Combinatorial Commutative Algebra and D-Branes

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Abstract

This is a survey paper written for Professor Ezra Miller’s Combinatorial Commutative Algebra course in the Spring of 2009 at Duke University. This survey is mostly based on the approach in the paper of Aspinwall [2]. The purpose of this survey is to explain how combinatorial commutative algebra techniques can be used to study D-branes on toric Calabi-Yau Varieties.

1 Introduction

The discovery of D-branes in string theory and its subsequent applications to mirror symmetry has led to some very interesting mathematics in the past decade. One of the earliest proposals for the use D-branes in mirror symmetry came from Kontsevich in his ICM lecture[10], where he proposed a relationship between the Fukaya Category (A-branes) and the derived category of coherent sheaves (B-branes). Since Kontsevich’s talk much work has been done on understanding these two categories as well as the mirror map between them. On the B-brane side, starting with papers by Douglas [12] and subsequent papers by Lawrence and Aspinwall [13], it has become increasingly clear that B-branes on a Calabi-Yau Variety $X$ should be thought of as $D^b(X)$, the bounded derived category of coherent sheaves on $X$. The purpose of this paper is to explain some connections between B-branes for a toric Calabi-Yau variety and combinatorial commutative algebra.

2 Toric Geometry

In this section we will review the construction of toric varieties starting from fans and then proceeding to Cox’s construction of the homogeneous coordinate ring associated to a toric variety [9]. The Cox construction is important from the combinatorial commutative algebra point of view because this construction gives a grading to the homogeneous coordinate ring associated to the toric variety. This general machinery will be applied specifically to toric Calabi-Yau varieties.
2.1 Fans

Toric varieties and their associated rings can be defined combinatorially using lattices and fans inside them. Start with a lattice $N$ of rank $d$ and its dual lattice $M = \text{Hom}(N, \mathbb{C})$, which is also of rank $d$, There are associated vector spaces $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ and $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$.

**Definition 2.1.** A strongly convex rational polyhedral cone $\sigma \subseteq N_{\mathbb{R}}$ is the set

$$\sigma = \{ \sum_{i} a_i v_i \mid a_i \geq 0 \}$$

where $\sigma$ is pointed and each $v_i \in N$. These will be referred to as cones in the future unless otherwise specified.

The combinatorial description of a toric variety uses these cones, glued together along faces, to form an object called a fan. All of the information regarding a toric variety can be read off of the fan. Before defining a fan precisely it is neccessary to define the faces of a cone, which will be the gluing data for affine covers of the toric variety. In order to define a face we will first define a dual cone.

**Definition 2.2.** Given a cone $\sigma$ there is a dual cone $\tilde{\sigma} \subseteq M_{\mathbb{R}}$ satisfying the following property:

$$\tilde{\sigma} = \{ v \in M_{\mathbb{R}} \mid \langle u, v \rangle \geq 0, \forall u \in \sigma \}$$

**Definition 2.3.** A face of a cone, $\tau \subseteq \sigma$ is the set of elements of $\sigma$ which vanish at an element $v$ of $\tilde{\sigma}$.

$$\tau = \{ u \in \sigma \mid \langle u, v \rangle = 0 \text{ for } v \in \tilde{\sigma} \}.$$ 

This definition for a face still satisfies the conditions for a cone. So a face is also a cone in $N_{\mathbb{R}}$.

