Binomial Exercises

Lecture I. Toric ideals

1. Exhibit a point configuration A whose affine semigroup $\mathbb{N}A$ does not consist of the intersection of the lattice $\mathbb{Z}A$ spanned by the columns of A with the real cone generated by A.

2. Prove that the affine toric variety Y_A for

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

is the affine cone over the product $\mathbb{P}^1 \times \mathbb{P}^1$ of two projective lines. What does this have to do with the geometry and combinatorics of a square?

3. Let A be any matrix for the quotient of $\mathbb{Z}^6$ modulo the sublattice $L \subseteq \mathbb{Z}^6$ spanned by

$$\begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \\ -1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ -1 \\ -1 \end{bmatrix}.$$

Show that the real cone $\mathbb{R}_{\geq 0}A$ generated by the affine semigroup $\mathbb{N}A$ is the cone over a triangular prism. Use any method you can think of (perhaps with the aid of a computer) to write down generators for the toric ideal I_A.

4. Let $L \subseteq \mathbb{Z}^n$ be an unsaturated sublattice.

   (i) Is it possible for $I_\rho \subseteq \mathbb{C}[\partial]$ to be prime for some choice of character $\rho : L \to \mathbb{C}^*$?

   (ii) What would happen if $\mathbb{C}[\partial]$ were replaced by $\Bbbk[\partial]$ for an algebraically closed field $\Bbbk$ of positive characteristic?

   (iii) What if $\Bbbk$ is allowed to have arbitrary characteristic, but is not required to be algebraically closed?

Binomial Exercises

Lecture II. Binomial primary decomposition

1. Let $A = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 3 & 2 & 1 & 0 \end{bmatrix}$. In the primary decomposition

$$\langle \partial_2^2 - \partial_1\partial_3, \partial_3^2 - \partial_2\partial_4 \rangle = \langle \partial_2^2 - \partial_1\partial_3, \partial_3^2 - \partial_2\partial_4, \partial_2\partial_3 - \partial_1\partial_4 \rangle \cap \langle \partial_2, \partial_3 \rangle$$

   determine whether each of the primary components is toral or Andean.

2. Let $\mathfrak{p}$ and $\mathfrak{q}$ be primes with $\mathfrak{p} \subseteq \mathfrak{q}$. Prove that if $\mathfrak{q}$ is Andean, then so is $\mathfrak{p}$.

3. Let $B = \begin{bmatrix} 1 & -5 & 0 \\ -1 & 1 & -1 \\ 0 & 3 & 1 \end{bmatrix}$. Enumerate all of the equivalence classes on $\mathbb{N}^3$ determined by $I(B)$. Can a computer help you do this? Draw a picture of every equivalence class.

4. Let $A = [5\ \ 4\ \ 3]$ and $B = \begin{bmatrix} -1 & 2 \\ 2 & -1 \\ -1 & -2 \end{bmatrix}$. Then $I(B) = \langle \partial_2^2 - \partial_1\partial_3, \partial_1^2 - \partial_2\partial_3^2 \rangle$. Let $I_{\rho,J} = \langle \partial_1, \partial_2 \rangle$, so that $J = \{3\}$ and $\rho : L \to \mathbb{C}^*$ for the lattice $L = \{0\}$. Note that $I_\rho = \langle 0 \rangle$.

   (i) What are all of the equivalence classes on $\mathbb{Z}^J \times \mathbb{N}^{\bar{J}} = \mathbb{N}^2 \times \mathbb{Z}$ determined by $I(B)$? (Hint: Project every equivalence class onto $\mathbb{N}^2$ and try to draw it there.)

   (ii) Determine every $u \in \mathbb{N}^3$ that lies in a finite equivalence class on $\mathbb{N}^2 \times \mathbb{Z}$.

   (iii) Use your answers to write down the $I_{\rho,J}$-primary component of $I(B)$.

   (iv) What is the primary decomposition of $I(B)$?

   A computer could give you hints by finding the answers to (iii) and (iv) directly. (A free beer to anyone who can get a computer to find the answers to (i) and (ii) directly, without using the combinatorial primary decomposition theorem. A published paper to anyone who can write down an algorithm to do it in general.)

