t]}} \varphi^{[t]}(\mathcal{F}_\bullet)$ (the tensor product over $S^{[t]}$ is via the inclusion $S^{[t]} \hookrightarrow S$). This implies that $\mathcal{F}_\bullet^{[t]}$ is indeed acyclic, since S is a free (and hence flat) $S^{[t]}$-module. For the $\mathbb{Z}^n$-graded shifts of the summands in $\mathcal{F}_\bullet$ to work out properly, $S^{[t]}$ must be graded by the sublattice $t\mathbb{Z}^n \subseteq \mathbb{Z}^n$ if S insists on being graded by $\mathbb{Z}^n$.

For each $t \geq 1$ there is an inclusion $\mathcal{F}_\bullet^{[t+1]} \to \mathcal{F}_\bullet^{[t]}$ of complexes via the homogeneous degree $\mathbf{0}$ maps $S(-(t+1)F) \to S(-tF)$ sending $1 \mapsto \mathbf{x}^F$. This makes $\{\mathcal{F}_\bullet^{[t+1]}\}$ into an inverse system of complexes. Just as with Koszul complexes, setting $\omega_S = S(\mathbf{1})$ and applying $\underline{\mathrm{Hom}}_S(-, \omega_S)$ to $\{\mathcal{F}_\bullet^{[t]}\}$ yields a directed system $\{\mathcal{F}^\bullet_{[t]}\}$, in which $S(tF - \mathbf{1}) \to S((t+1)F - \mathbf{1})$ via the natural inclusion $1 \mapsto \mathbf{x}^F$.

Assuming now that I is squarefree, $\mathcal{F}_\bullet$ can be chosen so that all of the summands $S(-F_j)$ of Definition 2.9 are in *squarefree degrees* $F_j \in \{0,1\}^n$. (This may not be obvious, but will follow from Lemma 3.2.) In this case $\mathcal{F}^\bullet = \underline{\mathrm{Hom}}_S(\mathcal{F}_\bullet, \omega_S)$ also has all of its summands in squarefree degrees, since $\underline{\mathrm{Hom}}_S(S(-F), \omega_S) = S(F - \mathbf{1})$. The reason for introducing Frobenius powers is the next lemma, to be applied in Section 4.

Lemma 2.10. *Let $\mathcal{F}_\bullet$ be a minimal free resolution of S/I and $\omega_S = S(\mathbf{1})$. If $\mathcal{F}^\bullet_{[t]} = \underline{\mathrm{Hom}}_S(\mathcal{F}_\bullet^{[t]}, \omega_S)$ then $\varinjlim_t \mathcal{F}^\bullet_{[t]} = \check{C}\mathcal{F}^\bullet$.*

Proof. $\mathcal{F}^\bullet$ is composed of maps $S(G - \mathbf{1}) \to S(F - \mathbf{1})$ between free modules generated in squarefree degrees. Since $\varinjlim S(tG - \mathbf{1}) = S(G - \mathbf{1})[\mathbf{x}^{-G}]$ and $\varinjlim S(tF - \mathbf{1}) = S(F - \mathbf{1})[\mathbf{x}^{-F}]$ are the corresponding localizations, it follows that $\varinjlim \mathcal{F}^\bullet_{[t]}$ is composed of natural inclusions $S(G - \mathbf{1})[\mathbf{x}^{-G}] \to S(F - \mathbf{1})[\mathbf{x}^{-F}]$. Now use Lemma 2.8. □

All of the $\mathbb{Z}^n$-graded shifts by $\mathbf{1}$ in the preceding are essential, because taking Čech hulls rarely commutes with such shifts. However, the restriction of minimality in Lemma 2.10 is unnecessary, as is clear from the proof: being generated in squarefree degrees will do. See Proposition 3.3 and Example 3.4 for examples.

Remark 2.11. The colimit $\varinjlim_t J_t$ of a directed system of $\mathbb{Z}^n$-graded modules is a quotient of $\bigoplus_t J_t$ by a $\mathbb{Z}^n$-graded submodule. It is therefore naturally $\mathbb{Z}^n$-graded.

Remark 2.12. In the context of Lemma 2.8, Yanagawa's notion of *straight hull* [Yan00] coincides with the Čech hull. The Čech hull is generalized in [HM00] to algebras graded by arbitrary semigroups, where it isn't exact. Its derived functors yield a spectral sequence of Ext modules converging to local cohomology, and are used to prove the finiteness for Bass numbers of graded local cohomology of finitely generated modules over simplicial semigroup rings. It is also shown that the local cohomology of some finitely generated module has infinite Bass numbers if the semigroup is not simplicial.

3. The Čech hull and local cohomology

3.1. **Generalized Čech complexes.** This section reviews the connection between the Čech hull and local cohomology [Mil00, Section 6] from a new point of view.

Let $\omega_S = S(-\mathbf{1})$ be the canonical module of S.

Definition 3.1. If $\mathcal{F}_{\bullet}$ is a free resolution of S/I and $\mathcal{F}^{\bullet} = \underline{\mathrm{Hom}}(\mathcal{F}_{\bullet}, \omega_S)$, define

$$\check{\mathcal{C}}^{\bullet}_{\mathcal{F}} = (\check{C}\mathcal{F}^{\bullet})(\mathbf{1})$$

to be the *generalized Čech complex* determined by $\mathcal{F}_{\bullet}$. When $\mathcal{F}_{\bullet}$ is minimal, $\check{\mathcal{C}}^{\bullet}_{\mathcal{F}}$ is called the *canonical Čech complex* for I, and is denoted by $\check{\mathcal{C}}^{\bullet}_{I}$.

The first task, Proposition 3.3, is to identify the usual Čech complex on generators for I as the generalized Čech complex for a certain (usually far from minimal) free resolution. Suppose $\mathbf{x}^{F_1}, \ldots, \mathbf{x}^{F_r} \in S$ are monomials (they don't need to be squarefree, yet). Given a subset $X \subseteq \{1, \ldots, r\}$, define F_X so that $\mathbf{x}^{F_X}$ is the least common multiple of the monomials $\mathbf{x}^{F_j}$ for $j \in X$. The *Taylor resolution* $\mathcal{T}_{\bullet}$ on $\mathbf{x}^{F_1}, \ldots, \mathbf{x}^{F_r}$ is

$$0 \leftarrow S \leftarrow \bigoplus_{j=1}^{r} S(-F_j) \leftarrow \cdots \leftarrow \bigoplus_{|X|=\ell} S(-F_X) \leftarrow \cdots \leftarrow S(-F_{\{1,\ldots,r\}}) \leftarrow 0,$$

where the map $S(-F_{X\setminus j}) \leftarrow S(-F_X)$ is $(-1)^{s-1}$ times the natural inclusion if j is the s^{th} element of X.

Lemma 3.2 ([Tay60]). *$\mathcal{T}_{\bullet}$ is a free resolution of $S/\langle \mathbf{x}^{F_1}, \ldots, \mathbf{x}^{F_r}\rangle$.*

Proof. The $\mathbb{Z}^n$-degree $\mathbf{b}$ piece of $\mathcal{T}_{\bullet}$ is zero unless $\mathbf{b} \in \mathbb{N}^n$, in which case $(\mathcal{T}_{\bullet})_{\mathbf{b}}$ is the reduced chain complex of the simplex whose vertices are $\{j \mid F_j \preceq \mathbf{b}\}$, with $\varnothing$ in homological degree 0. This chain complex has no homology unless $\mathbf{x}^{\mathbf{b}} \notin \langle \mathbf{x}^{F_1}, \ldots, \mathbf{x}^{F_r}\rangle$, when the only homology is k in homological degree 0. Thus $\mathcal{T}_{\bullet}$ is a free resolution of something. The image of the last map (to S) in $\mathcal{T}_{\bullet}$ is obviously $\langle \mathbf{x}^{F_1}, \ldots, \mathbf{x}^{F_r}\rangle$. □

The next result is needed for Proposition 3.7. It identifies the Čech complex not just as a limit of Koszul cochain complexes on generators for $I^{[t]}$, whose duals $\mathcal{K}^{[t]}_{\bullet}$ may not be acyclic, but as the result of applying the Čech hull and shift by $\mathbf{1}$ operations, which are exact, to a complex $\mathcal{T}^{\bullet} = \underline{\mathrm{Hom}}_S(\mathcal{T}_{\bullet}, \omega_S)$ whose dual is a *resolution* $\mathcal{T}_{\bullet}$ of S/I.

