Binomial Exercises

Lecture I. Toric ideals

1. Exhibit a point configuration $A$ whose affine semigroup $\mathbb{N}A$ does not consist of the intersection of the lattice $\mathbb{Z}A$ spanned by the columns of $A$ with the real cone generated by $A$.

2. Prove that the affine toric variety $Y_A$ for

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

is the affine cone over the product $\mathbb{P}^1 \times \mathbb{P}^1$ of two projective lines. What does this have to do with the geometry and combinatorics of a square?

3. Let $A$ be any matrix for the quotient of $\mathbb{Z}^6$ modulo the sublattice $L \subseteq \mathbb{Z}^6$ spanned by

$$\begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \\ -1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ -1 \\ -1 \end{bmatrix}.$$

Show that the real cone $\mathbb{R}_{\geq 0}A$ generated by the affine semigroup $\mathbb{N}A$ is the cone over a triangular prism. Use any method you can think of (perhaps with the aid of a computer) to write down generators for the toric ideal $I_A$.

4. Let $L \subseteq \mathbb{Z}^n$ be an unsaturated sublattice.

   (i) Is it possible for $I_\rho \subseteq \mathbb{C}[\partial]$ to be prime for some choice of character $\rho : L \to \mathbb{C}^*$?

   (ii) What would happen if $\mathbb{C}[\partial]$ were replaced by $\mathbb{k}[\partial]$ for an algebraically closed field $\mathbb{k}$ of positive characteristic?

   (iii) What if $\mathbb{k}$ is allowed to have arbitrary characteristic, but is not required to be algebraically closed?
Binomial Exercises

Lecture II. Binomial primary decomposition

1. Let

\[
A = \begin{bmatrix}
0 & 1 & 2 & 3 \\
3 & 2 & 1 & 0
\end{bmatrix}
\]

In the primary decomposition

\[
\langle \partial_2^2 - \partial_4 \partial_3, \partial_3^2 - \partial_2 \partial_4 \rangle = \langle \partial_2^2 - \partial_4 \partial_3, \partial_3^2 - \partial_2 \partial_4, \partial_2 \partial_3 - \partial_1 \partial_4 \rangle \cap \langle \partial_2, \partial_3 \rangle
\]

determine whether each of the primary components is toral or Andean.

2. Let \( p \) and \( q \) be primes with \( p \subseteq q \). Prove that if \( q \) is Andean, then so is \( p \).

3. Let

\[
B = \begin{bmatrix}
1 & -5 & 0 \\
-1 & 1 & -1 \\
0 & 3 & 1
\end{bmatrix}
\]

Enumerate all of the equivalence classes on \( \mathbb{N}^3 \) determined by \( I(B) \). Can a computer help you do this? Draw a picture of every equivalence class.

4. Let \( A = \begin{bmatrix} 5 & 4 & 3 \end{bmatrix} \) and \( B = \begin{bmatrix}
-1 & 2 \\
2 & -1 \\
-1 & -2
\end{bmatrix} \). Then \( I(B) = \langle \partial_2^2 - \partial_1 \partial_3, \partial_3^2 - \partial_2 \partial_3 \rangle \). Let \( I_{\rho,J} = \langle \partial_1, \partial_2 \rangle \), so that \( J = \{3\} \) and \( \rho : L \to \mathbb{C}^* \) for the lattice \( L = \{0\} \). Note that \( I_\rho = \langle 0 \rangle \).

(i) What are all of the equivalence classes on \( \mathbb{Z}^J \times \mathbb{N}^J = \mathbb{N}^2 \times \mathbb{Z} \) determined by \( I(B) \)?

(Hint: Project every equivalence class onto \( \mathbb{N}^2 \) and try to draw it there.)

(ii) Determine every \( u \in \mathbb{N}^3 \) that lies in a finite equivalence class on \( \mathbb{N}^2 \times \mathbb{Z} \).

(iii) Use your answers to write down the \( I_{\rho,J} \)-primary component of \( I(B) \).

(iv) What is the primary decomposition of \( I(B) \)?

A computer could give you hints by finding the answers to (iii) and (iv) directly. (A free beer to anyone who can get a computer to find the answers to (i) and (ii) directly, without using the combinatorial primary decomposition theorem. A published paper to anyone who can write down an algorithm to do it in general.)