**Definition 2.4.** A fan $\Sigma$ is a collection of cones in $N_{\mathbb{R}}$ which has the following properties.

1. For every cone in $\Sigma$ the faces of the cone are also in $\Sigma$.

2. Given two cones $\sigma_1$ and $\sigma_2$ the intersection $\sigma_1 \cap \sigma_2$ is a face of each of these cones.

From the lattice construction one can associate to each cone $\sigma$ a ring denoted $k[\tilde{\sigma} \cap M]$ which is the semigroup ring of the semigroup $\tilde{\sigma} \cap M$. Dualization of a cone is an inclusion-reversing operation, i.e., $\tau \subseteq \sigma \Rightarrow \tilde{\sigma} \subseteq \tilde{\tau}$. It can be checked that a face $\tau$ of the cone $\sigma$ corresponds to ring localization on the ring level: $k[\tilde{\tau} \cap M]$ is a localization of $k[\tilde{\sigma} \cap M]$. This fact gives a way of showing how the toric variety is constructed by gluing affine varieties together. The toric variety $X_\Sigma$ consists of affine varieties $\text{Spec}(k[\tilde{\sigma}_i \cap M])$ for each maximal dimension cone $\sigma_i$, glued together at faces of intersection. If $\tau_{ij} = \sigma_i \cap \sigma_j$ then as stated above, $\text{Spec}(k[\tilde{\tau}_{ij} \cap M])$ is the ring localization of both $\text{Spec}(k[\tilde{\sigma}_i \cap M])$ and $\text{Spec}(k[\tilde{\sigma}_j \cap M])$. Using standard algebraic
geometry constructions one can glue the affine varieties $\text{Spec}(k[\bar{\sigma}_i \cap M])$ along open subsets $\text{Spec}(k[\bar{\tau}_{ij} \cap M])$ to get an algebraic variety. A useful observation coming from the fan data is that every $r$ dimensional cone of the fan corresponds to a $(d-r)$-dimensional torus invariant subvariety of $X_{\Sigma}$. In particular the cone $\{0\}$, which sits inside every cone, corresponds to $T_N=\text{Spec}(k[M])$ which is the torus of this toric variety.

2.2 Homogeneous Coordinates

The fan construction gives a local construction of the toric variety. In order to give a global construction one can use the homogeneous coordinate ring and related GIT quotient given by Cox [9]. Starting with the fan $\Sigma$ associated to the toric variety $X_{\Sigma}$, look at the set, denoted $\Sigma(1)$, of one-dimensional cones of the fan. If $\rho_1, \rho_2, \ldots, \rho_n \in \Sigma(1)$ are the one-dimensional cones, which can also be thought of as torus invariant divisors, then a ring $S=k[x_1, x_2, \ldots, x_n]$ can be constructed where the coordinate $x_i$ corresponds to the one dimensional cone $\rho_i \in N$. Let $Z^{\Sigma(1)}$ denote the $n$-dimensional lattice where each $\rho_i$ corresponds to a fundamental lattice point which has a 1 in the $i$th spot and is zero everywhere else. There are two fundamental exact sequences which arise from $Z^{\Sigma(1)}$. The first exact sequence is

$$0 \rightarrow M \rightarrow Z^{\Sigma(1)} \xrightarrow{\Phi} A_{n-1}(X_{\Sigma}) \rightarrow 0,$$

where $A_{n-1}(X_{\Sigma})$ is the Chow group of $X_{\Sigma}$. The exact sequence works as follows: $m \in M$ maps to the $n$-tuple $(\langle m, \rho_1 \rangle, \ldots, \langle m, \rho_n \rangle)$ and some $n$-tuple $(a_1, a_2, \ldots, a_n)$ maps to $\sum_{i=1}^{n} a_i D_{\rho_i}$, where $D_{\rho_i}$ is the divisor associated to the cone $\rho_i$. The degree map for $m$ is $\deg(m) = \sum_{i=1}^{n} \rho_i a_i D_{\rho_i}$, which is why $m$ is trivial in $A_{n-1}(X_{\Sigma})$. The other exact sequence is the dual which we get by applying the functor $\text{Hom}(-, \mathbb{C}^*)$:

$$1 \rightarrow \text{Hom}(A_{n-1}(X_{\Sigma}), \mathbb{C}^*) \rightarrow \text{Hom}(Z^{\Sigma(1)}, \mathbb{C}^*) \rightarrow \text{Hom}(M, \mathbb{C}^*) \rightarrow 1,$$

which can be written as

$$1 \rightarrow G \rightarrow \mathbb{C}^{\Sigma(1)} \rightarrow T_N \rightarrow 1.$$

In the second exact sequence the group $G$ embeds in the torus $\mathbb{C}^{\Sigma(1)}$ so it acts on the space $\mathbb{C}^n$. If $r = n - d$ then $G \cong (\mathbb{C}^*)^r \times F$ where $F$ is a finite group. The action of this $(\mathbb{C}^*)^r$ on $(\mathbb{C})^n$ can be written as

$$(x_1, x_2, \ldots, x_n) \rightarrow (\lambda^{Q_1} x_1, \lambda^{Q_2} x_2, \ldots, \lambda^{Q_n} x_n)$$