Proposition 3.3. *If $I = \langle \mathbf{x}^{F_1}, \ldots, \mathbf{x}^{F_r}\rangle$ is generated by squarefree monomials, so $F_j \in \{0,1\}^n$ for all j, then $\check{\mathcal{C}}^{\bullet}_{\mathcal{T}}$ is the usual Čech complex*

$$0 \to S \to \bigoplus_{j=1}^{r} S[\mathbf{x}^{-F_j}] \to \cdots \to \bigoplus_{|X|=\ell} S[\mathbf{x}^{-F_X}] \to \cdots \to S[\mathbf{x}^{-F_{\{1,\ldots,r\}}}] \to 0$$

on the monomials $\mathbf{x}^{F_1}, \ldots, \mathbf{x}^{F_r}$ generating I.

Proof. Each original summand $S(-F)$ in $\mathcal{T}_{\bullet}$ turns into a corresponding summand $S(F-\mathbf{1}) = \underline{\mathrm{Hom}}_S(S(-F), \omega_S)$ in $\underline{\mathrm{Hom}}_S(\mathcal{T}_{\bullet}, \omega_S)$. Lemma 2.8 says $\check{C}(S(F-\mathbf{1})) \cong S[\mathbf{x}^{-F}](F-\mathbf{1})$, and this is isomorphic to $S[\mathbf{x}^{-F}](-\mathbf{1})$ because multiplication by $\mathbf{x}^F$ is a homogeneous degree F automorphism. Shifting by $\mathbf{1}$ yields the result. □

The proof of Proposition 3.3 says how to describe any generalized Čech complex $\check{\mathcal{C}}^{\bullet}_{\mathcal{F}}$ in more familiar terms: after choosing bases in the complex $\mathcal{F}^{\bullet} = \underline{\mathrm{Hom}}_S(\mathcal{F}_{\bullet}, \omega_S)$, simply replace each summand $S(F-\mathbf{1})$ by the localization $S[\mathbf{x}^{-F}](\mathbf{1})$.

Example 3.4. The usual Čech complex $\check{\mathcal{C}}^{\bullet}_{\mathfrak{m}} = \check{\mathcal{C}}^{\bullet}(x_1, \ldots, x_n)$ is the canonical Čech complex of the maximal $\mathbb{Z}^n$-graded ideal $\mathfrak{m}$. In other words, *the Cech hull of the Koszul complex is the Čech complex*, up to a $\mathbb{Z}^n$-graded shift by $\mathbf{1}$. This example, or more generally Proposition 3.3, is the reason for the term "Čech hull". □

Remark 3.5. The canonical Čech complex of I depends (up to isomorphism of $\mathbb{Z}^n$-graded complexes) only on I, not on any system of $\mathbb{Z}^n$-graded generators of I. The term "canonical Čech complex" refers both to this freedom from choices and the fact that it is, up to $\mathbb{Z}^n$-graded and homological shift, the Čech hull of a free resolution of the canonical module $\omega_{S/I}$ when S/I is Cohen-Macaulay.

3.2. **Local cohomology.** If one is convinced that the minimal free resolution of S/I is in any sense "better" than the Taylor resolution (or any other free resolution), then one should be equally convinced that the canonical Čech complex $\check{\mathcal{C}}^{\bullet}_I$ is similarly "better" than the usual Čech complex $\check{\mathcal{C}}^{\bullet}_{\mathcal{T}}$ on the minimal generators of I, by Proposition 3.3. Of course, the main use for the usual Čech complex is in defining the *local cohomology modules*

$$H^i_I(M) = H^i(M \otimes_S \check{\mathcal{C}}^{\bullet}_{\mathcal{T}}),$$

and one needs to be convinced that the canonical Čech complex is just as good at this local cohomology computation. In fact, this holds for any generalized Čech complex.

Theorem 3.6 ([Mil00])**.** *If $\mathcal{F}_{\bullet}$ is a free resolution of S/I and M is any S-module, then*

$$H^i_I(M) = H^i(M \otimes_S \check{\mathcal{C}}^{\bullet}_{\mathcal{F}}).$$

Although it is possible to give a concrete proof using Lemma 2.10 as in [Mil00, Theorem 6.2], a new and more conceptual proof is appropriate here, in anticipation of $\mathbb{Z}^n$-graded Greenlees-May duality in Section 5. The proof below pinpoints the relation between the canonical Čech complex and the usual Čech complex as being analogous to—and a consequence of—the relation between the minimal free resolution of S/I and the Taylor resolution. It rests on Proposition 3.7, which is the observation that propels the rest of the paper: Theorem 3.6 and Corollary 4.8 depend on Proposition 3.7, and Proposition 6.1 is modelled upon it.

In what follows, a *homology isomorphism* of complexes (also known as a quasi-isomorphism) is a map inducing an isomorphism on homology.

Proposition 3.7. *If $\mathcal{G}_{\bullet}$ and $\mathcal{F}_{\bullet}$ are free resolutions of S/I, then there is a homology isomorphism $\check{\mathcal{C}}^{\bullet}_{\mathcal{G}} \to \check{\mathcal{C}}^{\bullet}_{\mathcal{F}}$ between their corresponding generalized Čech complexes.*

Proof. If $\mathcal{F}^I_{\bullet}$ is a minimal free resolution of S/I, then there are homology isomorphisms $\mathcal{F}_{\bullet} \to \mathcal{F}^I_{\bullet}$ and $\mathcal{F}^I_{\bullet} \to \mathcal{G}_{\bullet}$ because $\mathcal{F}^I_{\bullet}$ is a split subcomplex of every free resolution of S/I [Eis95, Theorem 20.2]. Composing these, we have a homology isomorphism $\mathcal{F}_{\bullet} \to \mathcal{G}_{\bullet}$. Applying $\underline{\mathrm{Hom}}_S(-,\omega_S)$ yields a map $\mathcal{G}^{\bullet} \to \mathcal{F}^{\bullet}$ whose induced map on cohomology is an isomorphism: both have cohomology $\underline{\mathrm{Ext}}^{\bullet}_S(S/I,\omega_S)$. The desired homology isomorphism is therefore simply $(\check{C}\mathcal{G}^{\bullet})(\mathbf{1}) \to (\check{C}\mathcal{F}^{\bullet})(\mathbf{1})$, since the Čech hull and the shift by $\mathbf{1}$ are both exact. □

It's worth stating explicitly the following lemma, even though it is standard (so its proof, the Künneth spectral sequence, is omitted), because it will be used so often. A *bounded below* complex is one that has nonzero modules only in positive homological degrees; thus free resolutions of modules are bounded below.

Lemma 3.8. *If $\mathcal{L}_{\bullet} \to \mathcal{L}'_{\bullet}$ is a homology isomorphism of bounded below complexes of flat modules and M is any module, then $M \otimes \mathcal{L}_{\bullet} \to M \otimes \mathcal{L}'_{\bullet}$ is a homology isomorphism.*

Proof of Theorem 3.6. Proposition 3.7 produces a homology isomorphism $\check{\mathcal{C}}^{\bullet}_{\mathcal{T}(I)} \to \check{\mathcal{C}}^{\bullet}_{\mathcal{F}}$. Since both of the complexes $\check{\mathcal{C}}^{\bullet}_{\mathcal{T}(I)}$ and $\check{\mathcal{C}}^{\bullet}_{\mathcal{F}}$ are bounded below (and above), Lemma 3.8 says that $H^i_I(M) = H^i(M \otimes \check{\mathcal{C}}^{\bullet}_{\mathcal{T}(I)}) \to H^i(M \otimes \check{\mathcal{C}}^{\bullet}_{\mathcal{F}})$ is an isomorphism. □

Remark 3.9. If the module M is $\mathbb{Z}^n$-graded, then the isomorphism in Theorem 3.6 produces the natural $\mathbb{Z}^n$-grading on $H^i_I(M)$, since the right-hand side is still $\mathbb{Z}^n$-graded.

4. The Čech hull and completion

4.1. **Graded completion.** Suppose for the moment that I is an ideal in an ungraded commutative ring A. Just as Γ_I takes the direct limit of submodules annihilated by powers of I, the *I-adic completion* functor Λ^I takes the inverse limit of quotients annihilated by powers of I. More precisely, any A-module M has a filtration by submodules I^tM for $1 \leq t \in \mathbb{Z}$, giving rise to an inverse system $M/I^{t+1}M \to M/I^tM$. The I-adic completion $\Lambda^I(M)$ is then defined as the inverse limit of this system, $\varprojlim_t M/I^tM$.

In general, I-adic completion is neither left exact nor right exact (intuition from finitely generated modules over noetherian rings fails badly). However, one can still define the left derived functors $L^I_{\bullet}(M)$ of Λ^I by taking a free resolution of M and applying Λ^I to it. The catch is that the natural map from the zeroth homology $L^I_0(M)$ of the resulting complex to $\Lambda^I(M)$ need not be an isomorphism, as it would be if Λ^I were right exact.

