

15.

§ 3.6 Abstract vector spaces

Def: A vector space over \mathbb{R} is a set V with two operations

- vector addition: $u, v \in V \mapsto u+v \in V$
- scalar multiplication: $v \in V, c \in \mathbb{R} \mapsto cv \in V$

satisfying

- V is an abelian group $\left\{ \begin{array}{l} 1. u+v = v+u \quad \forall u, v \in V; \text{ [commutative]} \\ 2. (u+v)+w = u+(v+w) \quad \forall u, v, w \in V; \text{ [associative]} \\ 3. \exists 0 \in V \text{ with } 0+v = v \quad \forall v \in V; \\ 4. \text{ for each } v \in V \exists -v \in V \text{ with } v+(-v) = 0; \end{array} \right.$

axioms for "field"
are similar

and $\forall u, v \in V$ and $c, d \in \mathbb{R}$,

5. $c(dv) = (cd)v;$
6. $c(u+v) = cu+cv;$
7. $(c+d)v = cv+dv;$
8. $1v = v.$

E.g. 1. \mathbb{R}^n or any subspace

2. $\mathbb{R}^{m \times n}$ (or \mathbb{C} or \mathbb{Q} or \mathbb{F}_2)

3. I any set. $F(I) = \text{functions } I \rightarrow \mathbb{R} \Rightarrow F(I)$ is a vector space:

$$(cf)(t) = c f(t) \quad (f+g)(t) = f(t) + g(t)$$

Axiom check: not interesting; uses corresponding properties of \mathbb{R} .

0. \cdot \cdot \cdot \cdot \cdot \cdot \cdot
68. \cdot \cdot \cdot \cdot \cdot \cdot \cdot
e. \cdot \cdot \cdot \cdot \cdot \cdot \cdot
 π \cdot \cdot \cdot \cdot \cdot \cdot \cdot
 $\sqrt{2}$ \cdot \cdot \cdot \cdot \cdot \cdot \cdot

Think: real vectors with entries indexed by I

e.g. (i) $I = \{1, \dots, n\}$ $\begin{matrix} 1 \mapsto [x_1] \\ 2 \mapsto [x_2] \\ \vdots \\ n \mapsto [x_n] \end{matrix}$

$$\begin{matrix} (1,1)_N & & & \\ & x_{11} & \ddots & x_{mn} \\ & & \ddots & \\ & & & y_{(m,n)} \end{matrix}$$

(ii) $I = \{(i, j) \mid i \in \{1, \dots, m\} \text{ and } j \in \{1, \dots, n\}\}$

(iii) $I = V \Rightarrow F(V)$ is a vector space

dual vector space $V^* = \{ \text{linear functions } V \rightarrow \mathbb{R} \}$ better reflects structure of V

(iv) $I = \{1, 2, \dots\} \Rightarrow F(I) = \mathbb{R}^\omega \quad \omega = \#\text{positive integers}$

$$x \in \mathbb{R}^\omega \Rightarrow x = (x_1, x_2, \dots)$$

$$cx = (cx_1, cx_2, \dots)$$

$$x+y = (x_1+y_1, x_2+y_2, \dots)$$

Def: Let V be a vector space. $U \subseteq V$ is a subspace if $U \neq \emptyset$ and

$v+cw \in U \quad \forall v, w \in U \text{ and } c \in \mathbb{R}$. (same def as before)

Prop 6.1: $U \subseteq V$ subspace $\Rightarrow U$ is a vector space.

Pf: Already have + and scalar multiplication in V . Def $\Rightarrow u+v \in U$ and $c u \in U$ $\forall u, v \in U$ and $c \in \mathbb{R}$. What's needed: $\forall u, v, w \in U$ and $c, d \in \mathbb{R}$, every equality in the 8 axioms holds in U . But each one already holds in V ! \square

e.g. $u+v = v+u$ in V , but both sides lie in U

E.g. 4. $I \subseteq \mathbb{R}$ an interval $\Rightarrow \{ \text{continuous functions } I \rightarrow \mathbb{R} \} = C^0(I)$ is a vector space.