Where each $Q_i$ is a $r$-dimensional vector. The fact that it is the kernel of the map $\mathbb{C}^{\Sigma(1)} \rightarrow T_N$ induces conditions for each $\rho_i$, if we multiply the components of $\rho_i$ with $Q_i$ the sum $\sum_{j=1}^{n} \rho_i^j Q_j = 0$. This is the $G$ action on $\mathbb{C}^n$ which will be used to construct the GIT quotient later.
One consequence of these short exact sequences is that \( S = k[x_1, x_2, \ldots, x_n] \) is graded by the Chow group \( A_{n-1}(X) \), and this is explicitly given by the map \( \phi \). Cox also proved that any toric variety given from the fan data can be realized as a GIT quotient. Before giving the theorem the important Cox ideal needs to be defined.

**Definition 2.5.** Given a fan \( \Sigma \) and the set of maximal dimension cones \( \sigma_1, \sigma_2, \ldots, \sigma_n \in \Sigma(d) \) the ideal \( B_\Sigma = \langle \prod_{i \in \sigma_1} x_i, \prod_{i \in \sigma_2} x_i, \ldots, \prod_{i \in \sigma_n} x_i \rangle \) is the **Cox ideal**.

Notice that the Cox ideal is a square-free monomial ideal by construction. The Alexander dual of \( B_\Sigma \) is the Stanley-Reisner ideal \( I_\Sigma \) whose associated simplicial complex is precisely \( \Sigma \) when it is a simplicial fan. The \( \mathbb{Z}^n \)-Betti numbers of \( I_\Sigma \) will be important later in the paper.

**Theorem 2.1** ([9]). Fix a fan \( \Sigma \), a Cox ideal \( B_\Sigma \), and \( G = \text{Hom}(A_{n-1}(X_\Sigma), \mathbb{C}^*) \). Then

\[
X_\Sigma = \frac{\mathbb{C}^n - V(B_\Sigma)}{G}
\]

is the GIT quotient of \( \mathbb{C}^n - V(B_\Sigma) \) by the group \( G \).

The Cox construction allows us to take a toric variety \( X \) and associate an \( A_{n-1}(X) \)-graded ring \( S \) as well as a Cox ideal \( B_\Sigma \). Cox makes another observation in his paper which will be the starting point of being able to use combinatorial commutative algebra techniques: in future discussions the grading group will be \( A_{n-1}(X) \) for the toric variety \( X \).

**Theorem 2.2** ([9]). Let \( X \) be a toric variety with homogeneous coordinate ring \( S \).

1. If \( F \) is a finitely generated graded \( S \)-module, then the sheaf \( \tilde{F} \) is a coherent sheaf on \( X \).
2. If \( X \) is simplicial then every coherent sheaf on \( X \) is of the form \( \tilde{F} \) for some finitely generated graded \( S \)-module \( F \).
3. If \( X \) is simplicial and \( B_\Sigma \) is the Cox ideal then \( \tilde{F} = 0 \) if and only if there is some \( k \) such that \( B_\Sigma^k F = 0 \).

This theorem is the motivation for Aspinwall’s proposition about the bounded derived category of coherent sheaves on a toric variety \( X_\Sigma \).

**Proposition 2.1** ([2]). Let \( X_\Sigma \) be a smooth toric variety. Denote by \( D^b(X_\Sigma) \) the bounded derived category of coherent sheaves on \( X_\Sigma \), \( D^b(gr-S) \) the bounded derived category of finitely generated multigraded \( S \)-modules, and \( T_\Sigma \) the full subcategory generated by modules annihilated by a power of \( B_\Sigma \). Then

\[
D^b(X_\Sigma) = \frac{D^b(gr-S)}{T_\Sigma}.
\]

The homogeneous coordinate ring picture allows one to translate questions about coherent sheaves to questions about graded \( S \)-modules. The techniques of combinatorial commutative algebra can then be used to study questions about D-branes.
2.3 Toric Calabi-Yau

Having given the general view of toric varieties as fans and homogeneous coordinates, we can now specialize our discussion to toric Calabi-Yau varieties. In general a compact Calabi-Yau variety cannot be toric; if one wants to define toric Calabi-Yau conditions, the variety must be assumed to be non-compact. This means we have left the realm of Yau’s theorem which was proved for compact manifolds: it established the equality between the conditions of trivial canonical bundle and Ricci-flatness of the metric associated to the Kähler form. The Calabi-Yau condition which will be imposed in this paper will be the triviality of the canonical bundle.