Everything in the last two paragraphs (and everything in the next subsection, as well) makes complete (!) sense, mutatis mutandis, in the category of modules graded by some group or monoid over a similarly graded algebra. This includes the case of the $\mathbb{Z}^n$-graded polynomial k-algebra S and $\mathbb{Z}^n$-graded modules, to which we return for simplicity of exposition. The phrase "mutatis mutandis" here means only that the inverse limits defining the $\mathbb{Z}^n$-graded completion $\underline{\Lambda}^I$ must be taken in the category of $\mathbb{Z}^n$-graded objects and homogeneous maps *of degree zero*. Recall what this means for (say) an inverse system $\{\mathcal{F}^t\}$ of chain complexes of $\mathbb{Z}^n$-graded S-modules: the $\mathbb{Z}^n$-graded inverse limit ${}^*\varprojlim$ is defined by

$$
{}^*\varprojlim_t \mathcal{F}^t = \bigoplus_{\mathbf{b}\in\mathbb{Z}^n} \varprojlim_t \mathcal{F}^t_{\mathbf{b}}, \tag{1}
$$

where the ordinary inverse limits of the degree $\mathbf{b}$ pieces $\mathcal{F}^t_{\mathbf{b}}$ on the right are in the category of chain complexes of k-vector spaces.

Example 4.1. The polynomial ring S is $\mathbb{Z}^n$-graded complete with respect to its maximal ideal $\mathfrak{m} = \langle x_1, \ldots, x_n\rangle$, and hence with respect to every $\mathbb{Z}^n$-graded ideal I. Indeed, for any finitely generated module M, ideal I, and fixed $\mathbf{b}$, the inverse system $(M/I^{t+1}M)_{\mathbf{b}} \twoheadrightarrow (M/I^tM)_{\mathbf{b}}$ eventually stabilizes to become $M_{\mathbf{b}} \xrightarrow{\sim} M_{\mathbf{b}}$, so $\underline{\Lambda}^I(M) = M$. On the other hand, infinitely generated modules may behave differently:

1. Completion of the injective hull $S^\vee$ of k yields $\underline{\Lambda}^I(S^\vee) = 0$, since $I^tS^\vee = S^\vee$ for all nonzero ideals I^t. However, see Example 4.6 for $\underline{L}^I_{\bullet}(S^\vee)$.
2. As in the ungraded case, $\underline{\Lambda}^I(M) = 0$ if $IM = M$. This can't happen if $M \neq 0$ is finitely generated by Nakayama's Lemma, but can occur for localizations $M[\mathbf{x}^{-F}]$ (the notation is explained before Lemma 2.8).

Since graded modules behave so much like vector spaces, truly bad behavior of completion requires some infinite dimensionality. With one variable x, for instance, let $I = \langle x\rangle \subset k[x]$ and $M = \bigoplus_{b\in\mathbb{Z}} k[x](-b)$. The completion $\underline{\Lambda}^I(M)$ has the vector space $\underline{\Lambda}^I(M)_a = \prod_{b\leq a} k$ in $\mathbb{Z}$-degree a, with multiplication by x being inclusion. Observe, however, that when all of the direct summands are generated in the same $\mathbb{Z}$-degree, the module $\bigoplus_{b\in\mathbb{Z}} k[x] = \underline{\Lambda}^I(\bigoplus_{b\in\mathbb{Z}} k[x])$ is fixed by completion.

For comparison, the ungraded completion $\Lambda^I(\bigoplus_{b\in\mathbb{Z}} k[[x]])$ is even more infinite than $\underline{\Lambda}^I(M)$. The former contains any sequence $(x^{d_b})_{b\in\mathbb{Z}}$ in which, for each fixed d, there are finitely many b satisfying $d_b \leq d$, whereas the latter contains only semi-infinite sequences. □

The goal of this section is the computation in Theorem 4.4 of $\underline{L}^I_{\bullet}$ in terms of generalized Čech complexes.

4.2. **Telescopes and microscopes.** This section follows [GM92], but with gradings. Given a homogeneous map $\phi : Z^\bullet \to Y^\bullet$ of cochain complexes with differentials δ_z and δ_y, form the associated double complex $Y \oplus_\phi Z$ with $(Y^\bullet, \delta_y)$ in the 0^{th} row; $(Z^\bullet, -\delta_z)$ in row -1; and vertical differential $\phi : Z^\bullet \to Y^\bullet$. The *cone* or *cofiber* $\mathrm{cone}(\phi)$ is the total complex $\mathrm{Tot}\,(Y \oplus_\phi Z)$, which has $Y^i \oplus Z^{i+1}$ as its term in cohomological degree i and differential $\delta(y^i, z^{i+1}) = (\delta_y y^i + \phi z^{i+1}, -\delta_z z^{i+1})$. Defining $Z[t]^\bullet$ by $Z[t]^i = Z^{t+i}$ with differential $(-1)^t \delta_z$, there is a short exact sequence $0 \to Y \to \mathrm{cone}(\phi) \to Z[1] \to 0$ whose long exact cohomology sequence has connecting homomorphism ϕ.

Any cochain complex $Y^\bullet$ is also a chain complex $Y_\bullet$ with $Y^i = Y_{-i}$. Define the *fiber* of $\psi : Y_\bullet \to Z_\bullet$ by $\mathrm{fiber}(\psi) = \mathrm{cone}(-\psi)[-1]$. This is designed precisely so that if $\phi : \tilde{Z}^\bullet \to \tilde{Y}^\bullet$ is a map of cochain complexes of (say) k-vector spaces and $\psi : Y_\bullet \to Z_\bullet$ is its transpose, so $\mathrm{Hom}_k(\tilde{Y}^i, k) = Y_i$, then $\mathrm{fiber}(\psi)$ is the transpose of $\mathrm{cone}(\phi)$.

Now suppose $\{\phi_t : \mathcal{G}_t^\bullet \to \mathcal{G}_{t+1}^\bullet\}_{t\geq 1}$ is a sequence of cochain maps, homogeneous of degree zero, and define $\phi : \bigoplus \mathcal{G}_t^\bullet \to \bigoplus \mathcal{G}_t^\bullet$ by $\phi(x) = x - \phi_t(x)$ for $x \in \mathcal{G}_t^\bullet$. The *homotopy colimit* or *telescope* of the sequence $\{\phi_t\}_{t\geq 1}$ is $\mathrm{cone}(\phi)$, and denoted $\mathrm{Tel}\,(\mathcal{G}_t^\bullet)$. Although the projection $\mathrm{Tel}\,(\mathcal{G}_t^\bullet) \to \bigoplus \mathcal{G}_t^\bullet$ to the target of ϕ (corresponding to $Y^\bullet$ above) is not a morphism of cochain complexes, it becomes so after modding out by the image of ϕ. The resulting cochain map $\mathrm{Tel}\,(\mathcal{G}_t^\bullet) \to (\bigoplus \mathcal{G}_t^\bullet)/\mathrm{image}\,\phi = \varinjlim \mathcal{G}_t^\bullet$ is a homology isomorphism, identifying $H^i\mathrm{Tel}\,(\mathcal{G}_t^\bullet)$ as $\varinjlim_t (H^i \mathcal{G}_t^\bullet)$.

Dual to the telescope is the *homotopy limit* or *microscope* $\underline{\mathrm{Mic}}\,(\mathcal{G}_\bullet^t)$ of a sequence of homogeneous degree zero chain maps $\{\psi^t : \mathcal{G}_\bullet^{t+1} \to \mathcal{G}_\bullet^t\}_{t\geq 1}$. It is defined as $\mathrm{fiber}(\psi)$ for the map $\psi : \prod^* \mathcal{G}_\bullet^t \to \prod^* \mathcal{G}_\bullet^t$ sending the element $(x^t)_{t\geq 1} \mapsto (x^t - \psi^{t+1}(x^{t+1}))_{t\geq 1}$. The products here are in the category of graded modules, and thus do not agree with the usual products. They can be seen as special cases of ${}^*\varprojlim$; more concretely, $\prod^* \mathcal{G}_\bullet^t \subseteq \prod \mathcal{G}_\bullet^t$ is the submodule generated by arbitrary products of elements of the same homological degree *and* the same graded degree.

It is routine to verify the following relation between Tel and $\underline{\mathrm{Mic}}$ (check that the maps ϕ and ψ on the direct sum and product are dual, before actually taking the cone and fiber to get Tel and $\underline{\mathrm{Mic}}$).

