Math 501 Homework #4, Fall 2023

Instructor: Ezra Miller

Solutions by: ...your name...

Collaborators: ...list those with whom you worked on this assignment...

Due: 11:59pm on Saturday 4 November 2023

Exercises /60

1. Determine the class equations of the groups of order 12. /3
2. Prove that if p is prime then every group of order $2p$ is either cyclic or dihedral. /3
3. Given a group of order 30, show that it has a normal subgroup of order 3 or 5. /3
4. For permutations σ and π, must $\sigma\pi$ and $\pi\sigma$ have cycle decompositions of the same type? /3
5. What is the largest order of an element of S_7? /3
6. Prove that the symmetric group S_n is generated by $(12\cdots n)$ and (12). /3
7. In the free group F on x and y, show directly that the elements $u = x^2$, $v = y^2$, and $w = xy$ generate a free subgroup on u, v, and w. /3
8. In any group, show that $\langle a, b\rangle = \langle bab^2, bab^3\rangle$. /3
9. Does $y^{-7}x^4y^{16}x^5$ lie in the smallest normal subgroup (of any group) containing xy? /3
10. Let $X \subseteq G$ be a subset and R a (perhaps incomplete) set of relations on X in G. Show that the map $F_X \to G$ from the free group F_X induced by $X \subseteq G$ factors through F_X/N, where N is the normal subgroup of F_X generated by R. /3
11. Prove that the normal subgroup of the free group $F_{\{x,y\}}$ generated by the single commutator $xyx^{-1}y^{-1}$ is the entire commutator subgroup. /3
12. Does every finite group admit a presentation with a finite set of defining relations? /3
13. Let R be the ring of all continuous real-valued functions on the closed interval $[0, 1]$. Prove that the map $\varphi : R \to \mathbb{R}$ defined by $\varphi(f) = \int_0^1 f(t)dt$ is a homomorphism of additive groups but not a ring homomorphism. /3
14. Prove that the ring $M_2(\mathbb{R})$ of 2×2 matrices with real entries contains a subring isomorphic to the field $\mathbb{C}$ of complex numbers. /3
15. An element x in a commutative ring R is *nilpotent* if $x^n = 0$ for some $n \in \mathbb{N}$.
    (a) Prove that the set of nilpotent elements of R forms an ideal. (It is called the *nilradical* of R.) /3
    (b) What happens without the commutative hypothesis on R? /3
    (c) Bonus: Prove that the nilradical of a commutative ring equals the intersection of all of its prime ideals. /3

16. Let R be a commutative ring and $f \in R[x]$ a univariate polynomial over R.
    (a) Prove that f is nilpotent if and only if all of its coefficients are nilpotent in R. /3
    (b) Prove that f is a unit if and only if all of its coefficients are nilpotent in R except for its constant term, which is a unit of R. /3
17. Let $\varphi : R \to S$ be a homomorhpism of commutative rings. If $P \subset S$ is a prime ideal, then show that its preimage $\varphi^{-1}(P)$ is a prime ideal of R. /3