(rest of) today: \( A \in \mathbb{R}^{m \times n} \)

**Def:** The (system of) equation(s) \( Ax = b \) is **inhomogeneous** if \( b \neq 0 \); the corresponding equation(s) \( Ax = 0 \) is the associated **homogeneous** (system of) equation(s).

**Lemma:** For vectors \( x, y \in \mathbb{R}^n \) and scalar \( c \in \mathbb{R} \),
\[
A(x + cy) = Ax + cAy.
\]
**Pf:**
\[
A(x + cy) = (x_1 + cy_1)a_1 + \cdots + (x_n + cy_n)a_n
= x_1a_1 + \cdots + x_na_n + cy_1a_1 + \cdots + cy_na_n
= Ax + cAy.
\]

**Thm 5.3:** Assume \( Ax = b \) has a "particular" solution \( x_0 \).

\( v \) is a solution of \( Ax = b \) \( \iff \) \( v \) has the form
\[
v = x_0 + u
\]
for some solution \( u \) of \( Ax = 0 \).

**Pf:**
\( \iff: \)
\[
v = x_0 + u \implies Av = A(x_0 + u)
= Ax_0 + Au \quad \text{by Lemma}
= b + 0
= b \quad \implies v \text{ is a solution of } Ax = b.
\]
\( \Rightarrow: \) Assume \( \nu \) solves \( Ax = b \); i.e. assume \( Av = b \). Then \( v = x_0 + u \)
for some \( u \), namely \( u = v - x_0 \), and
\[
A(v - x_0) = Av - A x_0 \quad \text{by Lemma}
= b - b
= 0.
\]

**Corollary:** A **consistent** system \( Ax = b \) has a unique solution
\( \iff \) \( Ax = 0 \) has only the **trivial** solution \( x = 0 \).

**Prop 5.4:** \( Ax = 0 \) has unique solution \( \iff \) rank \( A = n \).

**Pf:** Equivalent: \( Ux = 0 \) \( \iff \) \( \text{rank } U = n \) for all \( U \) in reduced echelon form.
Why? If \( A \sim U \) then \( \text{sols } A = \text{sols } U \) and \( \text{rank } A = \text{rank } U \).

So let \( U \) be in r.e.f. Then

\[
\text{rank } U < n \Rightarrow U \text{ has } < n \text{ pivots} \Rightarrow \text{some column of } U \text{ has no pivot} \\
\Rightarrow \text{some variable is free} \Rightarrow Ux = 0 \text{ has (at least) } \mathbb{R}^m \text{-many solutions.}
\]

On the other hand, \( \text{rank } U = n \Rightarrow U = \begin{bmatrix}
1 \\

0 \\

\vdots \\
0 \\

\end{bmatrix} \Rightarrow \text{the only sols have } x_1 = 0, x_2 = 0, \ldots, x_n = 0.
\]

Geometrically, why should this (Prop 5.4) be?

\[
\begin{array}{cccc}
\mathbb{R}^m & \xrightarrow{A} & \mathbb{R}^n \\
\text{image} = C(A) & \text{dim} = \text{rank } A \\
\end{array}
\]

So \( \text{rank } A = n \) means \( A \) preserves dimension of \( \mathbb{R}^m \)!

\[
\begin{array}{cccc}
\mathbb{R}^m & \xrightarrow{A} & \mathbb{R}^n \\
\end{array}
\]

\( A \) sticks \( \mathbb{R}^n \) into \( \mathbb{R}^m \) without compression: \( x \neq y \Rightarrow Ax \neq Ay \)

In particular: \( x \neq 0 \Rightarrow Ax \neq A0 = 0 \).

Need more experience under our belts to do this justice.

\[
\begin{array}{cccc}
\mathbb{R}^2 & \xleftarrow{\text{or}} & \mathbb{R}^3 \\
\text{rank } = 1 & n = 1 & \sqrt{\text{rank } = 2} & n = 2 \\
\end{array}
\]

\( \mathbb{R}^2 \) vs. \( \mathbb{R}^3 \)

Q1. For which \( A \) does \( Ax = b \) have unique solution for all \( b \in \mathbb{R}^m \)?

Q2. """"""""""" is """"""""""" consistent """""""""""?

A. First look at the pictures: line misses \( \cdots \cdots \). Why? \( \text{rank } < m \)!

\[
\begin{array}{cccc}
\mathbb{R}^m & \xrightarrow{A} & \mathbb{R}^n \\
\text{general: for all } b \in \mathbb{R}^n, & \begin{array}{c}
\cdot b \text{ has the form } Ax \\
\cdot b \in C(A) \\
\cdot b \in \text{image of } A \\
\end{array} & \text{equivalent} \\
\end{array}
\]

\[
\begin{array}{cccc}
\mathbb{R}^m & \xrightarrow{A} & \mathbb{R}^n \\
\text{ie.} & \text{Prop: } Ax = b \text{ consistent for all } b \in \mathbb{R}^m & \text{iff } \text{rank } A = m. \\
\end{array}
\]

A2. Prop: \( Ax = b \) consistent for all \( b \in \mathbb{R}^m \) \( \Leftrightarrow \text{rank } A = m. \)

\[
\begin{array}{cccc}
\mathbb{R}^m & \xrightarrow{A} & \mathbb{R}^n \\
A1. & \text{Ax = b is consistent for all b and, by Cor,} & \begin{array}{c}
\cdot Ax = 0 \text{ has only the trivial solution } x = 0. \\
\Leftrightarrow \text{rank } A = n \\
\end{array} & \text{Prop.5.4} \\
\text{Prop.5.4} & \text{rank } A = m \\
\end{array}
\]

\[\Leftrightarrow \text{rank } A = m = n.\]
Def: A is non-singular (or invertible) if \( m = n = \text{rank} \, A \).
A is singular if \( m = n \) and \( \text{rank} \, A < n \).

E.g., The \( n \times n \) identity matrix \( I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \) is non-singular.

General: A non-singular \( \Rightarrow \) A has r.e.f. \( A \) \( \cong \) \( I_n \).

(Do in class if there is time:)

Application: curve fitting.

Given 3 points \((x_1, y_1), (x_2, y_2), (x_3, y_3) \in \mathbb{R}^2 \) with \( x_1, x_2, x_3 \) distinct, find a parabola \( y = ax^2 + bx + c \) through them.

(HW: \( v_1, v_2, v_3 \) not collinear \( \Rightarrow \) parabola exists and is unique.)

Answer: \( \begin{align*}
ax_1^2 + bx_1 + c &= y_1, \\
ax_2^2 + bx_2 + c &= y_2, \\
ax_3^2 + bx_3 + c &= y_3
\end{align*} \)

An inhomogeneous linear system!

Solution = coeffs \( a, b, c \) on parabola through \( v_1, v_2, v_3 \)

Class selects points; we all solve

(Ensure one pt. has \( x = 0 \), for ease of row reduction.)