Math 403 Homework #3, Spring 2022
Instructor: Ezra Miller

Solutions by: ...your name...
Collaborators: ...list those with whom you worked on this assignment...
(1 point for each of up to 3 collaborators who also list you)

Due: 5:00pm Tuesday 1 March 2022

Reading assignments

- for Tue. 22 February
  - [Treil, §6.3.3–§6.3.4] singular value decomposition
  - [Lax, Chap.7: §Norm of a Linear Map, §Spectral Radius]
  - [Lax, Chap.8: §Norm and Eigenvalues]
  - [Treil, §6.4] Applications of SVD: spectral radius, operator norm, condition number
- for Thu. 24 February
  - [Stewart–Sun, §II.2.1] matrix norms, consistency
  - [Stewart–Sun, §IV.1 through Thm 1.3] general perturbation theorems
- for Tue. 1 March
  - [Stewart–Sun, §IV.1.2 through Thm 1.6] Bauer–Fike theorem
  - [Lax, Appendix 7] Gershgorin’s Theorem
  - [Stewart–Sun, §IV.2 through §IV.2.1] Gershgorin theory
- for Thu. 3 March and Tue. 15 March
  - [Tapp, Chap.5] Lie algebras as tangent spaces to the identity
  - [Lax, Chap.9: Matrix-valued Functions, through §Matrix Exponential]
  - [Wikipedia, Exponential map (Lie theory)]

Exercises

1. Prove that a subset of a manifold is open if and only if its preimage under every chart is an open subset of a vector space. (Note: This can be used to specify the topology on a manifold without knowing what a topological space is, since vector spaces have the usual notion of “open”: what it means for a neighborhood of a point in a manifold to be open is well defined independent of which charts are used to verify openness.)

2. Prove that the sphere \( S^2 = \{ x \in \mathbb{R}^3 | ||x|| = 1 \} \) is a smooth manifold. Is it a rational algebraic variety over the field \( \mathbb{R} \)?

3. Fix a field \( F \). Prove that the unipotent lower-triangular matrices \( U^- \subseteq GL_n(F) \) map injectively to the set \( \mathcal{F}_n(F) \) of complete flags in \( F^n \) expressed as the set of orbits of the group \( B^+ \) acting on the right of \( GL_n(F) \); that is, \( U^- \rightarrow \mathcal{F}_n(F) = GL_n(F)/B^+ \).
4. Let $S_n \subseteq GL_n(F)$ be the set of permutation matrices. Prove that $\mathcal{F}_n(F)$ is a rational algebraic variety with atlas $\{ \pi_w : wU^\to \mathcal{F}_n(F) \mid w \in S_n \}$ naturally indexed by $S_n$. You will need to use that any level set of a polynomial with coefficients in $F$ is closed.

5. The standard complex structure on $\mathbb{R}^{2n}$ is the block-diagonal $2n \times 2n$ matrix $J_{2n}$ whose diagonal blocks are all \[ \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \] Prove that $A \in GL_{2n}(\mathbb{R})$ is complex-linear if and only if $A$ commutes with the standard complex structure: $AJ_{2n} = J_{2n}A$.

6. Prove that if $\lambda \in \mathbb{C}$ is an eigenvalue of a unitary matrix then $|\lambda| = 1$.

7. Show that the standard Hermitian inner product on $\mathbb{C}^n$ defines a distance $d(x, y) = ||x - y||$ on $\mathbb{C}^n$. Give an example of an isometry $\varphi$ of $\mathbb{C}^n$ such that $\varphi(0) = 0$ but $\varphi$ is not $\mathbb{C}$-linear.

8. Let $\varphi$ be an orthogonal transformation of a real inner product space $V$. Assume that $\varphi^2 = -I$. Show that $\text{dim} V$ is even, say $2n$. Moreover, prove that there exists a dimension $n$ subspace $W \subset V$ and an isometry $\psi : W \to W^\perp$ such that, in the decomposition $V = W \oplus W^\perp$, the operator $\varphi$ is given by the block matrix
\[
\begin{bmatrix}
0 & -\psi^* \\
\psi & 0
\end{bmatrix}
\]
(N.B. The result means that $\varphi$ can be thought of as multiplication by $i$ on a complex vector space whose real and imaginary parts are $W$ and $W^\perp$.)

9. True or false: the sum of two normal operators is normal. Justify.

10. Show that the space of positive (semi)definite real symmetric matrices is convex. Is the same true with “complex Hermitian” in place of “real symmetric”?

11. Orthogonally diagonalize the matrix $A = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$. Find all square roots of $A$; note that they are all self-adjoint.

12. Find a singular decomposition of the matrix $A = \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}$. Use it to find $\max_{\|x\| \leq 1} \|Ax\|$ and $\min_{\|x\|=1} \|Ax\|$, as well as the vectors where this maximum and minimum are attained. Describe geometrically the image under $A$ of the closed unit disk in $\mathbb{R}^2$.

13. Prove that the operator norm of a matrix $A$ coincides with the Frobenius norm of $A$ if and only if $A$ has rank at most 1.