**Definition 2.6.** A toric variety \( X_\Sigma \) is Calabi-Yau if the canonical bundle is trivial i.e., \( K_{X_\Sigma} \cong O_{X_\Sigma} \).

For a smooth toric variety the canonical bundle is found to be \( O(\sum_{i=1}^{n} -D_{\rho_i}) \). To show that this divisor is trivial amounts to showing that there is a hyperplane in \( N_\mathbb{R} \) containing all the rays \( \rho_i \in \Sigma(1) \).

**Proposition 2.2.** A toric variety is Calabi-Yau if and only if the set of \( \rho_i \) is contained in some hyperplane of \( N_\mathbb{R} \).

**Proof.** By definition \( X_\Sigma \) is Calabi-Yau when \( O(\sum_{i=1}^{n} -D_{\rho_i}) \) is trivial which occurs if and only if \( \sum_{i=1}^{n} -D_{\rho_i} \) is trivial in \( A_{n-1}(X_\Sigma) \). From the exact sequence above, a divisor is trivial if there is an \( m \in M\mathbb{R} \) such that \( D = \sum_{i=1}^{n} \langle \rho_i, m \rangle D_{\rho_i} \). So \( O(\sum_{i=1}^{n} -D_{\rho_i}) \) is trivial if and only if there exists an \( m \in M\mathbb{R} \) such that \( \langle \rho_i, m \rangle = -1 \), which occurs if and only if the \( \rho_i \) lie in a hyperplane of \( N_\mathbb{R} \).

One consequence of Proposition 2.2 is that it imposes conditions on the charge vectors \( \Phi_i \), which are the columns of the matrix \( \Phi \) in (2). If the vectors \( \Phi_i \), are put together to make an \( r \times n \) matrix called \( \Phi \) then the conditions \( \sum_{i=1}^{n} \rho_i^j Q_j = 0 \), along with the fact that the \( \rho_i \) lie in a hyperplane, force the rows of \( \Phi \) to equal zero. This amounts to the fact that the \( G \)-action on \( \mathbb{C}^n \) has some negative weights and some positive weights to balance out.

Since \( G=\text{Hom}(A_{n-1}(X_\Sigma), \mathbb{C}^*) \), the grading of \( S \) by \( A_{n-1} \) can be thought of as the grading associated to the \( G \)-action on \( \mathbb{C}^n \) (throughout the paper the grading of \( S \) will be denoted by \( D \)). In GIT, the quotient is found by looking at the invariant part of the ring, which is the zero-graded piece \( S_0 \) of this homogeneous ring. So the GIT quotient given by Cox can be thought of as \( X_\Sigma=\text{Spec}(S_0) \) which is some possibly singular Calabi-Yau toric variety.

In general one wants to desingularize a toric variety; Aspinwall [2] describes a canonical procedure which can generate partial crepant desingularizations of \( X_\Sigma \). The crepant desingularizations are special because the pullback of the canonical bundle under a crepant desingularization remains trivial. In general a desingularization of \( X_\Sigma \) comes from a subdivision of \( \Sigma \). A crepant resolution of the toric variety \( X_\Sigma \) comes from a regular subdivision of the
polytope $\mathcal{P} \cap \mathcal{N}$ associated to the fan $\Sigma$. One way to think of a regular subdivision of $\mathcal{P}$ is to imagine that it lies in a hyperplane of some higher dimensional space. Imagine taking some of the vertices of the polytope and varying the heights so that they rise out of the hyperplane. Every regular subdivision of $\mathcal{P}$ comes from taking the lower convex hull of these new vertices and projecting it back onto the hyperplane. It is known that a toric variety is completely desingularized when the subdivisions each have volume 1.

Every choice of regular division corresponds in physics language to a phase of $X_\Sigma$, which means it is a different crepant resolution of the original affine toric variety. In terms of the GIT quotient, the subdivision amounts to changing the Cox ideal. The subdivision, in effect, takes a cone and breaks it up into multiple cones. It can be seen, from the definition of the Cox ideal, that subdividing cones will increase the number of generators of the Cox ideal as well as change the monomial generators. In (3) the graded ring does not change, so $D^b(gr-S)$ stays the same’ only the subcategory $T_\Sigma$ changes due to the subdivision.