5* Let \( k \) be a field of characteristic \( p > 0 \). Assuming that \( I_{\rho,J} \) is minimal over a binomial ideal \( I \subseteq k[\partial] \), find a combinatorial characterization of the \( I_{\rho,J} \)-primary component of \( I \).

(It is known [Eisenbud & Sturmfels, Binomial ideals, Duke Math. J. 84 (1996), no. 1, 1–45] that this primary component is a binomial ideal.)

6. (i) Find a binomial prime \( I_{\rho,J} \) and an irreducible \( I_{\rho,J} \)-primary ideal that is not binomial.

(ii)* Characterize the irreducible \( I_{\rho,J} \)-primary binomial ideals in \( \mathbb{C}[\partial] \).
Binomial Exercises

Lecture III. Introduction to $D$-modules

1. Write $E = x_1 \partial_1 + \cdots + x_n \partial_n$ for the Euler operator, and let $f$ be a function of $x_1, \ldots, x_n$.
   
   (i) If $f$ is a polynomial, prove that it is homogeneous of degree $d$ if and only if $E(f) = d \cdot f$.
   
   (ii) Do you get more solutions to $E(f) = d \cdot f$ if $f$ is allowed to be a series involving positive and negative powers of some of the variables $x_i$? If there are more solutions, do any of them possess nonempty open subsets of $\mathbb{C}^n$ where they converge?
   
   (iii) In what kinds of modules might you look for solutions to $E f = \beta f$ for $\beta \in \mathbb{C}$?

2. The order of a differential operator $\phi \in D_n$ is its degree in $\partial_1, \ldots, \partial_n$ (i.e., think of $x_1, \ldots, x_n$ as having degree zero).
   
   (i) Show that taking the operators of order $\leq k$ for each $k$ induces an increasing filtration $0 = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_k \subset \cdots$ of $D$ whose associated graded ring is a commutative polynomial ring $\mathbb{C}[x, \xi]$ in $2n$ variables $x = x_1, \ldots, x_n$ and $\xi = \xi_1, \ldots, \xi_n$.
   
   (ii) Show that the order filtration in (i) descends to the quotient of $D$ by any left ideal.
   
   (iii) The symbol of $\phi \in D$ is the result of replacing $\partial$ with $\xi$ in the sum of all of its highest-order terms. Show that the associated graded module of $D/I$ is $D/s(I)$, where $s(I)$ is generated by the symbols of all of the operators in $I$.

Order filtrations reduce the computation of holonomic ranks to commutative algebra:

   (iv) Show that $\dim_{\mathbb{C}(x)} (\mathbb{C}(x) \otimes_{\mathbb{C}[x]} D/I) = \dim_{\mathbb{C}(x)} (\mathbb{C}(x) \otimes_{\mathbb{C}[x]} D/s(I))$.

Note: It is a fact that a quotient $D_n/I$ is holonomic if and only if $D_n/s(I)$ has Krull dimension $n$ as a module over the polynomial ring $\mathbb{C}[x, \xi]$.

3. Calculate the holonomic ranks of the following $D_1$-modules $\mathcal{M}$ by finding analytic solutions.
   
   (i) $\mathcal{M} = D_1/D_1(\partial^2 + 1)$
   
   (ii) $\mathcal{M} = D_1/D_1(x^2 \partial + 1)$

   In (ii), is there a point $p \in \mathbb{C}$ where $r_p \neq r$?

4. Do the previous exercise using Kashiwara’s theorem, by finding $\mathbb{C}(x)$-vector space bases. Or use Exercise 2(iv).

5. Find a holonomic module that is not regular holonomic.
Binomial Exercises

Lecture IV. Hypergeometric systems

1. What is the rational function \( r \) such that \( h_{\sigma + 1} = r(\sigma) \) for the Gauss hypergeometric series?

2. Provide the details in an argument showing that \( g(z_1 \partial z_1, \ldots, z_m \partial z_m)(z^\sigma) = g(\sigma)z^\sigma. \)

3. Let \( f(x) = \frac{x_3^{x_4}}{x_1 x_2} \binom{x_3 x_4}{x_1 x_2} \), where \( F(z) = 1 + \frac{ab}{c} z + \frac{a(a+1)b(b+1)}{c(c+1)} \frac{z^2}{2!} + \cdots. \)
   
