24.

Free modules/PID

<u>Def</u>: The <u>rank</u> of a free module F over a nonzero commutative ring is

$$\operatorname{rank} F = |\text{basis of } F|.$$

<u>Lemma</u>: does not depend on basis.

<u>Pf</u>: Suppose $F \cong \bigoplus_{s\in S} R$. Let $\mathfrak{p} \subseteq R$ be maximal.

$F/\mathfrak{p}F \cong \bigoplus_{s\in S} R/\mathfrak{p}$ is a vector space over $R/\mathfrak{p}$ of dim $|S|$. $\square$

<u>Thm</u>: Fix F free / PID R and a submodule $M \subseteq F$. Then M is free of rank $\leq$ rank F.

<u>Pf</u>: $F \cong \bigoplus_{\lambda\in\Lambda} Rx_\lambda$ for a basis $\{x_\lambda\}_{\lambda\in\Lambda}$.

$J \subseteq \Lambda \Rightarrow M_J \overset{\text{def}}{=} M \cap \bigoplus_{j\in J} Rx_j$ has the form

(*) and $M_J = \bigoplus_{j\in J} Ry_j$ for some $y_j \in M_J$ Warning: some of the y_j might be 0

or not. Order the set $\mathcal{Y}$ of

families $\{y_j\}_{j\in J}$ for which $\exists$ basis $\{x_\lambda\}_{\lambda\in\Lambda}$ satisfying (*)

by inclusion: $\{y_j\}_{j\in J} \subseteq \{y_j'\}_{j\in J'}$ if $J \subseteq J'$ and $y_j = y_j'$ $\forall j \in J$.

If $\mathcal{C}$ is a chain in $\mathcal{Y}$ then $\bigcup_{C\in\mathcal{C}} C \in \mathcal{Y}$ since any dependence relation involves only finitely many y_j.

Hence $\exists$ family $\{y_j\}_{j\in J}$ maximal in $\mathcal{Y}$. Want $J = \Lambda$.

Suffices: $k \in \Lambda \setminus J \Rightarrow \times$. Let $K = J \cup \{k\}$ and $M \hookrightarrow F \twoheadrightarrow Rx_k$ (π_k).

Then $\pi_k(M_K) = \langle a \rangle x_k \subseteq Rx_k$ since R is a PID.

But $\ker \pi_k|_{M_K} = M_J$, so

$$0 \to M_J \to M_K \to \pi_k(M_K) \to 0$$

is exact and splits because $\langle a \rangle x_k$ is free!

Thus $\{y_j\}_{j\in K} \in \mathcal{Y}$ if $y_k = ax_k$. $\times$

So $J = \Lambda$. $\square$

<u>Cor</u>: M finitely generated / PID R and $N \subseteq M$ submodule $\Rightarrow$ N f.g.

<u>Pf</u>: $f: R^n \twoheadrightarrow M \Rightarrow f^{-1}(N)$ free of rank $\leq n$

$\Rightarrow N$ f.g. since $f^{-1}(N) \twoheadrightarrow N$. $\square$