5.* Let $\Bbbk$ be a field of characteristic $p > 0$. Assuming that $I_{\rho,J}$ is minimal over a binomial ideal $I \subseteq \Bbbk[\partial]$, find a combinatorial characterization of the $I_{\rho,J}$-primary component of I. (It is known [Eisenbud & Sturmfels, *Binomial ideals*, Duke Math. J. **84** (1996), no. 1, 1–45] that this primary component is a binomial ideal.)

6. (i) Find a binomial prime $I_{\rho,J}$ and an irreducible $I_{\rho,J}$-primary ideal that is not binomial.

   (ii)* Characterize the irreducible $I_{\rho,J}$-primary binomial ideals in $\mathbb{C}[\partial]$.

Binomial Exercises

Lecture III. Introduction to D-modules

1. Write $E = x_1\partial_1 + \cdots + x_n\partial_n$ for the Euler operator, and let f be a function of $x_1, \ldots, x_n$.
   (i) If f is a polynomial, prove that it is homogeneous of degree d if and only if $E(f) = d \cdot f$.
   (ii) Do you get more solutions to $E(f) = d \cdot f$ if f is allowed to be a series involving positive and negative powers of some of the variables x_i? If there are more solutions, do any of them possess nonempty open subsets of $\mathbb{C}^n$ where they converge?
   (iii) In what kinds of modules might you look for solutions to $Ef = \beta f$ for $\beta \in \mathbb{C}$?

2. The *order* of a differential operator $\phi \in D_n$ is its degree in $\partial_1, \ldots, \partial_n$ (i.e., think of $x_1, \ldots, x_n$ as having degree zero).
   (i) Show that taking the operators of order $\leq k$ for each k induces an increasing filtration
   $$0 = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_k \subset \cdots$$
   of D whose associated graded ring is a commutative polynomial ring $\mathbb{C}[\mathbf{x}, \xi]$ in $2n$ variables $\mathbf{x} = x_1, \ldots, x_n$ and $\xi = \xi_1, \ldots, \xi_n$.
   (ii) Show that the *order filtration* in (i) descends to the quotient of D by any left ideal.
   (iii) The *symbol* of $\phi \in D$ is the result of replacing ∂ with ξ in the sum of all of its highest-order terms. Show that the associated graded module of D/I is $D/s(I)$, where $s(I)$ is generated by the symbols of all of the operators in I.

   Order filtrations reduce the computation of holonomic ranks to commutative algebra:

   (iv) Show that $\dim_{\mathbb{C}(\mathbf{x})} \big(\mathbb{C}(\mathbf{x}) \otimes_{\mathbb{C}[\mathbf{x}]} D/I\big) = \dim_{\mathbb{C}(\mathbf{x})} \big(\mathbb{C}(\mathbf{x}) \otimes_{\mathbb{C}[\mathbf{x}]} D/s(I)\big)$.

   Note: It is a fact that a quotient D_n/I is holonomic if and only if $D_n/s(I)$ has Krull dimension n as a module over the polynomial ring $\mathbb{C}[\mathbf{x}, \xi]$.

3. Calculate the holonomic ranks of the following D_1-modules $\mathcal{M}$ by finding analytic solutions.
   (i) $\mathcal{M} = D_1/D_1\langle \partial^2 + 1\rangle$
   (ii) $\mathcal{M} = D_1/D_1\langle x^2\partial + 1\rangle$

   In (ii), is there a point $p \in \mathbb{C}$ where $r_p \neq r$?

4. Do the previous exercise using Kashiwara's theorem, by finding $\mathbb{C}(\mathbf{x})$-vector space bases. Or use Exercise 2(iv).

5. Find a holonomic module that is not regular holonomic.

Binomial Exercises

Lecture IV. Hypergeometric systems

1. What is the rational function r such that $\frac{h_{\sigma+1}}{h_\sigma} = r(\sigma)$ for the Gauss hypergeometric series?

2. Provide the details in an argument showing that $g(z_1\partial_{z_1}, \ldots, z_m\partial_{z_m})(\mathbf{z}^\sigma) = g(\sigma)\mathbf{z}^\sigma$.

3. Let $f(\mathbf{x}) = \frac{x_3^{c-1}}{x_1^a x_2^b} F\Big(\frac{x_3x_4}{x_1x_2}\Big)$, where $F(z) = 1 + \frac{ab}{c}z + \frac{a(a+1)b(b+1)}{c(c+1)}\frac{z^2}{2!} + \cdots$.