Lemma 4.2. *If $\{\mathcal{G}_t^\bullet\}$ is a sequence of cochain complexes and M is any module, then $\underline{\mathrm{Mic}}\ \underline{\mathrm{Hom}}_S(\mathcal{G}_t^\bullet, M) = \underline{\mathrm{Hom}}_S(\mathrm{Tel}\,\mathcal{G}_t^\bullet, M)$. In particular, if $\mathcal{G}_t^\bullet = \mathcal{F}_t^\bullet(\mathbf{1}) = \underline{\mathrm{Hom}}_S(\mathcal{F}_\bullet^t, S)$ for a sequence of free resolutions $\mathcal{F}_\bullet^t$ of some ideals I_t, then*

$$\underline{\mathrm{Hom}}_S(\mathrm{Tel}\,\mathcal{F}_t^\bullet(\mathbf{1}), M) = \underline{\mathrm{Mic}}\,(\mathcal{F}_\bullet^t \otimes M).$$

Note that $\underline{\mathrm{Mic}}$ depends on the grading while Tel doesn't (hence the underlining). The association of microscopes to inverse sequences constitutes an exact functor on inverse sequences, since $\prod^*$ is exact. Furthermore, setting $\mathcal{G}_\bullet^t = \underline{\mathrm{Hom}}_S(\mathcal{G}_t^\bullet, S)$ for a system of free complexes $\mathcal{G}_t^\bullet$, sending $M \mapsto \underline{\mathrm{Mic}}\,(\mathcal{G}_\bullet^t \otimes M)$ is an exact functor of M.

The whole point of microscopes is that they compute $\underline{L}^I_\bullet$. To get the precise statement, say that two chains $\{I_t\}$ and $\{I'_t\}$ of ideals are *cofinal* if each I_t is contained in some $I'_{t'}$ and each $I'_{s'}$ is contained in some I_s. For instance, the Frobenius powers $\{I^{[t]}\}$ of Definition 2.9 are cofinal with the ordinary powers $\{I^t\}$.

Lemma 4.3 ([GM92, Proposition 1.1]). *Suppose $\{I_t\}$ is a chain of ideals cofinal with the powers $\{I^t\}$, and let $\mathcal{F}^t_\bullet$ be an inverse sequence of free resolutions of S/I_t, lifting the inverse sequence of surjections $S/I_{t+1} \twoheadrightarrow S/I_t$. If $\mathcal{E}_\bullet$ is a free resolution of some module M, then the total complex of the double complex $\underline{\mathrm{Mic}}\,(\mathcal{F}^t_\bullet \otimes \mathcal{E}_\bullet)$ is homology isomorphic to both $\underline{\mathrm{Mic}}\,(\mathcal{F}^t_\bullet \otimes M)$ and $\underline{\Lambda}^I(\mathcal{E}_\bullet)$. In particular,*

$$H_i\underline{\mathrm{Mic}}\,(\mathcal{F}^t_\bullet \otimes M) = \underline{L}^I_i(M).$$

Proof. For any module J, the short exact sequence for the fiber defining the microscope runs $0 \to \prod^*(\mathcal{F}^t_{1+\bullet} \otimes J) \to \underline{\mathrm{Mic}}\,(\mathcal{F}^t_\bullet \otimes J) \to \prod^*(\mathcal{F}^t_\bullet \otimes J) \to 0$. The final four terms of the resulting long exact sequence are

$$H_0\underline{\mathrm{Mic}}\,(\mathcal{F}^t_\bullet \otimes J) \to \textstyle\prod^*(S/I_t \otimes J) \to \prod^*(S/I_t \otimes J) \to H_{-1}\underline{\mathrm{Mic}}\,(\mathcal{F}^t_\bullet \otimes J) \to 0.$$

The $H_{-1}\underline{\mathrm{Mic}}$ term is zero because every map in the inverse system $\{S/I_t \otimes J\}$ is surjective [Wei94, Lemma 3.5.3]. If J is free, all of the higher terms vanish because each $\mathcal{F}^t_\bullet$ is a resolution, and the $H_0\underline{\mathrm{Mic}}$ term is $\underline{\Lambda}^I(J)$ by the cofinality assumption. Setting $J = \mathcal{E}_\bullet$ this shows that the homology of the double complex $\underline{\mathrm{Mic}}\,(\mathcal{F}^t_\bullet \otimes \mathcal{E}_\bullet)$ in the $\mathcal{F}$ direction is $\underline{\Lambda}^I(\mathcal{E}_\bullet)$.

Taking homology in the $\mathcal{E}$ direction yields $\underline{\mathrm{Mic}}\,(\mathcal{F}^t_\bullet \otimes M)$ by exactness of the microscope construction. The standard spectral sequence argument now applies. □

4.3. **Derived functors of completion.** The main result, Theorem 4.4, says that for many modules M, the left derived functors of $\mathbb{Z}^n$-graded completion can be calculated using the flat complex $\check{\mathcal{C}}^\bullet_{\mathcal{F}}$ that is approximated by the telescope $\mathrm{Tel}\,\underline{\mathrm{Hom}}_S(\mathcal{F}^{[t]}_\bullet, S)$. (Recall that $\mathcal{F}^{[t]}_\bullet$ is the Frobenius power of a free resolution of S/I as in Definition 2.9). This theorem is interesting for three reasons. First, $\check{\mathcal{C}}^\bullet_{\mathcal{F}}$ is never projective, so it doesn't seem agile enough to detect derived functors. Second, its finitely many indecomposable summands make $\check{\mathcal{C}}^\bullet_{\mathcal{F}}$ much smaller than either a microscope or a telescope: $\check{\mathcal{C}}^\bullet_{\mathcal{F}}$ is $\mathbb{Z}^n$-*finite* when $\mathcal{F}_\bullet$ is, meanining that its degree $\mathbf{b}$ pieces are finite k-vector spaces for all $\mathbf{b} \in \mathbb{Z}^n$ (note that $\check{C}$ preserves $\mathbb{Z}^n$-finiteness). Finally, it allows the use of *any* free resolution to make the generalized Čech complex. This last point is crucial for Corollary 4.8.

Theorem 4.4. *If $\mathcal{F}_\bullet$ is a free resolution of S/I and $M = J^\vee$ for some J (this includes all $\mathbb{Z}^n$-finite modules M), then*

$$\underline{L}^I_i(M) = H_i\underline{\mathrm{Hom}}_S(\check{\mathcal{C}}^\bullet_{\mathcal{F}}, M).$$

More precisely, $\underline{\mathrm{Hom}}_S(\check{\mathcal{C}}^\bullet_{\mathcal{F}}, M)$ is homology isomorphic to $\underline{\mathrm{Mic}}\,(\mathcal{F}^{[t]}_\bullet \otimes M)$.

Noteworthy is the case when $\mathcal{F}_\bullet = \mathcal{T}_\bullet$ is the Taylor resolution (Lemma 3.2), because $\check{\mathcal{C}}^\bullet_{\mathcal{T}}$ is then the usual Čech complex on squarefree generators for I, by Proposition 3.3.

Proof. Let $\mathcal{F}^\bullet_{[t]} = \underline{\mathrm{Hom}}_S(\mathcal{F}^{[t]}_\bullet, \omega_S)$, as usual. Assuming for the time being that $\mathcal{F}_\bullet$ is minimal, consider the following diagram:

$$\begin{array}{rl} \underline{\mathrm{Hom}}_S(\check{\mathcal{C}}^\bullet_I, J^\vee) = (\varinjlim_t \mathcal{F}^\bullet_{[t]}(\mathbf{1}) \otimes J)^\vee & \\ \downarrow & \\ (\mathrm{Tel}\,\mathcal{F}^\bullet_{[t]}(\mathbf{1}) \otimes J)^\vee & = \underline{\mathrm{Hom}}_S(\mathrm{Tel}\,\mathcal{F}^\bullet_{[t]}(\mathbf{1}), J^\vee). \end{array}$$

Both "=" symbols use Lemma 2.3, while the first uses also Lemma 2.10 and Definition 3.1. The downward map is obtained by tensoring the homology isomorphism $\mathrm{Tel}\,\mathcal{F}^\bullet_{[t]}(\mathbf{1}) \to \varinjlim_t \mathcal{F}^\bullet_{[t]}(\mathbf{1})$ with J, and then taking Matlis duals. That the downward map is itself a homology isomorphism is an application of Lemma 3.8. The diagram therefore represents a homology isomorphism $\underline{\mathrm{Hom}}_S(\check{\mathcal{C}}^\bullet_I, J^\vee) \to \underline{\mathrm{Mic}}(\mathcal{F}^{[t]}_\bullet \otimes M)$ by Lemma 4.2, and since the homology is $\underline{L}^I_\bullet(M)$ by Lemma 4.3, the result is proved when $\mathcal{F}_\bullet$ is minimal.

