Chapter 3: Vector spaces

\( \mathbb{R}^n \) is a (real) vector space (over \( \mathbb{R} \))
\( \mathbb{C}^n \) (complex)
\( \mathbb{Q}^n \) (rational)
\( \mathbb{F}_2 \) (binary)

**Def:** A subset \( V \subseteq \mathbb{R}^n \) is a **subspace** if
1. \( V \neq \emptyset \) and
2. \( v, w \in V \) and \( c \in \mathbb{R} \Rightarrow v + cw \in V \).

**Remark:**
1. can use instead: \( 0 \in V \) \( (v=w; c=1) \) but it’s not easier to check
2. \( \Rightarrow V \) closed under arbitrary linear combinations \( (Pf: \text{induction}) \)

**Examples**

1. \( V = \{ 0 \} \)
2. \( V = \mathbb{R}^n \)
3. line \( l = \text{span}(v) \) with \( v \neq 0 \)
4. hyperplane \( H = \{ x \in \mathbb{R}^n \mid Ax = 0 \} \) for some fixed \( a \in \mathbb{R}^n \)
5. the **nullspace** of any matrix \( A \in \mathbb{R}^{m \times n} \)
   \[
   N(A) = \{ x \in \mathbb{R}^n \mid Ax = 0 \} \}
   (= \text{kernel of } \mu_A ) = \text{sols } [A | 0]
   \]
   \( Pf: \) \( AO = 0 \) and \( \mu_A \) is linear!
6. \( \text{span}(v_1, \ldots, v_k) \) for any \( e \) \( [v_1, \ldots, v_k] \in \mathbb{R}^n \)
   \( = \text{C}(A) \) for \( A = [v_1, \ldots, v_k] \in \mathbb{R}^{n \times k} \)

\( Pf: \)
1. \( 0 = 0v_i \)
2. \( v = c_1v_1 + \ldots + c_kv_k \) and \( w = d_1v_1 + \ldots + d_kv_k \)
\( \Rightarrow v + cw = (c_1d_1v_1 + \ldots + c_kv_k) \in V \)

The point is that none of this is new; you already know what subspaces look like and how to describe all of them — yes, all:

**Thm:** Every subspace of \( \mathbb{R}^n \) has the form of \#5.

\( Pf: \) For \#6, keep finding "independent" vectors that increase dimension until you can’t anymore: if \( \exists v_i \neq 0 \) in \( V \), then \( \text{span}(v_i) \subseteq V \).

\begin{align*}
  v_2 \notin \text{span}(v_1) \\
  (v_1, v_2) \subseteq V \\
  & \vdots \\
  & \vdots \\
  & \text{stopping at } v_k \text{ for some } k \leq n
\end{align*}

because \( \text{span}(v_1, \ldots, v_k) = \text{C}(A) \) for \( A = [v_1, \ldots, v_k] \) is spanned by \( r \) vectors, where \( r = \text{rank } A \leq n \).

For \#5: find constraint equations, given \( v_1, \ldots, v_k \). \( \Box \)
Q. Is this a subspace?

1. 2. 3.

4. \( x_0 + \text{span}(v_1, \ldots, v_k) \) yes, if \( x_0 \in \text{span} \); no if not

5. \( U + V = \{ u + v \mid u \in U \text{ and } v \in V \} \) for subspaces \( U \) and \( V \)

\[
\begin{align*}
\text{Yes: } & 1.0 + 0 \in U + V; \quad 2. u + v \in U + V; \\
& u' + v' \in U + V \quad \Rightarrow u + v + c(u' + v') = (u + cu') + (v + cv') \in U + V.
\end{align*}
\]

Note: \( U + V \) is the smallest subspace containing \( U \) and \( V \).

E.g. \( x\)-axis + \( y\)-axis = \( xy\)-plane.