Pf: $\mathcal{F}(I)$ is a vector space. $0 \in C^0(I)$, and

$C^0(I)$ is a subspace: f, g continuous $\Rightarrow f+g$ and cf continuous.

$\hat{C}^0(I) = \{ \text{differentiable functions } I \rightarrow \mathbb{R} \}$ is a subspace: f, g differentiable $\Rightarrow f+g$ and cf "

$C^1(I) = \{ \text{continuously differentiable functions } I \rightarrow \mathbb{R} \}$

$C^2(I) = \{ f \in C^1(I) \mid f' \in C^1(I) \}$

$C^3(I) = \{ f \in C^2(I) \mid f' \in C^2(I) \}$

$C^\infty(I) = \bigcap_{i=0}^{\infty} C^i(I)$

5. $\mathcal{P} = \{ \text{polynomial functions } p(t) = a_k t^k + a_{k-1} t^{k-1} + \dots + a_1 t + a_0 \}$

\mathcal{P} subspace $\Rightarrow \mathcal{P}$ is a vector space.

Q. Is $\{ \text{polynomials of degree } k \}$ a vector space?

Ans: $(t^2 + 2t - 1) + (-t^2 + 2) = 2t + 1$ so No.

$\{ p \in \mathcal{P} \mid \deg p \leq k \}$ polynomials of degree $\leq k$

Q. $\dim \mathcal{P} = ?$

$\dim \mathcal{P}_k = ?$

Prop 6.3: \mathcal{P}_k has a basis $1, t, \dots, t^k$.

Pf: $\mathcal{P}_k = \text{span}(1, t, \dots, t^k)$ by definition.

But $p(t) = 0 \Rightarrow a_i = 0 \forall i$. Why?

1. $\deg p \leq k \Rightarrow p$ has $\leq k$ roots if $p \neq 0$.

2. $p(0) = 0 \Rightarrow a_0 = 0$

$$p'(0) = 0 \Rightarrow a_1 = 0$$

⋮

$$p^{(k)}(0) = 0 \Rightarrow k! a_k = 0 \Rightarrow a_k = 0. \square$$

span, linear (in)dependence, basis, dimension—all make sense for arbitrary vector spaces.

6. $V = \{f \in C^1(\mathbb{R}) \mid f' = f\} \subseteq C^1(\mathbb{R})$ subspace: $0' = 0$ ✓

$$e^x \in V,$$

$$\begin{aligned} (f+cg)' &= f' + cg' \\ &= f+cg \text{ if } f, g \in V. \end{aligned}$$

Claim: e^x is a basis for V .

Pf: Given $f \in V$, let $g(x) = f(x)e^{-x} = \frac{f(x)}{e^x}$.

Want: $g(x) = c$ constant, so $f(x) = ce^x$.

$$\begin{aligned} \text{Calculate: } g'(x) &= f'(x)e^{-x} + f(x)(-e^{-x}) \\ &= \underbrace{\left(f'(x) - f(x)\right)}_0 e^{-x} \\ &= 0. \end{aligned}$$

Thus $g \equiv c$, so $f(x) = ce^x$. \square

7. $\mathbb{R}^{n \times n}$

U

$\mathcal{U} = \left\{ \begin{bmatrix} * & & \\ & \ddots & \\ & & * \end{bmatrix} \right\}$ upper-triangular matrices

$\mathcal{L} = \mathcal{U}^T =$ lower-triangular matrices

D

$\mathcal{D} = \left\{ \begin{bmatrix} * & 0 & & \\ 0 & * & & \\ & & \ddots & \\ & & & * \end{bmatrix} \right\}$ diagonal matrices

$\left\{ \begin{bmatrix} * & 0 & & \\ 0 & * & & \\ & & \ddots & \\ & & & * \end{bmatrix} \right\}$