When $\mathcal{F}_\bullet$ isn't minimal, the top end of the downward map is still homology isomorphic to $(\check{\mathcal{C}}^\bullet_I \otimes J)^\vee$ by Lemma 2.10, Proposition 3.7, and Lemma 3.8. □

Remark 4.5. The assumption $M = J^\vee$ for some J in Theorem 4.4 is annoying, but it's unclear how to get rid of it. Anyway, even the restriction of $\mathbb{Z}^n$-finiteness isn't so bad. It allows for a significant number of infinitely generated modules, including Čech hulls of finitely generated modules and their localizations.

Example 4.6. Using Theorem 4.4, we compute $\underline{L}^I_\bullet(S^\vee) = (H^i\check{\mathcal{C}}^\bullet_I)^\vee$. Since $H^i\check{\mathcal{C}}^\bullet_I = (\check{C}\underline{\mathrm{Ext}}^i(S/I, \omega_S))(\mathbf{1})$, we conclude that

$$\underline{L}^I_i(S^\vee) = (\check{C}\underline{\mathrm{Ext}}^i_S(S/I, \omega_S))^\vee(-\mathbf{1}).$$

By definition, $\underline{L}^I_i(S^\vee)$ is therefore the Čech hull of the *Alexander dual* of $\underline{\mathrm{Ext}}^i(S/I, \omega_S)$ as defined in [Mil00, Röm00], and more cleanly denoted by $\check{C}\underline{\mathrm{Ext}}^i_S(S/I, \omega_S)^{\mathbf{1}}$. No matter the notation, the vanishing and nonvanishing of $\underline{L}^I_i(S^\vee)$ follows the same pattern as $\underline{\mathrm{Ext}}^i_S(S/I, \omega_S)$, since the latter can be recovered from the former. In particular, $\underline{L}^I_i(S^\vee)$ is nonzero precisely for $d = \mathrm{codim}(I)$ if and only if S/I is Cohen-Macaulay, in which case

$$\underline{L}^I_d(S^\vee) = \check{C}\omega^{\mathbf{1}}_{S/I}$$

is the Čech hull of the Alexander dual of the canonical module. Notice that neither an injective hull nor a localization can substitute for the Čech hull here in relating $\underline{L}^I_i(S^\vee)$ to the finitely generated module $\underline{\mathrm{Ext}}^i_S(S/I, \omega_S)^{\mathbf{1}}$. □

4.4. **Local homology.** Closely related to the left derived functors of completion $\underline{L}^I_\bullet$ is another collection of functors $\underline{H}^I_\bullet$. In the definition to come, $\mathcal{K}^\bullet_t$ is the Koszul cochain complex on $m^t_1, \ldots, m^t_r$. To be precise about homological and $\mathbb{Z}^n$-grading, $\mathcal{K}^\bullet_t$ is the tensor product $\bigotimes_{j=1}^r (S \to S(\deg m^t_j))$, with

S in cohomological degree 0 and $S(\deg m_j^t)$, which is generated in $\mathbb{Z}^n$-degree $-\deg(m_j^t)$, in cohomological degree 1.

Definition 4.7 ([GM92, Definition 2.4]). *Let I be generated by $m_1, \ldots, m_r$. The* local homology *of M at I is*

$$\underline{H}_i^I(M) = H_i \underline{\mathrm{Hom}}_S(\mathrm{Tel}\, \mathcal{K}_t^\bullet, M).$$

Of course, this definition works for any finitely generated graded ideal in a commutative ring with any grading when the m_j are arbitrary homogeneous elements. For instance, in the ungraded setting, Greenlees and May proved that $H_\bullet^I = L_\bullet^I$ under mild assumptions on I and the ambient ring. In the present $\mathbb{Z}^n$-graded context over S, this isomorphism of functors is an easy corollary to Theorem 4.4.

Corollary 4.8. *If $M = J^\vee$ for some J, then $\underline{L}_\bullet^I(M) \cong \underline{H}_\bullet^I(M)$.*

Proof. Start with the homology isomorphism $\mathrm{Tel}\, \mathcal{K}_t^\bullet \to \varinjlim \mathcal{K}_t^\bullet = \check{\mathcal{C}}_{\mathcal{I}}^\bullet$ to the Čech complex, tensor with J, and take Matlis duals to get a homology isomorphism $(\mathrm{Tel}\, \mathcal{K}_t^\bullet \otimes J)^\vee \leftarrow (\check{\mathcal{C}}_{\mathcal{I}}^\bullet \otimes J)^\vee$, by Lemma 3.8. Now apply Lemma 2.3 and Theorem 4.4 with $\mathcal{F}_\bullet = \mathcal{T}_\bullet$ being the Taylor resolution. □

The convenient feature of Theorem 4.4 for this corollary is that the microscopes there already compute $\underline{L}_\bullet^I$, rather than $\underline{H}_\bullet^I$ as in [GM92]. Thus, to get $\underline{H}_\bullet^I \cong \underline{L}_\bullet^I$, it suffices to compare telescopes, which is easier than comparing microscopes because direct limits behave better than inverse limits. This point will resurface in Section 6.

5. $\mathbb{Z}^n$-graded Greenlees-May duality

5.1. **The $\mathbb{Z}^n$-graded derived category.** The theorem of Greenlees and May is a remarkable adjointness between local cohomology and the left derived functors of completion. It greatly generalizes the local duality theorem of Grothendieck, and unifies a number of other dualities after it has been appropriately sheafified; see [AJL97] for a detailed explanation of these assertions. As with the ungraded versions in [AJL97], the $\mathbb{Z}^n$-graded version (Theorem 5.3) is most naturally stated in terms of the *derived category* $\underline{\mathcal{D}}$ of $\mathbb{Z}^n$graded S-modules. However, since hardly any of the machinery is used, all of the pertinent definitions and facts concerning $\underline{\mathcal{D}}$ can be presented from scratch, so this is done in the next few lines (without actually defining $\underline{\mathcal{D}}$).

If $\mathcal{G}$ is a bounded complex of $\mathbb{Z}^n$-graded S modules—so $\mathcal{G}$ is nonzero in only finitely many (co)homological degrees and has homogeneous maps of degree $\mathbf{0}$—then $\mathcal{G}$ represents an object in $\underline{\mathcal{D}}$. Every homogeneous degree $\mathbf{b}$ homomorphism ϕ of such complexes is a morphism of degree $\mathbf{b}$ in $\underline{\mathcal{D}}$. Moreover, if ϕ is a homology isomorphism then ϕ is an *isomorphism* (of degree $\mathbf{b}$) in $\underline{\mathcal{D}}$. The usual functors $\underline{\mathrm{Ext}}$, $H_I^\bullet$, and $\underline{L}_\bullet^I$ are replaced by their derived categorical versions $\mathbb{R}\underline{\mathrm{Hom}}$, $\mathbb{R}\Gamma_I$, and $\mathbb{L}\underline{\Lambda}^I$ as follows.

Suppose $\mathcal{G}$ is a fixed complex. The right derived functor $\mathbb{R}\underline{\mathrm{Hom}}(\mathcal{G}, -)$ is calculated on a complex $\mathcal{E}$ by applying $\underline{\mathrm{Hom}}(\mathcal{G}, -)$ to an injective resolution of $\mathcal{E}$. By definition, such an injective resolution is a complex $\mathcal{J}$ of injectives along with a homology isomorphism $\mathcal{E} \to \mathcal{J}$. The result of taking $\underline{\mathrm{Hom}}(\mathcal{G}, \mathcal{J})$ is a double complex whose total complex is defined to be $\mathbb{R}\underline{\mathrm{Hom}}(\mathcal{G}, \mathcal{E})$. It is a fact that $\mathbb{R}\underline{\mathrm{Hom}}(\mathcal{G}, \mathcal{E})$ doesn't depend (up to isomorphism in $\underline{\mathcal{D}}$) on the choice of injective resolution $\mathcal{J}$.

Alternatively, as with $\underline{\mathrm{Ext}}$, one also gets $\mathbb{R}\underline{\mathrm{Hom}}(\mathcal{G}, \mathcal{E})$ by taking a free resolution of $\mathcal{G}$ and applying $\underline{\mathrm{Hom}}(-, \mathcal{E})$ to it. Dually to an injective resolution, a free resolution of $\mathcal{G}$ is a homology isomorphism $\mathcal{F} \to \mathcal{G}$ from a complex of free modules to $\mathcal{G}$. Again, $\underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{E})$ yields a double complex, whose total complex $\mathrm{Tot}\,\underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{E})$ is isomorphic in $\underline{\mathcal{D}}$ to $\mathbb{R}\underline{\mathrm{Hom}}(\mathcal{G}, \mathcal{E})$, independent of the free resolution $\mathcal{F} \to \mathcal{G}$.

