Math 403 Homework #4, Spring 2018
Instructor: Ezra Miller
Solutions by: ...your name...
Collaborators: ...list those with whom you worked on this assignment...
Due: noon on Saturday 31 March 2018

Reading assignments
- for Fri. 23 March
  - [Lax, Chapter 16, p.237–240] entrywise positive matrices, Perron’s theorem
- for Wed. 28 March
  - [Lax, Chapter 16, p.240–245] stochastic and nonnegative matrices, Frobenius thm
- for Fri. 30 March
  - [Treil, §8.5] multilinear algebra, tensor product
  - [Wikipedia, Tensor product]
- for Wed. 4 April and Fri. 6 April
  - [Wikipedia, Exterior algebra]

Exercises

1. Prove that a manifold, defined as in class, is naturally a topological space. More precisely, what it means for a neighborhood of any given point in the manifold to be open is well defined independent of which charts are used to verify the openness.

2. Let $V$ be a vector space over $\mathbb{R}$ or $\mathbb{C}$ of finite dimension. Let $\varphi : V \rightarrow W$ be an injective homomorphism. Show that if $\mu$ is a norm on $W$ then $\mu \circ \varphi$ is a norm on $V$.

3. Let $\nu$ be a norm on $V = \mathbb{R}^n$ or $\mathbb{C}^n$ with unit $\nu$-ball $B_\nu = \{x \in V \mid \nu(x) \leq 1\}$. Prove that $B_\nu$ is closed, bounded, convex, and equilibrated: $x \in B_\nu$ and $|\alpha| \leq 1 \Rightarrow \alpha x \in B_\nu$. Also show that the origin lies interior to $B_\nu$. Conversely, if $B \subseteq V$ is closed, bounded, convex, equilibrated subset with 0 in its interior, then a norm $\nu_B : V \rightarrow \mathbb{R}$ with unit ball $B$ can be defined by $\nu_B(x) = \inf\{\alpha > 0 \mid \frac{1}{\alpha}x \in B\}$.

4. The Hahn–Banach theorem says that for any norm $\nu$ on $\mathbb{C}^n$ and any subspace $V \subseteq \mathbb{C}^n$, if $\varphi : V \rightarrow \mathbb{C}$ is a linear functional whose maximum magnitude $|\varphi(x)|$ for $x$ in the unit $\nu$-sphere of $V$ is $\alpha$, then $\varphi$ can be extended to a linear functional on $\mathbb{C}^n$ whose maximum magnitude on the unit $\nu$-sphere in $\mathbb{C}^n$ is $\alpha$. Show that the Hahn–Banach theorem is equivalent to the statement that separating hyperplanes exist for points outside of any closed, bounded, convex set that contains 0 in its interior.
5. Let $V$ be the vector space of real (or complex) absolutely convergent series. The function $x \mapsto \|x\|_p = (\sum_{i=1}^{\infty} |x_i|^p)^{\frac{1}{p}}$ for $p \geq 1$ is a norm on $V$. (You may assume this, although no complaints if you’d like to prove it.) Prove that $\|x\|_p$ is not topologically equivalent to $\|x\|_q$ if $p < q$.

6. Show that $\|AB\|_2 \leq \|A\|_2 \|B\|_2$ whenever the product $AB$ is defined.

7. Show that $\lim_{k \to \infty} A^k = 0$ if and only if its spectral radius satisfies $\rho(A) < 1$.

8. Show that if $\nu$ is a consistent norm on $\mathbb{C}^{n \times n}$, then $\lim_{k \to \infty} \nu(A^k)^{\frac{1}{k}} = \rho(A)$.

9. Prove or disprove, and salvage the statement as best you can in case it is false:
   If $\|\tilde{\lambda} - A\|_2 \geq \eta$ then there is an eigenvalue $\lambda$ of $A$ satisfying
   $$|\tilde{\lambda} - \lambda| \leq 2(\|A\|_2 + \eta^{-1})\eta^{-\frac{1}{2}}.$$

10. For $A \in \mathbb{C}^{m \times n}$, set $\|A\|_\infty = \max_{\|x\|_\infty = 1} \|Ax\|_\infty$. Prove that $\|A\|_\infty = \max_i \sum_j |a_{ij}|$.

11. Assume that the union of $m$ out of the $n$ Gerschgorin disks $G_i$ is disjoint from the other $n - m$ disks. Prove that the union of $m$ disks contains precisely $m$ of the eigenvalues.
   Hint: how do the eigenvalues move as $A$ proceeds along a straight line to $A$?

12. Let $G(A)$ be the union of the Gerschgorin disks of $A$. Show that the intersection $\bigcap_S G(S^{-1}AS)$ over all nonsingular matrices $S$ equals the spectrum of $A$.

13. Prove that the spectrum of $A$ is contained in $G(A) \cap G(A^\top)$. Illustrate with the $3 \times 3$ matrix with entries $a_{ij} = i/j$.

14. Fix the matrix

   $$A = \begin{bmatrix} 7 & -16 & 8 \\ -16 & 7 & -8 \\ 8 & -8 & -5 \end{bmatrix}.$$

   (i) Use Gerschgorin’s Theorem to say as much as you can about the locations of the eigenvalues of and the spectral radius of $A$.

   (ii) Consider $D^{-1}AD$ for a diagonal matrix $D$ to see if you can improve your estimates for the eigenvalue locations.

   (iii) Compute the actual eigenvalues and comment on the quality of your estimates in (i) and (ii).