Def: For any subset \( C \subseteq \mathbb{R}^n_{\text{col}} \) set \( C^\perp = \{ u \in \mathbb{R}^n_{\text{row}} \mid u \cdot x = 0 \ \forall \ x \in C \} \)

\[
\begin{align*}
\text{If } & R \subseteq \mathbb{R}^n_{\text{row}} \text{ set } \mathbb{R}^\perp = \{ x \in \mathbb{R}^n_{\text{col}} \mid u \cdot x = 0 \ \forall \ u \in R \}.
\end{align*}
\]

WARNING: The book calls \( C^\perp \) what I call \( (C')^\top \); book thinks every vector is a column! \( C^\perp \) in book: collection of columns will make even more sense when we get to row space \( R(A) \).

\( C^\perp \) for us: collection of rows

Prop: \( C^\perp \) is a subspace of \( \mathbb{R}^n_{\text{row}} \)

\( R^\perp \subset \subset \subset \mathbb{R}^n_{\text{col}} \)

Pf: 1. \( 0 \cdot x = 0 \).

2. \( v \cdot x = 0 \text{ and } w \cdot x = 0 \Rightarrow (v + cw) \cdot x = v \cdot x + cw \cdot x = 0 + c \cdot 0 = 0 \ \forall \ c \in \mathbb{R} \).

For \( R^\perp \), transpose the argument. \( \square \)

Lemma: \( V^\top \subseteq \mathbb{R}^n_{\text{row}} \) is a subspace \( \iff \) \( V \subseteq \mathbb{R}^n \) is a subspace.

Pf: Transpose is linear. \( \square \)

Check! \( (A + cB)^\top = A^\top + cB^\top \ \forall \ A, B \in \mathbb{R}^{m \times n} \)

Def: \( V \subseteq \mathbb{R}^n_{\text{col}} \) and \( W \subseteq \mathbb{R}^n_{\text{row}} \) are orthogonal, written \( V \bot W \),

if \( \langle V, W \rangle = 0 \). Equivalently:

- \( \langle v, w \rangle = 0 \ \forall \ v \in V \text{ and } w \in W \)
- \( W \subseteq V^\perp \)
- \( V \subseteq W^\perp \)
Also, \( V \in \mathbb{R}_{\text{col}}^n \) and \( W \in \mathbb{R}_{\text{col}}^n \):

\[
V \perp W \text{ if } V^T W = 0 \quad W^T \perp V
\]

E.g., \( C(A) = \text{column space of } A \)

\[
= \text{image of } \mu_A \quad \text{subspace as in #6.}
\]

\[
L(A) = \text{left nullspace of } A = \{ y \in \mathbb{R}_{\text{row}}^n \mid yA = 0 \}
\]

\[
= \text{kernel of } p_A : \mathbb{R}_{\text{row}}^n \rightarrow \mathbb{R}_{\text{row}}^n \quad y \mapsto yA
\]

\[
= \text{image of } \mu_A = N(A^T)^T
\]

Prop: \( L(A) = C(A)^\perp \).

Proof: \( y \in L(A) \Rightarrow yA = 0 \Rightarrow yA_{i, j} = 0 \quad \forall j = 1, \ldots, n \)

\[
\Rightarrow c_1 y_{1, v} + \cdots + c_n y_{n, v} = 0 \quad \forall c_1, \ldots, c_n \in \mathbb{R}
\]

\[
\Rightarrow \quad y_i (c_1 a_{i, 1} + \cdots + c_n a_{i, n}) = 0
\]

\[
\Rightarrow \quad y \in C(A)^\perp \quad \text{Hence } L(A) \subseteq C(A)^\perp
\]

But \( y \in C(A)^\perp \Rightarrow yA_{i, j} = 0 \forall j \) by def. (since \( a_{i, j} \in C(A) \))

\[
\Rightarrow \quad yA = 0 \quad \text{so } L(A) = C(A)^\perp
\]

Similarly:

Def: \( R(A) = \text{row space of } A = \text{span}(A_1, \ldots, A_m) \subseteq \mathbb{R}_{\text{row}}^n \)

Cor: \( N(A) = R(A)^\perp \).

Proof: Transpose previous Prop. \( L(A^T)^T = (C(A^T)^T)^\perp \)

WARNING: The book calls \( R(A)^T \) what I call \( R(A) \); book thinks every vector is a column!