14.

Groups of order 12

Fix
- a group G with $|G| = 12$
- $H \leq G$ a Sylow 2-subgroup $|H| = \cancel{?}\ 4$
- $K \leq G$ a Sylow 3-subgroup $|K| = \cancel{?}\ 3$

Sylow 3 $\Rightarrow$ H has 1 or 3 conjugates $\quad |3$ and $\equiv 1\ (2)$

K has 1 or 4 conjugates $\quad |4$ and $\equiv 1\ (3)$

$H \cong C_4$ or $H \cong C_2 \times C_2$ Klein 4-group

<u>Lemma</u>: $H \triangleleft G$ or $K \triangleleft G$ or both.

<u>Pf</u>: $K \ntriangleleft G \Rightarrow 4$ conjugates of K

$\Rightarrow$ 8 elements of order 3 in G

$\Rightarrow$ 4 elements remain.

Since H has no elements of order 3,

$H = G \setminus \{\text{elements of order } 3\}$.

$H \triangleleft G$ since same is true of gHg^{-1}. $\square$

Lemma $\Rightarrow$ $G = H \times K$ if both are normal

$H \rtimes K$ if $K \ntriangleleft G$

$K \rtimes H$ if $H \ntriangleleft G$

- $H \times K$: $G \cong C_4 \times C_3$ or $G \cong C_2 \times C_2 \times C_3$
- $H \rtimes K$: $G \circlearrowright X = \{\text{conjugates of } K\}$ by conjugation

$|X| = 4 \Rightarrow G \to S_4$

$X = \{K_1, \dots, K_4\}$

$$\ker(G \to S_4) = \underset{\underset{K_1}{\cup}}{N_G(K_1)} \cap \dots \cap \underset{\underset{K_4}{\cup}}{N_G(K_4)}$$

$N_G(K_i) = |G_{K_i}| = |G| / |\mathcal{O}_{K_i}| = 12/4 = 3$

$\Rightarrow N_G(K_i) = K_i$

$\Rightarrow \bigcap_{i=1}^{4} N_G(K_i) = \{1\}$

$\Rightarrow G \hookrightarrow S_4$.

But G ~~has~~ generated by 8 $\underbrace{\text{elements of order 3 in } S_4}_{\text{3-cycles}}$

(why? $|\text{subgroup}| \,\big|\, |G| = 12$)

$\Rightarrow$ every permutation in G is even

$\Rightarrow G = A_4$ since $|A_4| = 12$.

Note: $H \cong C_2 \times C_2$ is forced

- $K \rtimes H$: Let $K = \{1, y, y^2\}$.

$\operatorname{Aut} K \cong C_2 = \{id_K, (y \leftrightarrow y^2)\}$

Case 1: $\underset{\{1,x,x^2,x^3\}}{\overset{}{H}} \cong C_4 \Rightarrow \exists!$ nontrivial $H \to \operatorname{Aut} K$

$x \mapsto (y \leftrightarrow y^2)$

$x^2 \mapsto id_K$

$$G = \langle x, y \mid x^4 = 1,\ y^2 = 1,\ xyx^{-1} = y^2 \rangle.$$

Case 2: $H \cong C_2 \times C_2'$

$H \xrightarrow{\varphi} \operatorname{Aut} K$ nontrivial $\Rightarrow \exists!$ $\underset{u \neq 1}{u \in H}$ with $u : y \mapsto y$

$\langle u \rangle = \ker \varphi$; choose this to be C_2'

$$\left.\begin{array}{l} C_2 \hookrightarrow \operatorname{Aut} K \\ C_2' \text{ commutes with } K \end{array}\right\} \Rightarrow G = (K \rtimes C_2) \times C_2$$

$$\cong S_3 \times C_2$$

$$\cong D_6 \quad [M1\#1(c)].$$

Conclusion: #(groups of order 12) = 5.