Math 501 Homework #2, Fall 2017
Instructor: Ezra Miller

Solutions by: ...your name...

Collaborators: ...list those with whom you worked on this assignment...

Due: start of class on Wednesday 27 September 2017

Exercises

1. Determine the automorphism groups of the integers \( \mathbb{Z} \), the symmetric group \( S_3 \), and the cyclic group \( C_{10} \).

2. Find all subgroups of \( S_3 \) and determine which are normal.

3. Given two homomorphisms \( \varphi \) and \( \psi \) from a group \( G \) to \( G' \), let \( H \subseteq G \) be the subset where \( \varphi \) and \( \psi \) agree: \( H = \{ x \in G \mid \varphi(x) = \psi(x) \} \). Is \( H \) a subgroup of \( G' \)?

4. Prove that the center of any group is a normal subgroup.

5. If \( \varphi : G \rightarrow G' \) is a surjective homomorphism and \( N \trianglelefteq G \) is a normal subgroup, prove that the image \( \varphi(N) \trianglelefteq G' \) is also a normal subgroup.

6. Is the intersection \( R \cap R' \) of two equivalence relations in \( S \times S \) an equivalence relation on \( S \)? Is the union?

7. Prove that every group whose order is a power of a prime \( p \) contains an element of order \( p \).

8. Let \( \mathbb{F} \) be a field and \( W \) the solution set in \( \mathbb{F}^n \) of a system of homogeneous linear equations \( Ax = 0 \). Show that the solution set of any inhomogeneous system \( Ax = b \) is a coset of \( W \).

9. Prove that every index 2 subgroup is normal. Exhibit a non-normal index 3 subgroup.

10. Classify all groups of order 6. Hint: is there an element of order 6? Of order 3 but not of order 6? Or no element of order 3?

11. If \( G \) and \( G' \) are finite groups whose orders are relatively prime, prove that there is a unique homomorphism \( G \rightarrow G' \).

12. Fix subgroups \( H \) and \( K \) of a group \( G \). Prove that the intersection \( xH \cap yK \) of cosets is either empty or else is a coset of \( H \cap K \). Conclude that if \( H \) and \( K \) have finite index in \( G \) then so does \( H \cap K \).

13. Prove that a group of order 30 can have at most seven subgroups of order 5.
14. Fix a surjective group homomorphism \( \varphi : G \to G' \) with kernel \( K \). Show that the set of subgroups of \( G \) containing \( K \) and the set of all subgroups of \( G' \) are in bijection via the map \( H \mapsto \varphi(H) \). If \( H \trianglelefteq G \), must it be that \( \varphi(H) \trianglelefteq G' \)?

15. Is the symmetric group \( S_3 \) a direct product of nontrivial groups?

16. Prove that the product of two infinite cyclic groups is not cyclic. Is the same true without the word “infinite”?

17. Fix a group \( G \) whose order is \( |G| = ab \). Suppose that \( G \) has subgroups \( H \) and \( K \) with orders \( |H| = a \) and \( |K| = b \). Assume that \( |H \cap K| = 1 \). Prove that \( HK = G \). Is \( G \) isomorphic to the product group \( H \times K \)?

18. Suppose that a group \( G \) has a partition \( P \) with the property that for any pair of blocks \( A \) and \( B \) of the partition, the product \( AB \) is contained entirely within a block of \( P \). Let \( N \) be the block that contains the identity \( e \) of \( G \). Prove that \( N \trianglelefteq G \) and that \( P \) is the partition of \( G \) into the set of cosets of \( N \).

19. Let \( H = \{ \pm 1, \pm i \} \subset \mathbb{C}^\times \), the subgroup of fourth roots of unity. Describe the cosets of \( H \) in \( \mathbb{C}^\times \) explicitly (geometrically), and prove that \( \mathbb{C}^\times / H \cong \mathbb{C}^\times \).

20. Fix a group \( G \). Let \( N = \langle xyx^{-1}y^{-1} \mid x, y \in G \rangle \) be the subgroup of \( G \) generated by the commutators of pairs of elements of \( G \). Prove that \( N \) is normal and the quotient \( G/N \) is abelian. Moreover, show that any homomorphism \( G \to G' \) to an abelian group \( G' \) contains \( N \) in its kernel.

21. Assume that both \( H \) and \( K \) are normal subgroups of a group \( G \) and that \( |H \cap K| = 1 \). Prove that \( xy = yx \) for all \( x \in H \) and \( y \in K \). Hint: prove that \( xyx^{-1}y^{-1} \in H \cap K \).

22. Find a nonabelian group \( G \) and a proper normal subgroup \( N \) such that \( G/N \) is abelian.