

4.

## Homomorphisms

Def: A map  $\varphi: G \rightarrow G'$  of groups is a • homomorphism if  $\varphi(ab) = \varphi(a)\varphi(b)$   $\forall a, b \in G$

- isomorphism if also  $\varphi$  is bijective; write  $G \xrightarrow{\varphi^{-1} \text{ also isom.}} G'$  or  $G \cong G'$

### Examples

1.  $GL_n F \xrightarrow{\det} F^*$      $\det(AB) = \det A \det B$      $O_n \mathbb{R} \xrightarrow{\det} \{\pm 1\}$     ( $= \mathbb{Z}^*$ )
2. Fix  $P \in GL_n F$ . Then  $GL_n F \cong GL_n F$     conjugation by  $P$     change of basis
3.  $C_\infty = \text{infinite cyclic group } \langle a \rangle = \{ \dots, a^{-2}, a^{-1}, e, a, a^2, \dots \}$      $|a| = \infty$

$$\Rightarrow \mathbb{Z} \xrightarrow{\quad} C_\infty \quad \varphi(m+n) = a^{m+n} = a^m a^n = \varphi(m)\varphi(n)$$

composition in  $\mathbb{Z}$     composition in  $C_\infty$

4.  $\varphi: F^+ \rightarrow GL_n F$      $a \mapsto \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$      $\varphi(a+b) = ?$      $F = \text{field } \checkmark$      $F = \mathbb{Z} ? \checkmark$
5.  $\mathbb{Q}^n \xrightarrow{\varphi} \mathbb{Q}^m$     homomorphism  $\Leftrightarrow$  linear

Exercise: Prove there is a group homomorphism  $\mathbb{R}^n \rightarrow \mathbb{R}^m$  that isn't linear!

Lemma:  $G \cong G' \Rightarrow |G| = |G'|$ .

If  $\varphi: G \cong G'$  then  $|\varphi(a)| = |\varphi(a)| \quad \forall a \in G$ .  $\square$

Def: An isomorphism  $G \cong G$  is an automorphism of  $G$ .

E.g. conjugation by  $g \in G$  is an automorphism:  $a \mapsto gag^{-1}$     inverse?

$$gbg^{-1} \leftarrow b$$

E.g.  $\text{Aut } C_8 = ?$

$$C_8 = \langle a \rangle = \{e, a, a^2, a^3, a^4, a^5, a^6, a^7\}$$

orders: 1 8 4 8 2 8 4 8

$\varphi: C_8 \rightarrow \text{Aut } G$  determined by  $\varphi(a)$  since  $\varphi(a^n) = \varphi(a)^n \quad \forall n \in \mathbb{Z}$

$$|a| = 8 \Rightarrow |\varphi(a)| \in \{1, a^3, a^5, a^7\}.$$

Lemma:  $G$  abelian  $\Rightarrow$   $\varphi_n: G \rightarrow G$  is a homomorphism.  
 $g \mapsto g^n$

$$\text{Pf: } \varphi_n(gh) = (gh)^n = g^n h^n = \varphi(g)\varphi(h). \quad \square$$

Lemma  $\Rightarrow \Psi(a) = a^n$  homomorphism  $\forall n \in \{1, 3, 5, 7\}$ .  
 $\Rightarrow$  isomorphism, since bijective

Prop: If  $G \xrightarrow{\Psi} G'$  is a homomorphism then

(i)  $\varphi(e) = e'$  and  $\varphi(a^{-1}) = \varphi(a)^{-1} \quad \forall a \in G.$

$$(ii) \quad \text{im } \varphi = \{\varphi(a) \mid a \in G\} \text{ and}$$

$\ker \varphi = \{a \in G \mid \varphi(a) = e'\}$  are subgroups.

(iii)  $\ker \psi$  is a normal subgroup:  $a \in \ker \psi$  and  $g \in G \Rightarrow gag^{-1} \in \ker \psi$ .

closed under conjugation by  $G$

(iv)  $\varphi$  surjective  $\Leftrightarrow \text{im } \varphi = G'$ .

$\varphi$  injective  $\Leftrightarrow \ker \varphi = \{e\}$ .

$$\underline{\text{Pf:}} \quad (\text{i}) \quad \varphi(a a^{-1}) = \varphi(a) \varphi(a^{-1}) \quad a = e \Rightarrow \varphi(e) = \varphi(e)^{-1} \Rightarrow e' = \varphi(e)$$

$$\varphi(e) = e' \Rightarrow \varphi(a^{-1}) = \varphi(a)^{-1}$$

$$(ii) \quad a' = \varphi(a) \quad \text{and} \quad b' = \varphi(b) \quad \Rightarrow \quad a'b' = \varphi(a)\varphi(b) = \varphi(ab) \quad \checkmark$$

↓

$$(\alpha')^{-1} = \varphi(\alpha)^{-1} = \varphi(\alpha^{-1}). \quad \checkmark$$

$$\varphi(a) = e' \quad \text{and} \quad \varphi(b) = e' \quad \Rightarrow \quad \varphi(ab) = \varphi(a)\varphi(b) = e'e' = e'. \quad \checkmark$$

↓

$$\varphi(a^{-1}) = \varphi(a)^{-1} = (e')^{-1} = e'. \quad \checkmark$$

$$(iii) \quad a \in \ker \varphi \Rightarrow \varphi(a) = e' \Rightarrow \varphi(gag^{-1}) = \varphi(g)e'\varphi(g^{-1}) \\ = \varphi(g)\varphi(g^{-1})^{\textcolor{red}{\checkmark}} = e'. \quad \checkmark$$

(iv) surj. ✓

$$\varphi(a) \neq \varphi(b) \Leftrightarrow \varphi(a)\varphi(b)^{-1} \neq e'$$

$$\Leftrightarrow \varphi(a)\varphi(b^{-1}) = \varphi(ab^{-1}) \neq e'.$$

This is true  $\wedge a \neq b$  precisely when

$$\varphi(ab^{-1}) = e' \iff ab^{-1} = e \quad \forall a, b \in G$$

$$\text{i.e. } \varphi(c) = e' \Leftrightarrow c = e \quad \forall c \in G. \quad \square$$