3 Tilting Sheaves

The relationship (3) can only get one so far in terms of studying $D^b(X_\Sigma)$. For a given toric variety $X_\Sigma$ there are many finitely generated graded $S$-modules. The study of $D^b(X_\Sigma)$ can be simplified if one can find a special set of sheaves which generate all of $D^b(X_\Sigma)$. The term generate means that every element in the derived category comes from a direct sum and a special operation in the derived category called coning of these sheaves. The reason is that

Definition 3.1 ([2]). For a smooth Calabi-Yau variety $X$ over $\mathbb{C}$, a tilting sheaf of $D^b(X)$ is a coherent sheaf $\mathcal{M}$ on $X$ where

1. $\mathcal{M} = \mathcal{P}_1 \oplus \mathcal{P}_2 \oplus \ldots \mathcal{P}_n$ where each $\mathcal{P}_i$ is a simple invertible sheaf (for the purposes of the paper do not worry about the simple part);

2. $\text{Ext}^n(\mathcal{P}_i, \mathcal{P}_j) = 0$ for $\forall$ $i,j$ and $n > 0$;

3. The collection of $\mathcal{P}_i$ generate $D^b(X)$.

if a smooth variety $X$ has such a tilting sheaf $M$, the following theorem gives a relationship between $D^b(X)$ and the bounded derived category of $D^b(mod-A)$ finitely generated right modules over the endomorphism algebra $A = \text{End}(M)$. There has been a great deal of work done on finding tilting sheaves for toric varieties. In general not all toric varieties have tilting bundles. One of the points of [2] is to find such a set of bundles in the case of toric Calabi-Yau varieties.

Theorem 3.1 ([8] due to Baer and Bondal independently). Let $X$ be a smooth variety over $\mathbb{C}$. For a tilting sheaf $\mathcal{M}$ with algebra $A = \text{End}(\mathcal{M})$ the functor

$$\text{Hom}(\mathcal{M}, -) : \text{Coh}(X) \to \text{mod-A}$$

induces an equivalence of categories from $D^b(X)$ to $D^b(mod-A)$. The functor

$$(-) \otimes_A \mathcal{M} : \text{mod-A} \to \text{Coh}(X)$$

induces a equivalence of categories, from $D^b(mod-A)$ to $D^b(X)$, of $\text{Hom}(\mathcal{M}, -)$. 

6
3.1 Relationship with Quivers

Looking at the algebra $A=\text{End}(M)=\text{End}(P_1 \oplus P_2 \oplus \ldots \oplus P_n)$ more closely, it can be seen that elements of this non-commutative algebra come from elements of $\text{Hom}(P_i, P_j)$. One can can represent this geometrically using a combinatorial object called a quiver.

Definition 3.2 ([8]). A quiver $Q$ is a directed graph, which consist of a set $Q_0$ of vertices and a set $Q_1$ of arrows between the vertices. $Q$ comes with the maps $t,h$: $Q_1 \rightarrow Q_0$ which tell what vertex is the tail of each arrow and which is the head.

A path in the quiver consists of a collection of arrows $a_1, a_2, \ldots, a_k$ where the head of $a_{i+1}$ is equal to the tail of $a_i$. This can be used to create a path algebra of $Q$ denoted $kQ$. This algebra has a basis over $k$ consisting of all paths in $Q$. The multiplication of two paths $P_1$ and $P_2$ of $kQ$ is just the concatenation of path $P_1$ with path $P_2$ as long as the tail of $P_1$ is equal to the head of $P_2$ otherwise the product is just zero. Any path algebra $kQ$ can be generated from the arrows in $Q_1$. If a quiver has no cycles then this path algebra will be a finite dimensional vector space. The introduction of cycles in the quiver causes the path algebra to generate an infinite set of paths which eliminates the finite dimensional condition. In the case of a tilting sheaf $M$ one can associate a quiver as follows

1. For $M = P_1 \oplus P_2 \oplus \cdots \oplus P_n$ the associated quiver $Q$ has a vertex associated to each $P_i$.
2. For each pair $P_i$ and $P_j$ of vertices there are $\dim \mathbb{C} \text{Hom}(P_i, P_j)$ many arrows going from vertex $P_i$ to vertex $P_j$.

