Math 403: Advanced Linear Algebra
Spring 2021

Tue/Thu 12:00 - 13:15
403-01 Physics 154
703-01
403-02 online
703-02

Office hours: Mon 16:15-17:15 Zoom
Thu 13:15-14:30 Zoom from Physics 209 or outside

Safety
- 6 feet, including through doors
- masks, distance, wash hands on the way in
- politely point out noncompliance — including me!
- know where the exits are from the room and the building

Policies
- covered on Tue ⇒ fair game for HW or exam Tue
- collaboration/academic honesty: I have brought many cases to the Office of Student Conduct
- yes on HW, but...
- no on exams

Graded assignments
- HW — collaborate!
- Midterms — no collaboration or outside sources

"Index cards"
1. Name: Ezra Miller
2. Pronouns: he/him
3. Year: 41st grade
4. Major (or potential major): Math, Music
5. What you hope to get out of this course:
   students who know linear algebra, including the right way to think
6. Other Math courses you’ve taken
7. Hobbies: frisbee, gardening, beer
8. Something unique about yourself:
   X "I’m from MA"
   ✓ "I’m from HI — but I’m allergic to pineapple."

Now: place in Sakai Drop Box — this is a dry run for HW!

Fields

**Def**: A group is a set \( G \) with an associative binary operation
\[
\times: G \times G \rightarrow G \quad \text{(} g \times g' \text{)} \times g'' = g \times (g' \times g'')\]
\[(g, g') \mapsto g \times g'\]
that has an identity $e$ with $e * g = g = g * e \ \forall \ g \in G$

- an inverse $g^{-1}$ for each $g \in G$, so $g * g^{-1} = e$.

$G$ is abelian if $*$ is commutative: $g * h = h * g \ \forall \ g, h \in G$.

Example:

- $(\mathbb{R}, +)$
- $(\mathbb{C}, +)$
- $(\mathbb{Q}, +)$
- $(\mathbb{Z}, +)$

- $\mathbb{C}^\circ (\mathbb{R}^n \to \mathbb{R}, +)$
- $\text{Fun} (S \to \mathbb{A}, +)$

$A$ = any abelian group!

- $\left( \begin{smallmatrix} m \\ \times \ n \end{smallmatrix} \right)$ matrices with any of these, $+$
- $\left( \begin{smallmatrix} m \\ \times \ n \end{smallmatrix} \right)$ matrices with any of these, $\cdot$

non-abelian: $\{A \in \mathbb{R}^{2 \times 2} \mid \det A = 1\}$

Definition: A **field** is an abelian group $(F, +)$ with

- additive identity $0 \in F$ such that
  - $F^\ast = F \setminus \{0\}$ is an abelian group $(F^\ast, \cdot)$ and
  - multiplication distributes over addition $+: a \cdot (b + c) = a \cdot b + a \cdot c$.

Example:

- $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Q}$, $\mathbb{F}_2 = \{0, 1\}$, $\mathbb{F}_3 = \{0, 1, 2\}$, $\mathbb{F}_p = \{0, 1, \ldots, p-1\}$ for $p \in \mathbb{Z}$ prime
- $\mathbb{R}(i)$, $\mathbb{Q}(i)$

Math 221 works verbatim with any $F$ in place of $\mathbb{R}$, except for notions of length, angle, order ($a < b$), closeness (topology).

Definition: A **vector space** over $F$ is ... review from 221.

$(V, +)$ abelian group with an action of $F$:

- $F \times V \to V$
- $(\alpha, v) \mapsto \alpha v$

- distributes over $+$ on both sides
- associative: $\alpha (\beta v) = (\alpha \beta) v$
- $1 v = v \ \forall \ v \in V$.

Definition: A **homomorphism** of vector spaces is a linear map.

Example: $B$ is a basis for $V$

$\iff \{\text{functions } B \overset f \to W \iff \exists \exists \text{! homomorphisms } \varphi: V \to W \text{ with } \varphi|_B = f\}$

Definition: A **homomorphism** $\varphi: V \to W$ has

- kernel $\ker \varphi = \{v \in V \mid \varphi(v) = 0\} \subseteq V$
- image $\text{im } \varphi = \{\varphi(v) \mid v \in V\} \subseteq W$. 

subspaces
Thm (rank-nullity): \( \dim(\ker \Psi) + \dim(\im \Psi) = \dim V \). \( \text{rank-nullity} \)

Pf: Pick \( B' \) basis of \( \ker \Psi \).

\[ B'' \subseteq V \text{ with } B'' \leftrightarrow \Psi(B'') \text{ basis of } \im \Psi. \]

Then \( B = B' \cup B'' \) is a basis of \( V \).

\[ \# B' + \# B'' \]

requires proof, but it's a straightforward exercise.

Def: A homomorphism \( \Psi: V \rightarrow W \) is an \textit{isomorphism} if \( \Psi \) is \textit{injective and surjective} = \textit{bijective}.

\[ \Leftrightarrow \ker \Psi = 0 \quad \text{im } \Psi = W \]

E.g. \( \dim V = n \Rightarrow V \cong \mathbb{F}^n \).

Pf: Basis \( v_1, v_2, \ldots, v_n \) of \( V \Rightarrow v_i \mapsto e_i \) induces \( \cong: \ker \Psi = 0 \) because \( e_1, \ldots, e_n \) independent, and \( \text{im } \Psi = \mathbb{F}^n \) because \( e_1, \ldots, e_n \) span \( \mathbb{F}^n \). \( \square \)

E.g. \( \text{sols } \mathbb{R}^2(f'' + f = 0) \cong \mathbb{R}^2 = \text{span } \{ \sin, \cos \} \)?

\[ \sin x = \frac{e^{ix} - e^{-ix}}{2i}, \quad \cos x = \frac{e^{ix} + e^{-ix}}{2} \]

Def: An \textit{affine subspace} is a translate of a subspace = \textit{sols} (inhomogeneous linear system).

E.g. \( \mathbb{F}_3^4 \) has one rule: find an affine line in \( \mathbb{F}^4_3 \).

\[ \text{card } u \in \mathbb{F}_3^4 \]

\[ = (a, b, c, d). \quad v = (a', b', c', d') \Rightarrow \text{affine line } \overrightarrow{uv} \text{ is } \]

\[ L = v + \{ \lambda(u-v) \mid \lambda \in \mathbb{F}_3 \} = \{ v, u, -u-v \}. \]

Two possibilities:

(i) \( L \subseteq x_1 = a \) (if \( a' = a \));

(ii) \( x_1 \)-coordinates of all points in \( L \) are distinct (if \( a' \neq a \))

Hence the rule: "If two are and one isn't then it's not a set."

Q. What makes \( \mathbb{F}_3 \) so special?

A. Each pair of points (cards) yields a unique third in \( L \).