   (i) The series \( f(x) \) equals \( x^\gamma F(x^B) \) for some integer matrix \( B \) and vector \( \gamma \in \mathbb{C}^4. \) What are \( B \) and \( \gamma \)?
   (ii) Prove that \( \partial_3 \partial_4 f(x) = \partial_1 \partial_2 f(x). \)
   (iii) Prove that  
       \[
       \begin{align*}
       (x_1 \partial_1 + x_4 \partial_4) f(x) &= -af(x) \\
       (x_2 \partial_2 + x_4 \partial_4) f(x) &= -bf(x) \\
       (-x_3 \partial_3 + x_4 \partial_4) f(x) &= (1-c) f(x).
       \end{align*}
       \]
       (Hint: using Exercise 1 of Lecture III, it isn’t necessary to calculate any derivatives.)
   (iv) Conclude that \( f(x) \) satisfies the binomial Gauss system for the vector \( \beta = A\gamma, \) where \( A \) is any matrix for the left kernel of \( B, \) and \( \gamma \) is from part (i).

4. Consider the matrices 
   \[
   A = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 3 & 2 & 1 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 \\ -2 & 1 \\ 1 & -2 \\ 0 & 1 \end{bmatrix}.
   \]
   Verify that \( f(x) = x_1^{\beta_2/3} x_4^{\beta_1/3} \) is a solution of \( H_A(I(B), \beta) \) for every choice of \( \beta \in \mathbb{C}^2. \)
Binomial Exercises

Lecture V. Euler–Koszul homology

1. What conditions on a set of lattice points in \( \mathbb{Z}^3 \) guarantee that its Zariski closure in \( \mathbb{C}^3 \) is a 2-dimensional plane?

2. Let \( R \) be a noetherian \( A \)-graded ring that is finitely generated over its degree 0 piece. Prove that \( \text{qdeg}(M) \) for any finitely generated \( A \)-graded \( R \)-module \( M \) is a finite union of affine subspaces of \( \mathbb{C}^d \), each spanned by the degrees of some subset of the generators of \( R \), as follows.
   
   (i) Prove that \( M \) has a submodule isomorphic to an \( A \)-graded translate of a quotient by an \( A \)-graded prime (i.e., a submodule isomorphic as an ungraded module to \( R/p \), but generated in some possibly nonzero \( A \)-graded degree).
   
   (ii) Use noetherian induction to deduce that \( M \) has a finite filtration whose successive quotients are \( A \)-translates of quotients of \( R \) modulo prime ideals.
   
   (iii) Finish the proof by showing that the true degree set of a quotient \( R/p \) by a prime ideal \( p \) is an affine semigroup generated by the degrees of some of the generators of \( R \).

3. Let \( \mathcal{M} \) be an \( A \)-graded \( D \)-module, where \( \deg(x_j) = a_j \) and \( \deg(\partial_j) = -a_j \). Consider the action of \( E_i - \beta_i \) on \( \mathcal{M} \) determined by \( (E_i - \beta_i) \circ z = (E_i - \beta_i - \alpha_i) \cdot z \) for homogeneous elements \( z \in M \) of degree \( \alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{Z}^d \). Prove that this is a well-defined action of the commutative subalgebra \( \mathbb{C}[E - \beta] \subset D \) on \( \mathcal{M} \).

4. (i) Verify that \( \mathcal{K}_*(E - \beta; M) \) is a complex of \( D \)-modules.
   
   (ii) Check that \( \mathcal{H}_0(E - \beta; \mathbb{C}[\partial]/I) = D/H_A(I, \beta) \).

5. Prove that if \( I \subseteq J \subseteq \mathbb{C}[\partial] \) (think \( J = I_{\text{Andean}} \) ) are \( A \)-graded, then the natural surjection \( \mathbb{C}[\partial]/I \to \mathbb{C}[\partial]/J \) yields a map \( \mathcal{H}_0(E - \beta; \mathbb{C}[\partial]/I) \to \mathcal{H}_0(E - \beta; \mathbb{C}[\partial]/J) \) that is surjective.

6. (i) What is the volume of the convex hull in \( \mathbb{R}^3 \) of the origin and the columns of
   
   \[
   \begin{bmatrix}
   1 & 0 & 0 & 1 \\
   0 & 1 & 0 & 1 \\
   0 & 0 & -1 & 1
   \end{bmatrix}
   
   \]

   (ii) What is the rank of the binomial Gauss system?
   
   (iii) What has feathers and sounds like “Gauss”? (Hint: lieu, lieu, lieu.)