On the path algebra $kQ$ there may be some additional relationship between paths which can be placed in a left ideal denoted $\langle q \rangle$ of $kQ$. If we divide the path algebra by these relations $kQ/\langle q \rangle$ then it can be shown that for some $\langle q \rangle$, $A \cong kQ/\langle q \rangle$. This implies $D^b(\text{mod} - A)$ is equivalent to $D^b(\text{mod} kQ/\langle q \rangle)$ which is the derived category of finitely generated right $kQ/\langle q \rangle$ modules. One of the motivations of [2] is that these algebras $kQ/\langle q \rangle$ seem to be a non-commutative resolution of a singular toric Calabi-Yau variety. If one takes the center $Z(kQ/\langle q \rangle)$ and then $\text{Spec}(Z(kQ/\langle q \rangle))$ it can be seen that one gets some toric Calabi-Yau variety which is possibly singular. So this algebra $kQ/\langle q \rangle$ is giving a non-commutative resolution of the singular toric variety. The first step in this program of non-commutative resolutions involves finding a collection of tilting bundles for an arbitrary toric Calabi-Yau variety, which is where the techniques of combinatorial commutative algebra come into play.

4 Combinatorial Commutative Algebra

4.1 Local Cohomology

An advantage in using Cox’s construction is that invertible sheaves on a smooth $X_{\Sigma}$ can be thought of as graded rank 1 free $S$-module. These modules are of the form $S(\alpha)$ for $\alpha \in D$. Now the problem can be rephrased as, is there a tilting module $M = S(\alpha_1) \oplus S(\alpha_2) \oplus$
... $\oplus S(\alpha_n)$ whose sheaf $\widetilde{M}$ is the requisite tilting sheaf defined above? The first condition that needs to be satisfied is finding a collection $\{\alpha\} \subseteq D$ where $\text{Ext}^i_{X_S}(S(\alpha_i), S(\alpha_j)) = 0$ for all pairs $\alpha_i, \alpha_k \in \{\alpha\}$ and $i > 0$. For these invertible sheaves it is known that $\text{Ext}^i_{X_S}(S(\alpha), S(\beta)) \cong H^i(X_S, S(\alpha - \beta))$. Writing Ext in terms of cohomology is the proper setting in order to use local cohomology.

**Definition 4.1.** Given a $X_S$ with a graded $S$-module $M$ the total cohomology module for the coherent sheaf $\widetilde{M}$ is $H^i_{\text{tot}}(X_S, \widetilde{M}) = \bigoplus_{\delta \in D} H^i(X_S, \widetilde{M}(\delta))$. Moreover the $\delta$-graded piece is $H^i_{\text{tot}}(X_S, \widetilde{M})_{\delta} = H^i(X_S, \widetilde{M}(\delta))$.

In this definition the total cohomology module is graded by $D$. In particular, for any graded $S$-module $M$, $H^0_{\text{tot}}(X_S, \widetilde{M})_{\delta} = M_\delta$. This grading should give hints that this module is related to local cohomology. The precise relationship to local cohomology comes from [6] and [7] in the form of the following proposition.

**Proposition 4.1** (Prop. 2.3 in [7]). Given a smooth $X_S$ with homogeneous coordinate ring $S$, Cox ideal $B_S$ and graded $S$-module $M$, let $H^i_{B_S}(M)$ be the local cohomology module of $M$ support on $B_S$. Then:

1. there is an exact sequence of modules
   
   $0 \to H^0_{B_S}(M) \to M \to H^0_{\text{tot}}(X_S, \widetilde{M}) \to H^1_{B_S}(M) \to 0$;

2. for $i \geq 1$, $H^i_{\text{tot}}(X_S, \widetilde{M}) \cong H^{i+1}_{B_S}(M)$.

