Math 403 Homework #3, Spring 2018
Instructor: Ezra Miller

Solutions by: ...your name...
Collaborators: ...list those with whom you worked on this assignment...
Due: 11:59pm on Friday 9 March 2018

Reading assignments
• for Fri. 2 March
  – [Serge Lang, Linear Algebra, Chapter XII, §1–§2]
• for Wed. 7 March
  – [Serge Lang, Linear Algebra, Chapter XII, §3–§4]
• for Fri. 9 March
  – [Lax, Chapter 14, p.214–221] norms, equivalence, continuity, local compactness
  – [Stewart–Sun, §II.1] norms, equivalence, Hahn–Banach theorem
• for Wed. 21 March
  – [Stewart–Sun, §II.2.1] matrix norms, consistency
  – [Stewart–Sun, §IV.1 through Thm 1.3] general perturbation theorems
• for Fri. 23 March
  – [Stewart–Sun, §IV.1.2 through Thm 1.6] Bauer–Fike theorem
  – [Lax, Appendix 7] Gershgorin’s Theorem
  – [Stewart–Sun, §IV.2 through §IV.2.1] Gerschgorin theory

Exercises
1. True or false: the sum of two normal operators is normal. Justify. /3
2. Orthogonally diagonalize the matrix $A = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$. Find all square roots of $A$; note that they are all self-adjoint. /3
3. Find a singular decomposition of the matrix $A = \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}$. Use it to find $\max_{\|x\| \leq 1} \|Ax\|$ and $\min_{\|x\| = 1} \|Ax\|$, as well as the vectors where this maximum and minimum are attained. Describe geometrically the image under $A$ of the closed unit disk in $\mathbb{R}^2$. /3
4. Prove that the operator norm of a matrix $A$ coincides with the Frobenius norm of $A$ if and only if $A$ has rank 1. /3
5. Find the Lie algebra $\mathfrak{so}_n$ of the special orthogonal group $SO_n(\mathbb{R})$. /3
6. Fix a matrix group $G \subseteq M_n F$ over the field $F \in \{\mathbb{R}, \mathbb{C}\}$ and a matrix $A \in G$. Show that the tangent space $T_A(G)$ of $G$ at $A$ is $Ag$, and show that this equals $gA$.

7. Prove that $\det(e^A) = e^{\text{tr}(A)}$.

8. Explain why $\mathfrak{su}_n = \mathfrak{u}_n \cap \mathfrak{sl}_n(\mathbb{C})$. (You may use the previous exercise.)

9. Let $\gamma : (-\varepsilon, \varepsilon) \to GL_n(F)$ be a differentiable path, where $F \in \{\mathbb{R}, \mathbb{C}\}$. Use Cramer’s rule (or some other method) to show that the inverse path $t \mapsto \gamma(t)^{-1}$ is differentiable.

10. Sketch the convex hull of \[
\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} \right\} \text{ and } \left\{ \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}.
\]

11. Fix an operator $\varphi : \mathbb{R}^n \to \mathbb{R}^n$. If $X \subseteq \mathbb{R}^n$ is convex and $x$ is an extreme point of $X$, must $\varphi(x)$ be an extreme point of $\varphi(X)$? What if $\varphi$ is assumed invertible?

12. Prove that the intersection of finitely many closed halfspaces in $\mathbb{R}^n$ can have only finitely many extreme points. Must it have any extreme points at all? If it does have extreme points, must the intersection equal the convex hull of its extreme points?

13. Fix a column vector $b \in \mathbb{R}^n$ and an $n \times n$ real matrix $A$. Show that the set of solutions $\{x \in \mathbb{R}^n \mid Ax \leq b\}$ of the inequality $Ax \leq b$ is convex, where $a \leq b$ if $a_i \leq b_i$ for all $i$. What kind of convex set is it? (No need to provide full justification of this last bit.)

14. Show that the space of positive (semi)definite real symmetric matrices is convex. Is the same true with “complex Hermitian” in place of “real symmetric”? 

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