Def: Fix subspace \( W \subseteq V \).

1. A coset of \( W \) is an affine subspace \([v] = v + W\) for some \( v \in V \).

2. The quotient \( V/W \) is the set of cosets of \( W \).

\[ "V \text{ mod } W" \]

Prop: \( V/W \) is a vector space with

\[ [u] + [v] = [u + v] \quad \text{and} \quad \lambda [v] = [\lambda v]. \]

Pf: \((u + W) + (v + W) = (u + v) + W.

"add lines" since \( W + W = W \)

affine subspaces

\[ \lambda (v + W) = \lambda v + \lambda W = \lambda v + W \quad \text{if} \quad \lambda \neq 0, \quad \text{and} \]

\[ 0[v] = 0(v + W) = \{0\} \subseteq W = [0] = [0v]. \]

Cor: \( \dim V = \dim W + \dim V/W. \)

Pf: \( V \rightarrow V/W \) is a homomorphism (by Prop) with \( \ker = W \) and \( \text{im} = V/W. \)

Universal property of quotients

A homomorphism \( \overrightarrow{\varphi} \) on \( W \) is 0 on \( W \) \( \iff \) \( \varphi \) factors through \( V/W : V \rightarrow V/W \rightarrow U. \)

Pf: \exists \text{ well defined function } \rightarrow \text{ forget algebraic properties, like "homomorphism"}

\( V/W \rightarrow U \iff \text{each coset of } W \rightarrow \text{single point in } U \)

\[ \iff W \rightarrow \text{single point in } U. \]

\[ \varphi \text{ linear} \]

\[ \iff W \in \ker \varphi. \]

Def: Arbitrary homomorphism \( V \rightarrow W \) has

- \( \ker \subseteq V \)
- \( \text{im} \subseteq W \)
- \( \text{coker} \subseteq W/\text{im} \)
- \( \text{coimage} \subseteq V/\ker \)
- \( \forall \varphi \in \text{Hom}(V, W) \rightarrow \text{coker} \varphi = V/\ker \varphi \)

First Isomorphism Thm (requires proof!)

Pf: \( V \rightarrow \text{im} \) by def of im, so \( V \rightarrow V/K \rightarrow \text{im} \) by universal property of coker.

\[ \ker(V/K \rightarrow \text{im}) = \{ [v] \in V/K \mid v \mapsto 0 \} = [K] \text{ by def of ker,} \]

\[ = 0 \in V/K \Rightarrow V/K \rightarrow \text{im}. \]
Def: A sequence \( V_0 \xrightarrow{\varphi_1} V_1 \xrightarrow{\varphi_2} \cdots \xrightarrow{\varphi_r} V_r \) is exact if \( \ker \varphi_{i+1} = \im \varphi_i \) \( \forall i \).

E.g.

- \( 0 \to V \to V' \to 0 \) exact \( \iff V \cong V' \)
- \( 0 \to V \to V' \) exact \( \iff V \cong V' \)
- \( V \to V' \to 0 \) exact \( \iff V \cong V' \)
- \( 0 \to K \to V \to W \to W/I \to 0 \) is exact

- \( \text{exact here by FIT} \)
- \( O \to A \to B \to C \to 0 \) \( \iff \) \( A \cong B \) and \( C \cong B/A \)
- \( O \to A \to B \to B/A \to 0 \)

Def: \( V_* : \cdots \to V_{i-1} \to V_i \to V_{i+1} \to \cdots \) is a complex if \( V_{i-1} \to V_i \to V_{i+1} \) is exact \( \forall i \).

\( V_* \) has homology \( H_i V_* = \ker \varphi_{i+1} / \im \varphi_i \).

Lemma: \( \iff \im (V_{i-1} \xrightarrow{\varphi_{i-1}} V_i) \subset \ker (V_i \xrightarrow{\varphi_i} V_{i+1}) \)

measures how far a complex is from being exact. Problem: you don’t know many complexes yet. \( \text{(Do you?)} \)

E.g. algebraic topology simplices \( \cdot, \triangle, \bigtriangleup, \ldots \)

- of dim 0, 1, 2, 3, ...
- e.g. octahedron \( \bigtriangleup \rightarrow \) vector spaces \( \mathbb{F}_2 \)

\( V: \) basis = vertices \( E: \) basis = edges \( F: \) basis = faces

\( \partial_E (\bigtriangleup) = \) \( v \Rightarrow w \)
\( \partial_F (\bigtriangleup) = \) \( e_1 + e_2 + e_3 \)

Prop: \( C_* \) is a complex!

Pf: \( 2 \) (\( = 0 \)) ways to get from a simplex of dim \( i \) to a simplex of dim \( i-2 \). \( \square \)

Compute:

- \( H_0 C_* = \ker \partial_v / \im \partial_E = V/B \) \( B = \text{span} (v+w | \text{vertices } v, w) \) \( \dim H_0 = \geq 1 \)
- \( H_1 C_* = \ker \partial_E / \im \partial_F = \geq 0 \) exercise \( \text{(not assigned)} \)
- \( H_2 C_* = \ker \partial_F / \im \partial_E = \geq 0 \) \( \text{span} (f_1, \ldots, f_g) \) \( \dim H_2 = \geq 1 \)

Thm (rank-nullity): \( \sum_i (-1)^i \dim H_i = \sum_i (-1)^i \dim V_i \).

- \( \Rightarrow \dim K = \dim V + \dim I = 0 \) exact \( \Rightarrow \dim K = \dim V + \dim I = 0 \)

\( \text{Euler characteristic of } V \).

Cor: exercise! \( 6 - 12 + 8 = 1 - ? + 1 \) \( \Rightarrow ? = 0 \)

Pf of Thm: \( HW \)