Review def of manifold; e.g. \( X = \text{surface of Earth}, \) atlas = actual "rectangular" maps

\[ X = S^1 \]

Thm: \( G_k(F^n) \) is a rational algebraic variety of \( \dim k(n-k) \) with atlas
\[ \{ \pi_\sigma : F^{k \times n}_\sigma \rightarrow G_k(F^n) \mid \sigma \in \binom{[n]}{k} \}. \]

**Proof**:

1. \( \text{Prop } \Rightarrow X = \bigcup_{\alpha} X_\alpha : G_k(F^n) = \bigcup_{\sigma} G^\sigma_k \)
   - and \( G^\sigma_k \cong F^{k \times n}_\sigma \cong F^d \) for \( d = k(n-k) \)
2. Declare \( U \subset G_k(F^n) \) to be open \( \iff U \cap G^\sigma_k \) is open \( \forall \sigma \in \binom{[n]}{k} \). HW 2.16: well defined
3. Set \( F^{k \times n}_{\sigma, \tau} = \{ A \in F^{k \times n}_\sigma \mid A_\tau = [\text{cols of } A \text{ indexed by } \tau] \text{ is invertible} \} \)
   \[ = \pi^{-1}_\sigma (G^\sigma_k \cap G^\tau_k). \]
   *recall: [cols indexed by \( \sigma \)] is \( I_k \) for \( A \in F^{k \times n}_\sigma \)
   
   Then \( \pi^{-1}_\tau \circ \pi^{-1}_\sigma : F^{k \times n}_{\sigma, \tau} \rightarrow F^{k \times n}_{\tau, \sigma} \)
   
   \( A \mapsto \ ? \)

   Find matrix \( A' \) with \( [A'] = [A] \)

   easy: \( A' = A^{-1}_\tau A \)

   entries are rational functions of entries of \( A \).

So that's how grassmannians work. Let's all do an exercise together to see these methods in action.

**Def**: A (complete) flag in \( V \) is a chain
\( O = V_0 \subset V_1 \subset \cdots \subset V_{n-1} \subset V_n = V \)

of subspaces of \( V \) with \( \dim V_i = i \ \forall i \). Set
\( \mathcal{F}_n(F) = \{ \text{complete flags in } F^n \} \).

**Ex.** Express \( \mathcal{F}_n \) as a quotient.

- of what? How do you write down a flag? Do it, then quotient modulo choices.
  This a very general principle in math.

\[ V_1 = \langle v_1 \rangle \]
\[ V_2 = \langle v_1, v_2 \rangle \]
\[ \vdots \]
\[ V_i = \langle v_1, \ldots, v_i \rangle \]

\[ A = \begin{bmatrix} | & | \\ v_1 & \cdots & v_n \end{bmatrix} \in \text{GL}_n(F) \]

Why?

(Yes we're using columns now.)
by what?

\[ \langle v_1 \rangle = \langle \alpha v_1 \rangle \text{ for any } \alpha \in F^* \]

\[ \langle v_1, v_2 \rangle = \alpha \langle v_1 \rangle + \beta_1 \beta_2 \langle v_2 \rangle \text{ for any } \alpha \in F^*, \beta_1, \beta_2 \in F, \text{ and } F^* \]

\[ \langle v_1, v_2, v_3 \rangle = \alpha \langle v_1 \rangle + \beta_1 \beta_2 \langle v_2 \rangle + (\text{some replacement for } v_3) \in \text{span}(v_1, v_2, v_3) \setminus \text{span}(v_1, v_2), \]

so \( \alpha v_1 + \beta_1 v_2 + \beta_2 v_3 \) with \( \beta_2 \neq 0 \)

\[ A = AB \quad \text{for } B = \begin{bmatrix} F^* & F & F & \cdots & F \\ 0 & F^* & F & \cdots & F \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & F^* \\ 0 & 0 \cdots & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} F^* & * & \cdots & * \\ 0 & F^* & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 \cdots & 0 & F^* \\ 0 & 0 \cdots & 0 & 0 \end{bmatrix} \leq GL_n(F) \]

Def: \( B_n^+ = \{ \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix} \} \leq GL_n \) is the Borel subgroup.

Prop: \( J/F_n = GL_n/B_n^+ \).

Is it a manifold? A variety? Find

- "big" subset that is an open subset of a vector space
- enough copies to cover.

Assume A "generic". What does that mean? Don't know yet. Try; see what's needed.

Use \( b_{11} \) to make \( a_{11} = 1 \) needs "generic"

then \( b_{12}, \ldots, b_{1n} \) to cancel \( a_{12}, \ldots, a_{1n} \)

then \( b_{22} \) to make new \( a_{22} = 1 \)

then \( b_{23}, \ldots, b_{an} \) to cancel \( a_{23}, \ldots, a_{an} \)

Prop: \( U_n^- \hookrightarrow J/F_n = GL_n/B_n^+ \).

Pf: HW.

Thm: \( J/F_n \) is a rational algebraic variety with atlas

\( \{ wU_n^- \to GL_n/B_n^+ \mid w \text{ is a permutation matrix} \} \).

\[ AB_1 = \begin{bmatrix} a_{11} & -a_{12} & \cdots & -a_{1n} \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 \cdots & 0 & 1 \end{bmatrix} \]

\[ AB_1 = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ * & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ * & \cdots & * & 1 \end{bmatrix} \]

\[ AB_1B_2 = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ * & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ * & \cdots & * & 1 \end{bmatrix} \]

\[ \text{e } U_n^- \text{ unipotent subgroup} \]