# Contents

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Preface</td>
<td>vii</td>
</tr>
<tr>
<td><strong>I  Monomial Ideals</strong></td>
<td></td>
</tr>
<tr>
<td>1  Squarefree monomial ideals</td>
<td>3</td>
</tr>
<tr>
<td>1.1 Equivalent descriptions</td>
<td>3</td>
</tr>
<tr>
<td>1.2 Hilbert series</td>
<td>6</td>
</tr>
<tr>
<td>1.3 Simplicial complexes and homology</td>
<td>9</td>
</tr>
<tr>
<td>1.4 Monomial matrices</td>
<td>11</td>
</tr>
<tr>
<td>1.5 Betti numbers</td>
<td>14</td>
</tr>
<tr>
<td>Exercises</td>
<td>18</td>
</tr>
<tr>
<td>Notes</td>
<td>19</td>
</tr>
<tr>
<td>2  Borel-fixed monomial ideals</td>
<td>21</td>
</tr>
<tr>
<td>2.1 Group actions</td>
<td>21</td>
</tr>
<tr>
<td>2.2 Generic initial ideals</td>
<td>24</td>
</tr>
<tr>
<td>2.3 The Eliahou–Kervaire resolution</td>
<td>27</td>
</tr>
<tr>
<td>2.4 Lex-segment ideals</td>
<td>33</td>
</tr>
<tr>
<td>Exercises</td>
<td>39</td>
</tr>
<tr>
<td>Notes</td>
<td>40</td>
</tr>
<tr>
<td>3  Three-dimensional staircases</td>
<td>41</td>
</tr>
<tr>
<td>3.1 Monomial ideals in two variables</td>
<td>42</td>
</tr>
<tr>
<td>3.2 An example with six monomials</td>
<td>44</td>
</tr>
<tr>
<td>3.3 The Buchberger graph</td>
<td>47</td>
</tr>
<tr>
<td>3.4 Genericity and deformations</td>
<td>49</td>
</tr>
<tr>
<td>3.5 The planar resolution algorithm</td>
<td>53</td>
</tr>
<tr>
<td>Exercises</td>
<td>58</td>
</tr>
<tr>
<td>Notes</td>
<td>60</td>
</tr>
<tr>
<td>4  Cellular resolutions</td>
<td>61</td>
</tr>
<tr>
<td>4.1 Construction and exactness</td>
<td>62</td>
</tr>
<tr>
<td>4.2 Betti numbers and K-polynomials</td>
<td>65</td>
</tr>
<tr>
<td>4.3 Examples of cellular resolutions</td>
<td>67</td>
</tr>
<tr>
<td>4.4 The hull resolution</td>
<td>71</td>
</tr>
<tr>
<td>4.5 Subdividing the simplex</td>
<td>76</td>
</tr>
</tbody>
</table>
5 Alexander duality 81
  5.1 Simplicial Alexander duality 81
  5.2 Generators versus irreducible components 87
  5.3 Duality for resolutions 91
  5.4 Cohull resolutions and other applications 95
  5.5 Projective dimension and regularity 100

Exercises 104
Notes 105

6 Generic monomial ideals 107
  6.1 Taylor complexes and genericity 107
  6.2 The Scarf complex 110
  6.3 Genericity by deformation 115
  6.4 Bounds on Betti numbers 119
  6.5 Cogeneric monomial ideals 122

Exercises 125
Notes 126

II Toric Algebra 127

7 Semigroup rings 129
  7.1 Semigroups and lattice ideals 129
  7.2 Affine semigroups and polyhedral cones 133
  7.3 Hilbert bases 137
  7.4 Initial ideals of lattice ideals 142

Exercises 146
Notes 148

8 Multigraded polynomial rings 149
  8.1 Multigradings 149
  8.2 Hilbert series and K-polynomials 153
  8.3 Multigraded Betti numbers 157
  8.4 K-polynomials in nonpositive gradings 161
  8.5 Multidegrees 165

Exercises 170
Notes 172

9 Syzygies of lattice ideals 173
  9.1 Betti numbers 173
  9.2 Laurent monomial modules 176
  9.3 Free resolutions of lattice ideals 181
  9.4 Genericity and the Scarf complex 187

Exercises 189
Notes 190
### 10 Toric varieties
  10.1 Abelian group actions ........................................... 191
  10.2 Projective quotients ............................................. 194
  10.3 Constructing toric varieties ..................................... 198
  10.4 Toric varieties as quotients ..................................... 203
    Exercises .......................................................... 207
    Notes .............................................................. 208

### 11 Irreducible and injective resolutions
  11.1 Irreducible resolutions .......................................... 209
  11.2 Injective modules ................................................. 212
  11.3 Monomial matrices revisited ..................................... 215
  11.4 Essential properties of injectives ............................... 218
  11.5 Injective hulls and resolutions .................................. 221
    Exercises .......................................................... 225
    Notes .............................................................. 227

### 12 Ehrhart polynomials
  12.1 Ehrhart from Hilbert .............................................. 229
  12.2 Dualizing complexes ............................................. 232
  12.3 Brion’s Formula .................................................. 236
  12.4 Short rational generating functions ............................. 241
    Exercises .......................................................... 245
    Notes .............................................................. 246

### 13 Local cohomology
  13.1 Equivalent definitions ........................................... 247
  13.2 Hilbert series calculations ...................................... 253
  13.3 Toric local cohomology ........................................... 256
  13.4 Cohen–Macaulay conditions ....................................... 262
  13.5 Examples of Cohen–Macaulay rings ................................. 266
    Exercises .......................................................... 268
    Notes .............................................................. 269

### III Determinants
  14 Plücker coordinates ................................................. 273
  14.1 The complete flag variety ....................................... 273
  14.2 Quadratic Plücker relations .................................... 275
  14.3 Minors form sagbi bases ......................................... 279
  14.4 Gelfand–Tsetlin semigroups ..................................... 284
    Exercises .......................................................... 286
    Notes .............................................................. 287
15 Matrix Schubert varieties 289
  15.1 Schubert determinantal ideals 290
  15.2 Essential sets 294
  15.3 Bruhat and weak orders 295
  15.4 Borel group orbits 299
  15.5 Schubert polynomials 304
    Exercises 308
    Notes 309

16 Antidiagonal initial ideals 311
  16.1 Pipe dreams 312
  16.2 A combinatorial formula 315
  16.3 Antidiagonal simplicial complexes 318
  16.4 Minors form Gröbner bases 323
  16.5 Subword complexes 325
    Exercises 328
    Notes 329

17 Minors in matrix products 331
  17.1 Quiver ideals and quiver loci 331
  17.2 Zelevinsky map 336
  17.3 Primality and Cohen–Macaulayness 341
  17.4 Quiver polynomials 343
  17.5 Pipes to laces 348
    Exercises 351
    Notes 352

18 Hilbert schemes of points 355
  18.1 Ideals of points in the plane 355
  18.2 Connectedness and smoothness 359
  18.3 Haiman’s theory 363
  18.4 Ideals of points in \(d\)-space 368
  18.5 Multigraded Hilbert schemes 373
    Exercises 377
    Notes 377

References 379

Glossary of notation 397

Index 401