12. Q. \( A \in \mathbb{R}^{m \times n} \Rightarrow \text{dim } C(A) = ? \)

Ans: \( n \), unless there's some coincidence.

\[ \text{rank } A \]

\[ n \]

\[ m \]

\[ A \]

\[ \text{injective} \Rightarrow \text{no coincidence} \]

\[ \text{surjective} \Rightarrow \text{target covered} \]

Q \( \Leftrightarrow \) Q': Given \( n \) vectors \( v_1, \ldots, v_n \in \mathbb{R}^m \), what is \( \text{dim span}(v_1, \ldots, v_n) \)?

coincidence: span has dim less than possible

Def: vectors \( v_1, \ldots, v_k \) are linearly dependent if \( c_1v_1 + \cdots + c_kv_k = 0 \) for some \( c_1, \ldots, c_k \in \mathbb{R} \) not all 0.

Prop: \( v_1, \ldots, v_k \) linearly dependent \( \Leftrightarrow \) \( v_i \in \text{span}(v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_k) \) for some \( i \).

\[ \text{Pf: } \Rightarrow: c_1v_1 + \cdots + c_kv_k = 0 \text{ with } c_i \neq 0 \Rightarrow v_i = -\frac{1}{c_i} \sum_{j \neq i} c_j v_j \in V_i. \]

\[ \Leftarrow: v_i = \sum_{j \neq i} c_j v_j \Rightarrow \sum_{j=1}^k c_j v_j = 0, \text{ where } c_i = -1. \]

E.g. Are \( \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 8 \\ 1 \\ -1 \end{bmatrix} \) linearly dependent?

Sol:

\( x_1v_1 + x_2v_2 + x_3v_3 = 0 \Leftrightarrow x \in N(A) \)

dependence relation \( \Leftrightarrow \) also \( x \neq 0 \)

\[ \begin{bmatrix} 3 & -1 & 8 \\ 0 & 1 & 1 \\ 1 & -4 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \Rightarrow \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} \in N(A) \setminus \{0\}, \text{ so: yes.} \]

Moral: \( N(A) \setminus \{0\} = \text{linear dependence relations on columns of } A \)

n - m if \( n > m \), unless there is some coincidence!

A. \( n - m \) if \( n > m \), unless there is some coincidence!

0 if \( n \leq m \).
\(a_1, \ldots, a_n\) linearly independent in \(\mathbb{R}^m\)
\[\iff \text{if } x_1 a_1 + \cdots + x_n a_n = 0 \text{ then } x_i = 0 \forall i \]
\(N(A) = 0\) for \(A = \begin{bmatrix} a_1 & \cdots & a_n \end{bmatrix}\)
\(0\) can be expressed uniquely as a linear combination of \(a_1, \ldots, a_n\)
\(b \in \text{span}(a_1, \ldots, a_n) = C(A)\) is uniquely a linear combination of \(a_1, \ldots, a_n\)
\(Ax = 0\) has only one solution \(\iff [A \mid b]\) consistent
\(Ax = b\) has only one solution when \(b \in C(A)\)
\(Ax = b\) has at most one solution \(\forall b \in \mathbb{R}^m\)
\(M_A\) is injective
\(M_A\) does not decrease dimension
\(\text{rank } A = n\)
\(\mathbb{R}^m \xrightarrow{M_A} \mathbb{R}^n\)
\(\Leftrightarrow\) no column of \(A\) lies in the span of the others \(Ax \xrightarrow{\implies} x\)

**Prop 3.2:** Assume \(v_1, \ldots, v_k\) are linearly independent.
Then \(v_1, \ldots, v_k, v\) is linearly independent \(\iff v \notin \text{span}(v_1, \ldots, v_k)\), 
linearly dependent \(\iff v \in \text{span}(v_1, \ldots, v_k)\)

**Pf:** \(\Rightarrow\) Suppose \(c_1 v_1 + \cdots + c_k v_k + cv = 0\). Then \(c \neq 0\) since \(c = 0 \Rightarrow c_1 v_1 + \cdots + c_k v_k = 0 \Rightarrow c_1 = \cdots = c_k = 0\) because \(v_1, \ldots, v_k\) are linearly independent.
So \(v = -\frac{1}{c}(c_1 v_1 + \cdots + c_k v_k)\).
\(\Leftarrow\): Previous prop. (Doesn't need \(v_1, \ldots, v_k\) linearly independent.) \(\Box\)

**E.g.** Is \(0\) linearly independent? \(\No: 1 \cdot 0 = 0\)

**E.g.** Is \(\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}\) linearly independent? \(\No: 1 \cdot v_1 - 1 \cdot v_2 = 0. \Rightarrow \text{need multisets technically}\)

**E.g.** Is \(\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\) \(\notin \text{span}(v_1, v_2)\) but \(v_1, v_2, v\) not linearly independent.
\(v_1, v_2, v\) Why doesn't this contradict Prop.?

**E.g.** Prove that \(A v_1, \ldots, A v_k\) are linearly independent if \(v_1, \ldots, v_k\) are linearly independent and \(A \in \mathbb{R}^{m \times n}\) has rank \(n\).

**Sol.** Suppose \(c_1 A v_1 + \cdots + c_k A v_k = 0\). Then \(A (c_1 v_1 + \cdots + c_k v_k) = 0\), so \(c_1 v_1 + \cdots + c_k v_k = 0\) because \(M_A\) is injective. Thus \(c_1 = \cdots = c_k = 0\) since \(v_1, \ldots, v_k\) are linearly independent.