Observe that if either $\mathcal{G}$ is already a complex of free modules or $\mathcal{E}$ is already a complex of injectives, then $\underline{\mathrm{Hom}}(\mathcal{G}, \mathcal{E})$ represents the right derived functor, and the $\mathbb{R}$ may be left off. The relation between $\mathbb{R}\underline{\mathrm{Hom}}(\mathcal{G}, \mathcal{E})$ and $\underline{\mathrm{Ext}}$ modules is seen when $\mathcal{G} = G$ and $\mathcal{E} = E$ are both modules: the usual notions of free and injective resolutions of modules may be regarded as homology isomorphisms as above, and $\underline{\mathrm{Ext}}^{\bullet}(G, E)$ is the cohomology of $\mathbb{R}\underline{\mathrm{Hom}}(G, E)$ (calculated either way).

The discussion above works just as well with $\mathbb{R}\Gamma_I$ in place of $\mathbb{R}\underline{\mathrm{Hom}}(\mathcal{G}, -)$ (injective resolution of "$-$" required) and with $\mathbb{L}\underline{\Lambda}^I$ in place of $\mathbb{R}\underline{\mathrm{Hom}}(-, \mathcal{E})$ (free resolution of "$-$" required), except that no double complexes appear, so there's no need to take any total complexes.

Remark 5.1. Proposition 3.7 says that $\check{\mathcal{C}}^{\bullet}_{\mathcal{F}}$ and $\check{\mathcal{C}}^{\bullet}_{\mathcal{T}(I)}$ are isomorphic in $\underline{\mathcal{D}}$. Lemma 4.3 says that $\underline{\mathrm{Mic}}(\mathcal{F}^t_{\bullet}, M) \cong_{\underline{\mathcal{D}}} \mathbb{L}\underline{\Lambda}^I(M)$; in fact, M there can be replaced by a bounded complex $\mathcal{E}$, since Lemma 3.8 works with bounded $\mathcal{E}$ in place of M. All resolutions of bounded complexes over S can be chosen bounded, so we restrict to bounded complexes to avoid technical issues.

As in the previous remark, the next lemma says that certain objects are isomorphic in $\underline{\mathcal{D}}$. Its proof uses the standard spectral sequence arguments, and is omitted. The symbols $\xrightarrow{\approx}$ and $\xleftarrow{\approx}$ denote homology isomorphisms, so as to keep track of their directions (and thus avoid getting too steeped in $\underline{\mathcal{D}}$, where $\cong_{\underline{\mathcal{D}}}$ would suffice for both).

Lemma 5.2. *Suppose $\mathcal{F} \xrightarrow{\approx} \mathcal{G}$ and $\mathcal{E} \xrightarrow{\approx} \mathcal{J}$. Then:*

1. *If $\mathcal{F}$ is free then $\underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{E}) \xrightarrow{\approx} \underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{J})$.*
2. *If $\mathcal{J}$ is injective then $\underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{J}) \xleftarrow{\approx} \underline{\mathrm{Hom}}(\mathcal{G}, \mathcal{J})$.*

5.2. **The duality theorem.**

Theorem 5.3. *If $\mathcal{G}$ and $\mathcal{E}$ are any bounded complexes of $\mathbb{Z}^n$-graded S-modules, then*

$$\mathbb{R}\underline{\mathrm{Hom}}(\mathbb{R}\Gamma_I(\mathcal{G}), \mathcal{E}) \cong_{\underline{\mathcal{D}}} \mathbb{R}\underline{\mathrm{Hom}}(\mathcal{G}, \mathbb{L}\underline{\Lambda}^I(\mathcal{E})).$$

Proof. Let $\mathcal{E} \to \mathcal{J}$ be an injective resolution, $\mathcal{T}_\bullet^{[t]}$ the Frobenius power of a Taylor resolution for I on squarefree generators, and $\mathcal{T}_{[t]}^\bullet = \underline{\mathrm{Hom}}\,(\mathcal{T}_\bullet^{[t]}, \omega_S)$, as usual. Calculate:

$$\begin{array}{rcl} \mathbb{L}\underline{\Lambda}^I(\mathcal{E}) & \cong & \underline{\mathrm{Mic}}\,(\mathcal{T}_\bullet^{[t]} \otimes \mathcal{E}) \\ & \xrightarrow{\approx} & \underline{\mathrm{Mic}}\,(\mathcal{T}_\bullet^{[t]} \otimes \mathcal{J}) \\ & \cong & \underline{\mathrm{Hom}}\,(\mathrm{Tel}\,\mathcal{T}_{[t]}^\bullet(\mathbf{1}), \mathcal{J}) \\ & \xleftarrow{\approx} & \underline{\mathrm{Hom}}\,(\check{\mathcal{C}}_{\mathcal{T}}^\bullet, \mathcal{J}). \end{array} \tag{2}$$

The first $\cong$ is by Lemma 4.3, where Remark 5.1 has been used to justify replacing M by a complex $\mathcal{E}$; the $\xrightarrow{\approx}$ uses the exactness of formation of microscopes when the inverse system is flat; the second $\cong$ uses Lemma 4.2; and Eq. (2) is by Lemma 5.2.2.

Observe that Eq. (2) is a complex of injectives (proof: the functor of "$-$" given by $\underline{\mathrm{Hom}}\,(-, \underline{\mathrm{Hom}}\,(\text{flat}, \text{injective})) = \underline{\mathrm{Hom}}\,(- \otimes \text{flat}, \text{injective})$ is exact). Therefore

$$\begin{array}{rcl} \mathbb{R}\underline{\mathrm{Hom}}\,(\mathcal{G}, \mathbb{L}\underline{\Lambda}^I(\mathcal{E})) & \cong_{\mathcal{D}} & \underline{\mathrm{Hom}}\,(\mathcal{G}, \underline{\mathrm{Hom}}\,(\check{\mathcal{C}}_{\mathcal{T}}^\bullet, \mathcal{J})) \\ & \cong & \underline{\mathrm{Hom}}\,(\mathcal{G} \otimes \check{\mathcal{C}}_{\mathcal{T}}^\bullet, \mathcal{J}) \\ & \cong_{\mathcal{D}} & \mathbb{R}\underline{\mathrm{Hom}}\,(\mathcal{G} \otimes \check{\mathcal{C}}_{\mathcal{T}}^\bullet, \mathcal{E}), \end{array}$$

and the result is a consequence of the standard isomorphism $\mathcal{G} \otimes \check{\mathcal{C}}_{\mathcal{T}}^\bullet \xrightarrow{\approx} \mathbb{R}\Gamma_I(\mathcal{G})$ (which is proved by applying Γ_I to an injective resolution of $\mathcal{G}$). □

Remark 5.4. The restrictions on M that appear in Theorem 4.4 and Corollary 4.8 don't appear in Theorem 5.3 because we allowed ourselves to replace M by an injective resolution $\mathcal{J}$: the homology isomorphism in Eq. (2) follows without the dualization argument used for Theorem 4.4. The fact that $\check{\mathcal{C}}_{\mathcal{T}}^\bullet$ isn't free means we can't *a priori* replace $\mathcal{J}$ by $\mathcal{E}$ as in Lemma 5.2.1 (or M as in Theorem 4.4).

$\mathbb{Z}^n$-graded local duality is a special case of Theorem 5.3.

Corollary 5.5 (Local duality with monomial support [Mil00])**.** *Suppose $\mathcal{F}_\bullet$ is a minimal free resolution of S/I, and let $\check{\mathcal{C}}_{\mathcal{F}}^\bullet$ be the generalized Čech complex determined by $\mathcal{F}_\bullet$. For arbitrary $\mathbb{Z}^n$-graded modules M,*

$$H_I^i(M)^\vee \cong H_i \underline{\mathrm{Hom}}_S(M, (\check{\mathcal{C}}_{\mathcal{F}}^\bullet)^\vee).$$

In particular, if S/I is Cohen-Macaulay of codimension d and $\omega_{S/I}^{\mathbf{1}}$ is the Alexander dual of the canonical module $\omega_{S/I}$ (Example 4.6), then

$$H_I^i(M)^\vee \cong \underline{\mathrm{Ext}}_S^{d-i}(M, \check{C}\omega_{S/I}^{\mathbf{1}}).$$

Proof. Setting $\mathcal{E} = S^\vee$ and $\mathcal{G} = M$ in Theorem 5.3 yields the first displayed equation by Example 4.6 (note that $S^\vee$ is $\mathbb{Z}^n$-finite). By Lemma 2.4 $(\check{\mathcal{C}}_{\mathcal{F}}^\bullet)^\vee$ is a complex of injectives (decreasing in homological degree). Under the

Cohen-Macaulay hypothesis, Example 4.6 also says the homology of $(\check{\mathcal{C}}^{\bullet}_{\mathcal{F}})^\vee$ is $\check{C}\omega^{\mathbf{1}}_{S/I}$, in homological degree d. □

The usual $\mathbb{Z}^n$-graded Grothendieck-Serre local duality theorem (support taken on $\mathfrak{m} = \langle x_1, \ldots , x_n\rangle$) is a special case of Corollary 5.5, by Example 2.5. Another consequence is the theorem in combinatorial commutative algebra relating local cohomology of S with support on I to local cohomology of S/I with maximal support. The combinatorial interpretation is as an equality of $\mathbb{Z}^n$-graded Hilbert series whose coefficients are Betti numbers of certain simplicial complexes related to the *Stanley-Reisner simplicial complex* of I.

