Reading assignments in [Vakil]
- by Tuesday 6 September: §2.1, §2.2, §2.3, §2.4
- by Thursday 8 September: §2.6
- by Thursday 15 September: §2.7, §2.5

Exercises: In [Vakil], exercises have labels C.S.N, for “Chapter C, Section S, Exercise N”, where C, S ∈ ℤ+ and N ∈ A, . . . , Z. Exercises marked “[essential]” are essential.

2.2.C
2.2.G (a) [essential]
   (b)
   (c) Demonstrate that 2.2.F is a special case of part (a) by considering the projection Y × X → X.

2.2.I
2.3.C [essential] (Note: Most commonly, sheaf-hom is denoted using some form of calligraphy or math italics, such as \( \mathcal{H}om(F, G) \), since \( \text{Hom}(F, G) \) is most often interpreted as the group of homomorphisms \( F \to G \) between objects \( F \) and \( G \) in the category of sheaves.)

2.3.J [essential]
2.4.E [essential]
2.4.M
2.4.P [essential]
2.6.A
2.6.B
2.6.C
2.6.F [The part about the global section functor not being exact is required]
2.6.G [essential]
Non-book Exercises: These exercises give an inkling of the flexibility of sheaf theory.

1. Fix a real vector space $Q$ of finite dimension and a cone $C \subseteq Q$, meaning an additive submonoid of $Q$ (so $0 \in C$ and $C + C \subseteq C$). Assume $C$ has only the trivial unit $0$. Prove that the relation on $Q$ that sets $q \preceq q'$ if $q' \in q + C$ constitutes a partial order. Prove that every partial order on $Q$ such that $p \preceq q \Rightarrow p + r \preceq q + r$ for all $r \in Q$ arises this way.

**Definition.** Fix a partially ordered real vector space $Q$ whose positive cone $C$ is closed in the usual topology and contains all positive real rescalings of itself. An *upset* in $Q$ is a subset $U$ closed under addition by $C$, so $U + C \subseteq U$.

- The *conic topology* on $Q$ consists of the upsets that are open in the ordinary topology.
- The *Alexandrov topology* consists of all of the upsets in $Q$.

To avoid confusion when it might occur, write

- $Q^{\text{con}}$ for the set $Q$ with the conic topology,
- $Q^{\text{ale}}$ for the set $Q$ with the Alexandrov topology, and
- $Q^{\text{ord}}$ for the set $Q$ with its ordinary topology.

In the situation of this definition, prove the following.

2. The identity on $Q$ yields continuous maps of topological spaces

$$
\iota : Q^{\text{ord}} \to Q^{\text{con}} \quad \text{and} \quad j : Q^{\text{ale}} \to Q^{\text{con}}.
$$

3. Any sheaf $\mathcal{F}$ on $Q^{\text{ord}}$ pulled back from $Q^{\text{con}}$ has natural maps on stalks

$$
\mathcal{F}_q \to \mathcal{F}_{q'} \quad \text{for } q \preceq q' \text{ in } Q.
$$

4. Similarly, any sheaf $\mathcal{G}$ on $Q^{\text{ale}}$ has natural maps on stalks

$$
\mathcal{G}_q \to \mathcal{G}_{q'} \quad \text{for } q \preceq q' \text{ in } Q.
$$

5. If sheaves $\mathcal{F}$ on $Q^{\text{ord}}$ and $\mathcal{G}$ on $Q^{\text{ale}}$ are both pulled back from the same sheaf $\mathcal{E}$ on $Q^{\text{con}}$, then the diagrams of vector spaces indexed by $Q$ in items 3 and 4 are the same.

6. The pushforward functor $j_*$ is exact, and $j_*j^{-1}\mathcal{E} \cong \mathcal{E}$.

Note: the diagrams in items 3 and 4 are called $Q$-modules. The functor in item 4 from sheaves on $Q^{\text{ale}}$ to $Q$-modules is an equivalence of categories.

References