Math 403 Homework #1, Spring 2018
Instructor: Ezra Miller
Solutions by: ...your name...
Collaborators: ...list those with whom you worked on this assignment...
Due: 11:59pm Wednesday 31 January 2018

Reading assignments

- for Fri. 12 January
  - [Hefferon, p.144–145] field axioms
  - [Cornell, “Fields” (about 1/3 of the way through 4330-week1.pdf)]: more on fields
  - [Climenhaga, §7.2, §9.1] isomorphism, rank-nullity

- for Fri. 19 January
  - [Lax, Chapter 1] quotients
  - [Climenhaga, §5.1] quotients
  - [Cornell, “Quotient Spaces” (4330-week5.pdf)] universal properties
  - [Cornell, “Exact Sequences” (about halfway through 4330-week5.pdf)]

- for Wed. 24 January
  - [Lax, Chapter 2] duality
  - [Climenhaga, §5.1, §8.2] duality, transpose
  - [Treil, §5.1–5.5] Hermitian inner products, aijoints; §5.2–5.4 should be mostly review

- for Fri. 26 January
  - [Lax, Appendix 15] Jordan canonical form (very short proof)
  - [Hefferon, §5.IV.1] characteristic polynomial and minimal polynomial
  - [Treil, §9.3 and §9.5] generalized eigenspaces and Jordan canonical form

- for Wed. 31 January
  - [Wikipedia, Grassmannian] as much as you’re willing; thru §3 (“as a set”) at least
  - [Wikipedia, Group action] get a feel for the concept; no need to overdo it

- for Fri. 2 February
  - [multivariable calculus, any source]: open & closed sets; differentiability; chain rule
    \[ D(f \circ g) = Df \circ Dg \]; inverse function theorem: \( Df \) invertible \( \Rightarrow f \) locally invertible
  - [Wikipedia, Topological space] get a feel; through §5.1 (Metric spaces), say
  - [Wikipedia, Differentiable manifold] read up through §2.1 (Atlases)

- for Wed. 7 February
  - [Lax, 7.Isometry, p.87–89] isometry, orthogonal group
Exercises

1. (Freshman’s Dream): The characteristic of a field $F$ is the smallest positive integer $p$ such that the sum $1 + \cdots + 1$ of $p$ multiplicative identities is 0 in $F$. Prove that $p$ is prime if it is finite. If $F$ has characteristic $p$, show that $(a + b)^p = a^p + b^p$ for $a, b \in F$.

2. Fix a vector space $V$ and a subspace $W \subseteq V$ over a field $F$. Let $\pi : V \to V/W$ be the projection homomorphism given by $\pi(v) = v + W$. Write $X$ for the set of all subspaces of $V$ that contain $W$, and write $Y$ for the set of all subspaces of $V/W$. Prove that $\pi$ induces a bijection between these two sets, with

$$X \to Y$$

$$L \mapsto \pi(L) = \{ \pi(v) \mid v \in L \}$$

and

$$Y \to X$$

$$M \mapsto \pi^{-1}(M) = \{ v \in V \mid \pi(v) \in M \}.$$ 

3. Show that giving an exact sequence $\cdots \to V_{i-1} \to V_i \to V_{i+1} \to \cdots$ is the same as giving a collection of short exact sequences $0 \to K_i \to V_i \to K_{i+1} \to 0$, one for each $i$. (The long exact sequence is said to be constructed by splicing the short exact sequences together.)

4. Rank-nullity theorem for exact sequences: Given an exact sequence

$$0 \to V_0 \to V_1 \to \cdots \to V_r \to 0,$$

prove that $\sum_{i=0}^r (-1)^i \dim V_i = 0$.

5. Rank-nullity for arbitrary complexes: Given a complex $0 \to V_0 \to V_1 \to \cdots \to V_r \to 0$, write $B_i = \text{im}(V_{i-1} \to V_i)$ and $Z_i = \ker(V_i \to V_{i+1})$. Prove that

$$\sum_{i=0}^r (-1)^i \dim V_i = \sum_{i} (-1)^i \dim H_i,$$

where $H_i = Z_i/B_i$ is the $i$th homology of the complex. [In Exercise 4, $B_i = Z_i = K_i$, so $H_i = Z_i/B_i = 0$ for all $i$.]

6. Two elements $u$ and $v$ in a vector space $V$ are congruent modulo a subspace $W \subseteq V$, written $u \equiv v \pmod W$, if $u + W = v + W$. Show that congruence modulo $W$ is an equivalence relation on $V$, meaning that it is

- reflexive: $v \equiv v \pmod W$ for all $v \in V$;
- symmetric: if $u \equiv v \pmod W$, then $v \equiv u \pmod W$; and
- transitive: if $u \equiv v \pmod W$ and $v \equiv x \pmod W$ then $u \equiv x \pmod W$. 

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7. Give an example of three subspaces \( Y_1, Y_2, \) and \( Y_3 \) in \( \mathbb{R}^2 \) such that \( Y_1 + Y_2 + Y_3 = \mathbb{R}^2 \) and \( Y_i \cap Y_j = \{0\} \) for all \( i \neq j \), but \( \mathbb{R}^2 \) is not the direct sum of \( Y_1, Y_2, \) and \( Y_3 \).

8. Prove that if \( V \) is a vector space and \( W \subseteq V \) is a subspace, then \( W \) has a complement: a subspace \( U \subseteq V \) such that \( V = W \oplus U \). Hint: \( V/W \) has a basis; lift it back to \( V \).

9. For vectors \( x = (1, 2i, 1 + i) \) and \( y = (i, 2 - i, 3) \), compute
   
   (a) \( \langle x, y \rangle, ||x||^2, ||y||^2, \) and \( ||y|| \);
   
   (b) \( \langle 3x, 2iy \rangle \) and \( \langle 2x, ix + 2y \rangle \);
   
   (c) \( ||x + 2y|| \). [Use parts (a) and (b) for this.]

10. Prove that for vectors in an inner product space, \( ||x + y||^2 = ||x||^2 + ||y||^2 + 2Re\langle x, y \rangle \).

11. For any \( m \times n \) complex matrix \( A \), prove that \( \ker(A^*A) = \ker(A) \).

12. Prove that if \( P \) is self-ajoint (that is, \( P^* = P \)) and idempotent (that is, \( P^2 = P \)) then \( P \) is the matrix for an orthogonal projection.

13. If \( V \) is a vector space over \( \mathbb{C} \) of dimension \( n \), then it is also a vector space over \( \mathbb{R} \) of dimension \( 2n \). (If this isn’t clear to you, then write down a proof.) Given a Hermitian inner product \( \langle x, y \rangle \) on \( V \) as a complex vector space, show that the real part \( \langle x, y \rangle_{\mathbb{R}} = \text{Re}(\langle x, y \rangle) \) is an inner product on \( V \) as a real vector space.

14. Find all possible Jordan forms of linear transformations with characteristic polynomial \( (t - 1)^2(t + 2)^2 \).

15. Find all possible Jordan forms of linear transformations with characteristic polynomial \( (t - 2)^3(t + 1) \) and minimal polynomial \( (t - 2)^2(t + 1) \).

16. How many similarity classes are there for \( 3 \times 3 \) matrices whose only eigenvalues are \(-3 \) and \( 4 \)?

17. Prove or disprove: two \( n \times n \) matrices are similar if and only if they have the same characteristic polynomial and minimal polynomial.