Corollary 5.6 ([Mus00a, Ter99, Mil00])**.** *For a squarefree monomial ideal I,*

$$H^i_I(\omega_S) \cong \check{C}(H^{n-i}_{\mathfrak{m}}(S/I)^\vee) \cong \check{C}\underline{\mathrm{Ext}}^{\,i}_S(S/I, \omega_S).$$

Proof. The second isomorphism is by usual local duality, and the first is the Matlis dual of Corollary 5.5 with $M = \omega_S = S(-\mathbf{1})$, using Example 4.6. □

Judging from the relation between the usual local duality theorem and Serre duality for projective schemes, it seems clear, in view of the connections made in [Cox95, EMS00, Mus00b] between local cohomology with monomial support in polynomial rings and sheaf cohomology on toric varieties, that graded Greenlees-May duality has something to do with Serre duality on toric varieties. It would be interesting to see exactly how the details work out. In particular, what will be the role played by the *cellular flat complexes* related to toric varieties in [Mil00, Example 6.6] and [MP01, Proposition 5.4, Examples 6.3 and 8.14]?

6. Graded noetherian rings

Let $I = \langle \alpha_1, \ldots , \alpha_r\rangle$ be a finitely generated graded ideal in a commutative ring A graded by a commutative monoid. The $\mathbb{Z}^n$-graded methods of this paper suggest a transparent proof of GM duality in this general graded case, at least when A is noetherian. This is not to say that the graded case doesn't follow with care from known proofs for proregular sequences in arbitrary commutative rings [GM92, AJL97]. Rather, the innovation here is the simplicity of the proof; the consideration of a grading is done "for the record", because it requires no extra effort at this point.

The interesting part about the proofs in Sections 4 and 5 is that they work with the direct limits associated to telescopes, whereas previous methods in [GM92, AJL97] worked with the inverse limits associated to microscopes. The direct limit technique is applied below; it has the advantage that the colimits are exact, and the adjunction between $\mathbb{R}\Gamma_I$ and $\mathbb{L}\Lambda^I$ gets reduced directly to the adjunction between $\underline{\mathrm{Hom}}$ and $\otimes$.

In general, the adjointness of $H^{\bullet}_I$ and $\underline{H}^I_{\bullet}$ is essentially by definition, while the identification of the latter with $\underline{L}^I_{\bullet}$ requires hypotheses. The proof here relies on two facts about noetherian graded rings. The first, whose standard proof is omitted, is that the Čech complex $C^\bullet(\alpha_1, \dots, \alpha_r)$ is isomorphic in the derived category to $\mathbb{R}\Gamma_I A$ (apply $-\otimes C^\bullet(\alpha_1, \dots, \alpha_r)$ to a graded injective resolution). The second is contained in the next subsection.

6.1. **General analogue of Čech hull colimits.** The main feature of the Čech hull that made it useful earlier was its expression as a direct limit, resulting in the homology isomorphisms of Proposition 3.7. Although the combinatorial construction is lost with general gradings, the direct limit still makes sense, and the homology isomorphism survives, thanks to the next proposition. Let

$$\mathcal{K}^t_\bullet = \mathcal{K}_\bullet(\alpha_1^t, \dots, \alpha_r^t) = \bigotimes_{j=1}^{r} \big(A(-\deg \alpha_j^t) \xrightarrow{\alpha_j^t} A\big)$$

be the Koszul chain complex whose tensor factors are in homological degrees 1 and 0.

Proposition 6.1. *Suppose that A is arbitrary, but for every $t \geq 1$ the ideal $I^{[t]} = \langle \alpha_1^t, \dots, \alpha_r^t \rangle$ has a resolution by finite-rank free A-modules. Then there is a morphism $\{\rho^t\} : \{\mathcal{K}^t_\bullet\} \to \{\mathcal{F}^t_\bullet\}$ of inverse systems of complexes (indexed by $t \geq 1$) in which*

1. *$\mathcal{F}^t_\bullet$ is a resolution of $A/I^{[t]}$ by finite rank free modules for all $t \geq 1$;*
2. *the maps $\phi^t : \mathcal{F}^{t+1}_\bullet \to \mathcal{F}^t_\bullet$ lift the surjections $A/I^{[t+1]} \twoheadrightarrow A/I^{[t]}$;*
3. *the maps on tensor factors given by the identity in homological degree 0 and multiplication by α_j in homological degree 1 determine the maps $\kappa^t : \mathcal{K}^{t+1}_\bullet \to \mathcal{K}^t_\bullet$; and*
4. *each map $\rho^t : \mathcal{K}^t_\bullet \to \mathcal{F}^t_\bullet$ induces an isomorphism on homology in degree 0.*

Under these conditions, the transpose direct system $\{\rho_t\} : \{\mathcal{F}_t^\bullet\} \to \{\mathcal{K}_t^\bullet\}$ obtained by applying $\underline{\mathrm{Hom}}_A(-, A)$ to $\{\rho^t\}$ induces a homology isomorphism

$$\varinjlim_{t} \mathcal{F}_t^\bullet \xrightarrow{\approx} \varinjlim_{t} \mathcal{K}_t^\bullet.$$

Proof. All complexes of free modules appearing in this proof will be assumed to have finite rank in each homological degree, by the hypothesis on the ideals $I^{[t]}$. Conditions 1 and 2 can be forced upon any list $\{\tilde{\mathcal{F}}^t_\bullet\}$ of resolutions for the quotients $A/I^{[t]}$. Moreover, the acyclicity of $\tilde{\mathcal{F}}^t_\bullet$ and the freeness of $\mathcal{K}^t_\bullet$ imply that maps $\tilde{\rho}^t : \mathcal{K}^t_\bullet \to \tilde{\mathcal{F}}^t_\bullet$ as in condition 4 exist and are unique up to homotopy [Wei94, Porism 2.2.7]. Although $\{\tilde{\rho}^t\}$ may not *a priori* constitute a morphism of inverse systems, the uniqueness up to homotopy can be used to remedy this, by constructing $\mathcal{F}^t_\bullet$, ϕ^t, and ρ^t inductively, starting with a resolution $\mathcal{F}^1_\bullet$ of A/I and a chain map ρ^1 as in condition 4.

Having defined $\mathcal{F}^t_\bullet$ and ρ^t, choose a resolution $\tilde{\mathcal{F}}^{t+1}_\bullet$ of $A/I^{[t+1]}$ and a map $\tilde{\rho}^{t+1} : \mathcal{K}^{t+1}_\bullet \to \tilde{\mathcal{F}}^{t+1}_\bullet$ as in condition 4, and let $\tilde{\phi}^t : \tilde{\mathcal{F}}^{t+1}_\bullet \to \mathcal{F}^t_\bullet$ be any lift of the surjection $A/I^{[t+1]} \to A/I^{[t]}$. Then $\tilde{\phi}^{t+1}\tilde{\rho}^{t+1}$ is homotopic to $\rho^t\kappa^t$. Letting a lowered "t" index denote transpose, so (for instance) $\rho_t : \mathcal{F}_t^\bullet \to \mathcal{K}_t^\bullet$ is obtained by applying $\underline{\mathrm{Hom}}_A(-,A)$ to ρ^t, it follows that $\tilde{\rho}_{t+1}\tilde{\phi}_{t+1}$ is homotopic to $\kappa_t\rho_t : \mathcal{F}_t^\bullet \to \mathcal{K}_{t+1}^\bullet$.

Any choice of homotopy induces a map $\rho_{t+1} : \mathrm{cyl}(\tilde{\phi}_t) \to \mathcal{K}_{t+1}^\bullet$ from the *mapping cylinder* of $\tilde{\phi}_t$ to $\mathcal{K}_{t+1}^\bullet$. (See [Wei94, Section 1.5] for definitions and generalities concerning mapping cylinders; only the properties of $\mathrm{cyl}(\tilde{\phi}_t)$ required here are presented below.) The map ρ_t is produced essentially by [Wei94, Exercise 1.5.3], and satisfies:

(i) There is an inclusion $\phi_t : \mathcal{F}_t^\bullet \hookrightarrow \mathrm{cyl}(\tilde{\phi}_t)$ of complexes, and $\rho_{t+1}\phi_t = \kappa_t\rho_t$.
(ii) $\tilde{\mathcal{F}}_{t+1}^\bullet$ injects into $\mathrm{cyl}(\tilde{\phi}_t)$, and the composite $\tilde{\mathcal{F}}_{t+1}^\bullet \to \mathrm{cyl}(\tilde{\phi}_t) \xrightarrow{\rho_{t+1}} \mathcal{K}_{t+1}^\bullet$ is $\tilde{\rho}_{t+1}$.
(iii) The inclusion $\tilde{\mathcal{F}}_{t+1}^\bullet \hookrightarrow \mathrm{cyl}(\tilde{\phi}_t)$ is a homotopy equivalence, and the composite $\mathcal{F}_t^\bullet \to \mathrm{cyl}(\tilde{\phi}^t) \to \tilde{\mathcal{F}}_{t+1}^\bullet$ is just $\tilde{\phi}_t$.