   (i) The series $f(\mathbf{x})$ equals $\mathbf{x}^\gamma F(\mathbf{x}^B)$ for some integer matrix B and vector $\gamma \in \mathbb{C}^4$. What are B and γ?

   (ii) Prove that $\partial_3\partial_4 f(\mathbf{x}) = \partial_1\partial_2 f(\mathbf{x})$.

   (iii) Prove that
   $$\begin{aligned} (x_1\partial_1 + x_4\partial_4)f(\mathbf{x}) &= -af(\mathbf{x}) \\ (x_2\partial_2 + x_4\partial_4)f(\mathbf{x}) &= -bf(\mathbf{x}) \\ (-x_3\partial_3 + x_4\partial_4)f(\mathbf{x}) &= (1-c)f(\mathbf{x}). \end{aligned}$$
   (Hint: using Exercise 1 of Lecture III, it isn't necessary to calculate any derivatives.)

   (iv) Conclude that $f(\mathbf{x})$ satisfies the binomial Gauss system for the vector $\beta = A\gamma$, where A is any matrix for the left kernel of B, and γ is from part (i).

4. Consider the matrices
$$A = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 3 & 2 & 1 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 \\ -2 & 1 \\ 1 & -2 \\ 0 & 1 \end{bmatrix}.$$
   Verify that $f(\mathbf{x}) = x_1^{\beta_2/3}x_4^{\beta_1/3}$ is a solution of $H_A(I(B), \beta)$ for every choice of $\beta \in \mathbb{C}^2$.

Binomial Exercises

Lecture V. Euler–Koszul homology

1. What conditions on a set of lattice points in $\mathbb{Z}^3$ guarantee that its Zariski closure in $\mathbb{C}^3$ is a 2-dimensional plane?

2. Let R be a noetherian A-graded ring that is finitely generated over its degree 0 piece. Prove that $\mathrm{qdeg}(M)$ for any finitely generated A-graded R-module M is a finite union of affine subspaces of $\mathbb{C}^d$, each spanned by the degrees of some subset of the generators of R, as follows.

   (i) Prove that M has a submodule isomorphic to an A-graded translate of a quotient by an A-graded prime (i.e., a submodule isomorphic as an ungraded module to $R/\mathfrak{p}$, but generated in some possibly nonzero A-graded degree).

   (ii) Use noetherian induction to deduce that M has a finite filtration whose successive quotients are A-translates of quotients of R modulo prime ideals.

   (iii) Finish the proof by showing that the true degree set of a quotient $R/\mathfrak{p}$ by a prime ideal $\mathfrak{p}$ is an affine semigroup generated by the degrees of some of the generators of R.

3. Let $\mathcal{M}$ be an A-graded D-module, where $\deg(x_j) = \mathbf{a}_j$ and $\deg(\partial_j) = -\mathbf{a}_j$. Consider the action of $E_i - \beta_i$ on $\mathcal{M}$ determined by $(E_i - \beta_i) \circ z = (E_i - \beta_i - \alpha_i) \cdot z$ for homogeneous elements $z \in M$ of degree $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{Z}^d$. Prove that this is a well-defined action of the commutative subalgebra $\mathbb{C}[E - \beta] \subset D$ on $\mathcal{M}$.

4. (i) Verify that $\mathcal{K}_\bullet(E - \beta; M)$ is a complex of D-modules.

   (ii) Check that $\mathcal{H}_0(E - \beta; \mathbb{C}[\partial]/I) = D/H_A(I, \beta)$.

5. Prove that if $I \subseteq J \subseteq \mathbb{C}[\partial]$ (think $J = I_{\text{Andean}}$) are A-graded, then the natural surjection $\mathbb{C}[\partial]/I \to \mathbb{C}[\partial]/J$ yields a map $\mathcal{H}_0(E - \beta; \mathbb{C}[\partial]/I) \to \mathcal{H}_0(E - \beta; \mathbb{C}[\partial]/J)$ that is surjective.

6. (i) What is the volume of the convex hull in $\mathbb{R}^3$ of the origin and the columns of

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}?$$

   (ii) What is the rank of the binomial Gauss system?

   (iii) What has feathers and sounds like "Gauss"? (Hint: lieu, lieu, lieu.)