

8.

Group actions

General principle: $\text{Aut}(\text{anything}) \cap \text{symmetries}(\text{anything}) = G$ is a group

$\begin{cases} g: X \xrightarrow{\sim} X \\ g': X \rightarrow X \\ \text{id}: X \rightarrow X \end{cases} \Rightarrow g' \circ g: X \rightarrow X$

$g^{-1}: X \rightarrow X$ $x \in X \Rightarrow gx \in X$

$g'gx \in X$ $1x \in X$

often algebraic, in this class
but there's no real difference

Def: A group action (or operation) of G on a set S is a map

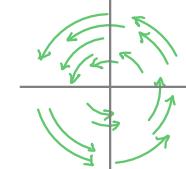
$$\begin{aligned} G \times S &\rightarrow S & \text{satisfying} && (i) 1s = s \\ (g, s) &\mapsto gs & && (ii) g'(gs) = (g'g)s \quad \forall g, g' \in G \text{ and } s \in S \end{aligned}$$

Terminology: S is a G -set with a left action (right action $S \times G \rightarrow S$)

E.g. 1. $S = F^n$ $G = GL_n F$ algebraic? geometric?

$$\begin{array}{c} \text{point} \\ \text{line} \\ \text{plane} \\ \vdots \end{array} \quad \begin{array}{c} SL_n F \\ O_n F \\ SO_n F \end{array}$$

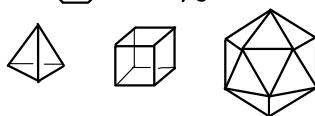
$$\text{e.g. } SO_2 \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}^2$$



2. $G = S_n$ $S = \{1, \dots, n\}$ neither: preserves no algebraic or geometric structure

3. $S = \text{points in } \mathbb{R}^n$ $G = \text{rigid motions or translations}$ geometric

4. $S = \text{polygon}$ $G = \text{cyclic, dihedral,}$



$$\begin{array}{c} A_4 \\ S_4 \\ S_4 \times C_2 \\ H_3 \end{array}$$

geometric

5. $S = \mathbb{C}$ $G = C_2 = \{1, r\}$ $r \cdot \alpha = \bar{\alpha}$ algebraic? geometric?

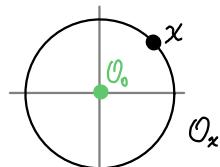
6. $S = \mathbb{Z}/10\mathbb{Z}$ $G = C_4 = \text{Aut}(\mathbb{Z}/10\mathbb{Z})$ algebraic

Notation: $\lambda_g: S \rightarrow S$ left multiplication by g Q. $\lambda_g \circ \lambda_{g'} = ?$ A. $\lambda_1 = \text{id}_S$

7. $S = F^{n \times}$ $G = GL_n F$ left actions $\lambda_g(s) = gs$ $\lambda'_g(s) = sg^{-1}$
right actions $\rho_g(s) = sg$ $\rho'_g(s) = g^{-1}s$

Def: Let G act on S . The orbit of $s \in S$ is $\mathcal{O}_s = \{gs \mid g \in G\} = Gs$.

E.g. 1. $G = SO_2 \mathbb{R}$ $S = \mathbb{R}^2$



$$G_0 = SO_2 \mathbb{R}$$

$$G_x = \text{id}_{\mathbb{R}^2} \quad \forall x \neq 0$$

2. $\mathcal{O}_i = \{1, \dots, n\}$ only one orbit: transitive action

	$G = \text{rigid motions}$	$G = \text{translations}$
points	transitive	transitive
lines	transitive	$\mathcal{O}_l = \{l' \subseteq \mathbb{R}^n \mid l' \parallel l\}$ parallel class of l
planes	transitive $G_s = \{e\} \forall s$	" $G_l = \{\text{translations parallel to } l\}$

4. skip fixes l as a set but maybe not pointwise

$$5. \mathcal{O}_\alpha = \{\alpha, \bar{\alpha}\} \text{ if } \alpha \notin \mathbb{R} \quad G_\alpha = \{1\}$$

$$\mathcal{O}_\alpha = \{\alpha\} \text{ if } \alpha \in \mathbb{R} \quad G_\alpha = \{1, \alpha\}$$

Lemma: The orbits of G on S partition S .

Pf: Write $s \sim s'$ if $s' = gs$ for some $g \in G$. Show it's an equivalence relation. \square

Note: G acts transitively on each orbit.

Def: The stabilizer of $s \in S$ is $G_s = \{g \in G \mid gs = s\} \leqslant G$.

Lemma: $gs = hs \Leftrightarrow g^{-1}hs = s \Leftrightarrow g^{-1}h \in G_s$. \square

E.g. 1. $G_0 = \text{Sp}_{\mathbb{R}^2}$ 5. $G_\alpha = \{1\}$ 3. $G_s = \{e\} \forall s$
 $G_x = \text{id}_{\mathbb{R}^2} \forall x \neq 0$ 5. $G_\alpha = \{1, \alpha\}$ 3. $G_l = \{\text{translations parallel to } l\}$

General: bigger orbit \Leftrightarrow smaller stabilizer

Lemma: $H \leqslant G \Rightarrow G$ acts transitively on G/H via $g \cdot aH = gaH$

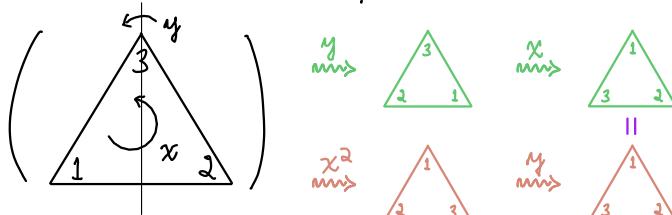
Pf: $(ba^{-1})aH = bH$. \square

$$C \in G/H \Rightarrow gC = \{gc \mid c \in C\}$$

Q. $G_{1+H} = ?$ $H!$

Note: $hH \Rightarrow hH = H$, but h doesn't act trivially on H .

E.g. $G = D_3 = \text{symmetries}$



$$= \langle x, y \mid x^3 = 1, y^2 = 1, xy = yx^2 \rangle$$

$$H = \{1, y\}$$

$$G/H = \left\{ \begin{array}{l} \{1, y\} = C_1 \\ \{x, xy\} = C_2 \\ \{x^2, x^2y\} = C_3 \end{array} \right\}$$

$$x: \begin{matrix} & C_3 & \\ & \curvearrowleft & \curvearrowright \\ C_1 & \curvearrowright & C_2 \end{matrix} \quad (1 \ 2 \ 3)$$

$$y: \begin{matrix} & C_1 & \\ & \curvearrowleft & \curvearrowright \\ C_2 & \curvearrowright & C_3 \end{matrix} \quad (2 \ 3)$$

$$yx^2 = xy$$

$$yxy = x^2$$