The transpose $\rho^{t+1} : \mathcal{K}_\bullet^{t+1} \to \underline{\mathrm{Hom}}_A(\mathrm{cyl}(\tilde{\phi}_t), A) =: \mathcal{F}_\bullet^{t+1}$ satisfies the required conditions, with $\phi^t : \mathcal{F}_\bullet^{t+1} \to \mathcal{F}_\bullet^t$ being the transpose of the map ϕ_t in (i) and (iii).

The induced map on direct limits is a homology isomorphism because

$$\varinjlim \underline{\mathrm{Ext}}_A^i(A/I^{[t]}, A) = \varinjlim H^i(\mathcal{F}_t^\bullet) = H^i \varinjlim (\mathcal{F}_t^\bullet) \to H^i \varinjlim (\mathcal{K}_t^\bullet) = H_I^i(A)$$

is the canonical isomorphism on the local cohomology module $H_I^i(A)$; note that the graded module $\underline{\mathrm{Ext}}_A^i(A/I^{[t]}, A)$ is naturally isomorphic to the usual ungraded module $\mathrm{Ext}_A^i(A/I^{[t]}, A)$ because $A/I^{[t]}$ is finitely presented. □

Remark 6.2. Like the generalized Čech complexes of Definition 3.1, the object $\varinjlim \mathcal{F}_t^\bullet$ of Proposition 6.1 is a complex of flat modules. This is because the colimits are taken over directed systems of complexes of free modules, so complexes of flat modules result by a theorem of Govorov and Lazard [Eis95, Theorem A6.6].

It is unclear whether the methods of Proposition 6.1 can be made to apply when the only assumption is that $(\alpha_1, \ldots, \alpha_r) \subset A$ is a proregular sequence [GM92, AJL97]. In general, under what conditions will $\langle \alpha_1^t, \ldots, \alpha_r^t \rangle$ have finite Betti numbers as an A-module for infinitely many $t \geq 1$? Perhaps characteristic $p > 0$ criteria are possible.

6.2. **Graded noetherian Greenlees-May duality.** In the following theorem, the derived category statements concern complexes $\mathcal{G}$ and $\mathcal{E}$ that are bounded, for simplicity. The proof is nearly the same as that of Theorem 5.3, but is repeated in full to be self-contained, and for notation's sake.

Theorem 6.3 ([GM92, AJL97])**.** *Let A be a noetherian ring graded by a monoid and $I \subset A$ a graded ideal. Then $\underline{H}_\bullet^I(M) \cong \underline{L}_\bullet^I(M)$ for any graded A-module M. If $\mathcal{G}$ and $\mathcal{E}$ are bounded complexes of graded A-modules, then*

$$\mathbb{R}\underline{\mathrm{Hom}}(\mathbb{R}\Gamma_I(\mathcal{G}), \mathcal{E}) \cong \mathbb{R}\underline{\mathrm{Hom}}(\mathcal{G}, \mathbb{L}\underline{\Lambda}^I(\mathcal{E})).$$

Proof. Let the ideals $I^{[t]}$, the free resolutions $\mathcal{F}^t_\bullet$, their transposes $\mathcal{F}_t^\bullet$, the Koszul chain complexes $\mathcal{K}^t_\bullet$, and their transposes $\mathcal{K}_t^\bullet$ be as in Proposition 6.1. If $\mathcal{E} \to \mathcal{J}$ is an injective resolution, then

$$\begin{aligned}
&(3) & \mathbb{L}\underline{\Lambda}^I(\mathcal{E}) &\cong \underline{\mathrm{Mic}}\,(\mathcal{F}^t_\bullet \otimes \mathcal{E}) \\
&(4) & &\xrightarrow{\approx} \underline{\mathrm{Mic}}\,(\mathcal{F}^t_\bullet \otimes \mathcal{J}) \\
&(5) & &\cong \underline{\mathrm{Hom}}\,(\mathrm{Tel}\,\mathcal{F}_t^\bullet, \mathcal{J}) \\
&(6) & &\xleftarrow{\approx} \underline{\mathrm{Hom}}\,(\varinjlim \mathcal{F}_t^\bullet, \mathcal{J}) \\
&(7) & &\xleftarrow{\approx} \underline{\mathrm{Hom}}\,(\varinjlim \mathcal{K}_t^\bullet, \mathcal{J}) \\
&(8) & &\xrightarrow{\approx} \underline{\mathrm{Hom}}\,(\mathrm{Tel}\,\mathcal{K}_t^\bullet, \mathcal{J}) \\
&(9) & &\xleftarrow{\approx} \underline{\mathrm{Hom}}\,(\mathrm{Tel}\,\mathcal{K}_t^\bullet, \mathcal{E}).
\end{aligned}$$

Eq. (3) follows by replacing M and its free resolution $\mathcal{E}_\bullet$ in Lemma 4.3 with a complex $\mathcal{E}$ and a free resolution of it (the proof is the same). Eq. (4) uses the exactness of formation of microscopes when the inverse system is flat. Eq. (5) is by Lemma 4.2. Eq. (6) is because $\mathcal{J}$ is injective and $\mathrm{Tel}\,\mathcal{F}_t^\bullet \xrightarrow{\approx} \varinjlim \mathcal{F}_t^\bullet$, and similarly for Eq. (8). Proposition 6.1 implies Eq. (7) (this is the key step!). Finally, Eq. (9) uses the fact that $\mathrm{Tel}\,\mathcal{K}_t^\bullet$ is free.

When $\mathcal{E} = M$ is a module, the conclusion of Eq. (9) implies that $\underline{L}^I_\bullet(M) \cong \underline{H}^I_\bullet(M)$, by definition. To get the derived categorical statement, observe first that Eq. (7) is a complex of injectives because the functor given by $\underline{\mathrm{Hom}}\,(-, \underline{\mathrm{Hom}}\,(\text{flat}, \text{injective})) = \underline{\mathrm{Hom}}\,(- \otimes \text{flat}, \text{injective})$ is exact. Therefore

$$\begin{aligned}
\mathbb{R}\underline{\mathrm{Hom}}\,(\mathcal{G}, \mathbb{L}\underline{\Lambda}^I(\mathcal{E})) &\cong \underline{\mathrm{Hom}}\,(\mathcal{G}, \underline{\mathrm{Hom}}\,(\varinjlim \mathcal{K}_t^\bullet, \mathcal{J})) \\
&\cong \underline{\mathrm{Hom}}\,(\mathcal{G} \otimes \varinjlim \mathcal{K}_t^\bullet, \mathcal{J}) \\
&\cong \mathbb{R}\underline{\mathrm{Hom}}\,(\mathcal{G} \otimes \varinjlim \mathcal{K}_t^\bullet, \mathcal{E}) \\
&\cong \mathbb{R}\underline{\mathrm{Hom}}\,(\mathbb{R}\Gamma_I \mathcal{G}, \mathcal{E}),
\end{aligned}$$

the second isomorphism being at the level of complexes while the rest are in the derived category. The last isomorphism is the standard isomorphism for noetherian graded rings, mentioned at the beginning of Section 6. □

The above proof essentially requires $\mathcal{F}^t_\bullet = \underline{\mathrm{Hom}}\,(\underline{\mathrm{Hom}}\,(\mathcal{F}^t_\bullet, A), A)$ to be its own double dual (e.g., in Eq. (5)), thus using again the finite rank conditions.

Remark 6.4. Looking back at Theorem 4.4, under what circumstances can the injective resolution $\mathcal{J}$ in Eq. (5) be replaced by a module M? I.e., when will $\underline{L}^I_i(M)$ be isomorphic to $H_i\underline{\mathrm{Hom}}_A(\varinjlim_t \mathcal{F}_t^\bullet, M)$ (Remark 6.2) or $\underline{\mathrm{Hom}}_A(\varinjlim_t \mathcal{K}_t^\bullet, M)$, rather than having to replace the colimits by projective approximations in the form of telescopes?

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