Math 403 Homework #5, Spring 2022
Instructor: Ezra Miller
Solutions by: ...your name...
Collaborators: ...list those with whom you worked on this assignment...
(1 point for each of up to 3 collaborators who also list you)
Due: 5:00pm Tuesday 5 April 2022

Reading assignments
- for Thu. 24 March
  - [Treil, §8.5] multilinear algebra, tensor product
  - [Wikipedia, Tensor product]
- for Tue. 29 March and Thu. 31 March
  - [Wikipedia, Exterior algebra]

Exercises
1. For \( P \in \mathbb{R}^{n \times n} \) with \( P > 0 \), set \( \Gamma(P) = \{ \lambda \in \mathbb{R}_{>0} \mid Px \leq \lambda x \text{ for some nonzero } x \geq 0 \} \). Show that the dominant eigenvalue \( \lambda(P) \) satisfies \( \lambda(P) = \min_{\lambda \in \Gamma(P)} \lambda \).

2. In class, we stated the theorem that for \( P > 0 \) a stochastic \( n \times n \) matrix with dominant eigenvector \( v \), and \( x \geq 0 \) any nonzero vector, \( P^k x \rightarrow \alpha v \) as \( k \rightarrow \infty \) for some real \( \alpha > 0 \). But we only proved it when \( P \) is diagonalizable. Complete the proof.

3. Prove that \( P \) has a dominant positive eigenvalue if \( P \geq 0 \) and \( P^k > 0 \) for some \( k > 0 \).

4. Prove or disprove: the set of stochastic \( n \times n \) matrices is compact and convex.

5. Use the universal property of tensor products to prove commutativity: there is a unique isomorphism \( V \otimes W \rightarrow W \otimes V \) such that \( v \otimes w \mapsto w \otimes v \) for all \( v \in V \) and \( w \in W \).

6. Use the universal property of tensor products to prove associativity: there is a unique isomorphism \( (U \otimes V) \otimes W \rightarrow U \otimes (V \otimes W) \) such that \( (u \otimes v) \otimes w \mapsto u \otimes (v \otimes w) \) for all \( u \in U \), \( v \in V \), and \( w \in W \). Hint: You can either use the universal property to produce the map or check that the two parenthesizations have the same universal property regarding bilinear maps on \( (U \times V) \times W = U \times (V \times W) \) and appeal to “abstract nonsense”: universal constructions are unique up to unique isomorphism.

7. Prove that homomorphisms \( \varphi : V \rightarrow V' \) and \( \psi : W \rightarrow W' \) result in a canonical homomorphism \( \varphi \otimes \psi : V \otimes W \rightarrow V' \otimes W' \). Given matrices for \( \varphi \) and \( \psi \), write down a matrix for \( \varphi \otimes \psi \). Note: your answer will depend on how you order the basis of \( V \otimes W \).
8. Prove that a homomorphism \( \varphi : V \to W \) results in a canonical homomorphism \( \wedge^r \varphi : \wedge^r V \to \wedge^r W \). Given a matrix for \( \varphi \), write down a matrix for \( \wedge^r \varphi \). Note: make no attempt to draw a matrix; just describe its entries as labeled by pairs of basis vectors.

9. Prove that tensor products commute with direct sums: if \( I \) is any (finite or infinite) index set and \( V = \bigoplus_{i \in I} V_i \), then there is a natural isomorphism \( V \otimes W \to \bigoplus_{i \in I} V_i \otimes W \).

10. Construct a natural map \( V^* \otimes W^* \to (V \otimes W)^* \). Show that it is injective. If one of \( V \) and \( W \) has finite dimension, show that the map is an isomorphism.

11. Prove the existence of a bilinear map \( \wedge^r V \times \wedge^s V \to \wedge^{r+s} V \) taking
\[
(v_1 \wedge \cdots \wedge v_r, v'_1 \wedge \cdots \wedge v'_s) \mapsto v_1 \wedge \cdots \wedge v_r \wedge v'_1 \wedge \cdots \wedge v'_s.
\]
Write \( \omega = v_1 \wedge \cdots \wedge v_r \) and \( \omega' = v'_1 \wedge \cdots \wedge v'_s \), so \( \omega \wedge \omega' = v_1 \wedge \cdots \wedge v_r \wedge v'_1 \wedge \cdots \wedge v'_s \).
Show that \( \omega' \wedge \omega = (-1)^{rs} \omega \wedge \omega' \).