### 4.2 Calculating Vanishing Ext Groups

The local cohomology module of interest in this paper will be $H^i_{B_S}(S)$. From 4.1, the vanishing Ext condition for tilting modules can be reinterpreted as finding the gradings $\delta$ where $H^k_{B_S}(S)_{\delta} = 0$ for $k \geq 2$. In [2] the condition imposed is even stronger by requiring that $k = 0$ and $k = 1$ also have vanishing local cohomology. In order to narrow down which $\delta$ have vanishing local cohomology, one can try to isolate this region in $D$ by eliminating the regions which have non-vanishing local cohomology. There is a theorem in [6] which will provide such a method (we will provide the version of this result given in [2]). First take the minimal $\mathbb{Z}^n$ graded resolution of $I_S$ and look at the Betti numbers $b_{k, \alpha}$ of this resolution. Since $I_S$ is a Stanley-Reisner ideal, the Betti numbers will only be binary vectors in $\mathbb{Z}^n$.

There is associated map from $\mathbb{Z}^n$ to the binary vectors $\{0, 1\}^n$.

**Definition 4.2.** The binary map $\Xi : \mathbb{Z}^n \to \{0, 1\}^n$ replaces the entry $v_i$ of $v = (v_1, v_2, \ldots, v_n) \in \mathbb{Z}^n$ with 0 if $v_i$ is non-negative, and with 1 if $v_i$ is negative.

Notice that the binary map really only depends on the signs of vectors in $\mathbb{Z}^n$ so entire octants in $\mathbb{Z}^n$ map to the same binary vector. It is important to point out that $H^i_{B_S}(S)$ actually has two gradings. The first comes from $D$, as previously discussed but there is also the fine $\mathbb{Z}^n$ grading. The two gradings are related by the degree map $\Phi$ in (2). The theorem in [6] will be applied the $\mathbb{Z}^n$ grading of the local cohomology modules.
Theorem 4.1 ([6]). $H^i_{B_S}(S_v)$ for $v \in \mathbb{Z}^n$ is nonzero if and only if there is, for a minimal $\mathbb{Z}^n$ graded resolution of $I_\Sigma$, a non-zero Betti number $b_{k,\alpha}$ where $\Sigma(v) = \alpha$.

So for a minimal resolution of $I_\Sigma$ the region $\Xi^{-1}(\alpha)$ will be excluded as possible candidate degrees. Whatever regions in $\mathbb{Z}^n$ which still exist can be mapped by $\Phi$ in (2) to $D$ and this will be the set of candidate degrees $\delta$. The actual goal is to find $\alpha_i$ and $\alpha_j$ such that $\delta$ is the difference of these two degrees. The previous procedure generates an infinite set of $\delta$ so it may seem that finding a finite set of $\alpha_i$ may be impossible, but there is another condition which can be imposed. The vanishing of $\text{Ext}$ must be symmetric so both $\delta$ and $-\delta$ have to be candidate degrees. This puts enough constraints so that there will only be a finite number of $\delta$ which also satisfy the previous condition. In [2] Aspinwall proves that there is a finite set of collections $\{\alpha_i\}$ which satisfy these conditions.

Theorem 4.2 ([2]). Given a smooth $X_\Sigma$ the number of $\delta$ where $H^i_{B_S}(S)_\delta = 0$ and $H^i_{B_S}(S)_{-\delta} = 0$ is finite. Furthermore, the number of choices of $\{\alpha_i\}$ is finite.

Aspinwall’s method gives a very constructive method for determining a finite set of candidates $\{\hat{S}(\alpha)\}$ which will generate $D^b(X_\Sigma)$. In [2] it is shown that for $X_\Sigma$ where the group for the GIT quotient is $\mathbb{C}^*$, the set $\{\hat{S}(\alpha)\}$ can be found, and it depends only on the collection $\Sigma(1)$ and not on the fan structure of $\Sigma$.

5 Conclusion

The study of D-branes on Calabi-Yau manifolds is an exciting and interesting subject to both mathematicians and physicists. There is much that is still not completely understood in this area. Hopefully this paper elucidates some of the connections between toric varieties, combinatorial commutative algebra and D-branes. One of the fundamental open questions raised in [2] is whether one can find $\{\hat{S}(\alpha)\}$ which depends only on $\Sigma(1)$ for a general toric variety. While it has been shown for many cases, it is not known if this is the case for general toric Calabi-Yau varieties. The tools of combinatorial commutative algebra could be helpful in providing an answer to this question. There are certainly many more questions about D-branes on toric Calabi-Yau manifolds which remain to be understood.

References


