Banach spaces: complete normed vector spaces over \( F = \mathbb{R} \) or \( \mathbb{C} \)

Cauchy sequences converge

**Def:** A norm on \( V / F \) is \( \nu : V \rightarrow \mathbb{R}_+ \) with:
- positive-definite: \( \nu(x) = 0 \iff x = 0 \)
- homogeneous: \( \nu(\lambda x) = |\lambda| \nu(x) \)
- subadditive: \( \nu(x+y) \leq \nu(x) + \nu(y) \)
- triangle inequality

**E.g.:**
- Manhattan: \( p=1 \cdot \|x\|_1 = |x_1| + \cdots + |x_n| \)
- Euclidean: \( p=2 \cdot \|x\|_2 = (|x_1|^2 + \cdots + |x_n|^2)^{1/2} \)

\[ \lim_{p \to \infty} \norm{x}_p = \max_{i=1}^n |x_i| \]
\[ \norm{x}_p = (|x_1|^p + \cdots + |x_n|^p)^{1/p} \]

**Def:** A metric space \( X \) with a distance \( d : X \times X \rightarrow \mathbb{R}_+ \) such that \( \forall x, y \in X \)
- \( d(x, y) = 0 \iff x = y \) separates
- \( d(x, y) = d(y, x) \) symmetric
- \( d(x, y) \leq d(x, z) + d(z, y) \) \( \forall x \in X \)

**E.g.:** norm \( \nu \) induces distance \( d_\nu(x, y) = \nu(x-y) \), such as

**Euclidean metric** \( \|x-y\|_2 \) on \( \mathbb{F}^n \)

**Manhattan metric** \( \|x-y\|_1 \) on \( \mathbb{F}^n \)

All norms “pretty much feel the same”. In what sense?

**Def:** A topology on a set \( S \) is a collection \( U \) of subsets called open sets such that
- any union of open sets is open
- any finite intersection of open sets is open
- \( S \) and \( \emptyset \) are open

**E.g.:** usual topology on \( \mathbb{F}^n \): \( U \in U \iff B_\varepsilon(x) \subseteq U \forall x \in U \) and \( \varepsilon = \varepsilon_x \ll 1 \)

More generally: metric \( d \) on \( X \) \( \longrightarrow \) topology on \( X \) with \( U \) open \( \iff B_\varepsilon^d(x) \subseteq U \forall x \in U \) and \( \varepsilon = \varepsilon_x \ll 1 \)

**Def:** \( B \in U \) is a base for the topology if \( U \in U \Rightarrow U = \bigcup_{B \in B} B \) for some \( B' \subseteq B \)

**E.g.:** \( \{ B_\varepsilon^\nu(x) \mid x \in V \text{ and } \varepsilon \in \mathbb{R}_+ \} \)

**Prop:** \( X \subseteq S \) closed \( \iff S \setminus X \) open.

Prop: \( S \setminus X \) open and \( \{x_k\} \subseteq X \) if \( \{x_k\} \) is eventually in \( U \) open \( \forall x \), meaning \( \exists N_k \in \mathbb{N} \) with \( x_k \in U \forall k \geq N_k \).

\( X \subseteq S \) is closed if \( x \in X \) whenever \( \{x_k\} \rightarrow x \) in \( S \) with \( \{x_k\} \subseteq X \).

S\setminus X \text{ not open } \Rightarrow \exists y \in S \setminus X \text{ such that } V \text{ open } \forall y \in V \setminus \{x\}. \text{ Then } \{x\}_{y \in \mathcal{U}} \rightarrow y \notin X. \)
Prop: Any norm \( \nu \) on \( \mathbb{F}^n \) is continuous in the Euclidean metric.

\[ \text{Pf: } \text{Given } \varepsilon > 0, \text{ need } \delta \text{ so that } |\nu(x) - \nu(y)| < \varepsilon \text{ whenever } \|x-y\| < \delta. \]

Subadditivity \( \Rightarrow \nu(x) \leq \nu(x-y) + \nu(y) \) and \( \nu(y) \leq \nu(y-x) + \nu(x) \)

\[ \Rightarrow \nu(x) - \nu(y) \leq \nu(x-y), \quad \nu(y) - \nu(x) \leq \nu(y-x), \text{ so} \]

\[ |\nu(x) - \nu(y)| \leq \nu(x-y) = \nu(\sum_{i=1}^{n} (x_i - y_i) e_i) \leq \sum_{i=1}^{n} |x_i - y_i| \nu(e_i) \]

\[ \text{Pick } \delta = \frac{\varepsilon}{\|\nu\|_2} \]  

Q. why? A. Cauchy-Schwarz!

Def: Norms \( \nu \) and \( \mu \) on \( V = \mathbb{F}^n \) are (topologically) equivalent, written \( \nu \sim \mu \), if

\[ \exists \alpha, \beta \in \mathbb{R}_{>0} \text{ with } \alpha \nu(x) \leq \mu(x) \leq \beta \nu(x) \quad \forall x \in V. \]

\[ \text{Interpretation: } \nu \sim \mu \iff B^\nu_{\beta \nu}(x) \subseteq B^\mu_\alpha(x) \subseteq B^\nu_{\alpha \nu}(x) \quad \forall x \in V \]

\[ \iff \text{every } \varepsilon \text{-ball base for the } \mu \text{-topology is a base for the } \nu \text{-topology} \]

E.g. \[ \|x\|_2 \leq \|x\|_1 \leq \sqrt{n} \|x\|_2 \]

\[ \frac{1}{\sqrt{n}} \|x\|_2 \leq \|x\|_{\infty} \leq \|x\|_2 \]

\[ \text{Pf: exercise (not assigned)} \]

\[ \|x\|_{\infty} \leq \|x\|_1 \leq n \|x\|_\infty \]

Lemma: \( \sim \) is an equivalence relation.

\[ \text{Pf: symmetric: } \frac{1}{\beta} \mu(x) \leq \nu(x) \leq \frac{1}{\alpha} \mu(x), \]

transitive: exercise.

Reflexive: \( \alpha = \beta = 1 \).

Thm: \( \mu, \nu \) norms on \( V = \mathbb{F}^n \Rightarrow \nu \sim \mu \).

\[ \text{Pf: By Lemma, need only check } \nu = \|\cdot\|_2. \text{ Can assume } x \neq 0. \]

\[ \text{(x) with } y = 0 \text{ and } \mu \text{ instead of } \nu \Rightarrow \mu(x) \leq \|x\|_2 \|\nu\|_2 \quad \text{for } v = (\mu(e_1), \ldots, \mu(e_n)) \]

\[ \Rightarrow \text{ take } \beta = \|\nu\|_2. \]

Set \( \alpha = \min \{ \mu(x) \mid \|x\|_2 = 1 \} \), which exists by Prop because sphere \( S^{n-1} \) is closed and bounded.

Then \( \mu(x) = \mu(\|x\|_2, \|x\|_2) = \|x\|_2 \mu(\|x\|_2) \)

\[ \geq \|x\|_2 \alpha. \]

Def: norm on \( V \) dual to \( \nu \) on \( V \) is \( \nu^*(\Psi) = \max_{\nu(x) = 1} |\Psi(x)|. \)

well defined since \( S_{\nu} = \{ x \in V \mid \nu(x) = 1 \} \) is closed and bounded by Thm.