A \in \mathbb{R}^{m \times n} be \mathbb{R}^{m}

Q. What operations on \([A \mid b]\) preserve sol set \(S\)?

A. (i) Swap any pair of rows.
   (ii) Multiply a row by a scalar \(\neq 0\).
   (iii) Replace any row by its sum with a multiple of another row.

Def (i), (ii), and (iii) are elementary row operations.

Theorem 4.1 Applying any sequence of elementary row ops to \([A \mid b]\) results in a system with the same solution set.

Pf: (i) merely lists the equations \(A_i x = b_i\) in a different order.
   For (ii), note that \(A_i x = b_i \iff c(A_i x) = c b_i \iff (cA_i) x = c b_i\).
   For (iii), let \([A' \mid b']\) be the system obtained from \([A \mid b]\) by replacing the row \([A_i \mid b_i]\) with \([A_i + cA_j \mid b_i + cb_j]\).

Let \(S' = \text{sols of } A'x = b'\). Then \(S \subseteq S'\)

because \((A_i + cA_j)x = A_i x + cA_j x = b_i + cb_j\) when \(x \in S\).

Aside: need \(S = S'\), not just \(S \subseteq S'\).
\[ S \subseteq S' \text{ and } S' \subseteq S. \]

But \(A_i = A'_i - cA'_j\) (!) so also \(S' \subseteq S\).

Finally, since each of (i), (ii), (iii) preserves \(S\), any sequence of them does, as well. \(\square\)

e.g. \(Ax = b\) for \(A = \begin{bmatrix} 3 & -2 & 2 & 9 \\ 2 & -2 & -2 & -4 \end{bmatrix}\) and \(b = \begin{bmatrix} 4 \\ 6 \end{bmatrix}\).
$$\begin{bmatrix} A \mid b \end{bmatrix} = \begin{bmatrix} 3 & -2 & 2 & 9 & 4 \\ 2 & 2 & -2 & -4 & 6 \end{bmatrix}$$

A_1 \rightarrow A_1 - A_2

$$\begin{bmatrix} 1 & -4 & 4 & 13 & -2 \\ 2 & -2 & -4 & 6 \end{bmatrix}$$

A_2 \rightarrow A_2 - 2A_1

$$\begin{bmatrix} 1 & -4 & 4 & 13 & -2 \\ 0 & 10 & -10 & -30 & 10 \end{bmatrix}$$

A_2 \rightarrow A_2 - 2A_1

$$\begin{bmatrix} 1 & -4 & 4 & 13 & -2 \\ 0 & 1 & -1 & -3 & 1 \end{bmatrix}$$

A_1 \rightarrow A_1 + 4A_2

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & -1 & -3 & 1 \end{bmatrix} = \begin{bmatrix} A' \mid b' \end{bmatrix}$$

$$\Rightarrow A'x = b'$$ has same solution $S$ as $A'x = b'$.

$$\Rightarrow S = \{ x \in \mathbb{R}^4 \mid x_1 + 3x_4 = 2, x_2 - x_3 - 3x_4 = 1 \}$$

In particular,

$$x_3 = x_4 = 0 \Rightarrow x_1 = 2, x_1 = 1 \Rightarrow \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \in S.$$

A particular solution $x_0$

Think of as parameters

general solution: $x_3, x_4$ free variables — can take on any values

$$x_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -x_4 \\ 3x_4 \\ 0 \\ 0 \end{bmatrix}$$

$x \in S \Leftrightarrow x = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ x_3 \\ 3x_4 \\ 0 \end{bmatrix}$

Def: A matrix is in echelon form if

1. the leading (leftmost nonzero) entries progress to the right from each row to the next; and
2. all 0 rows are at the bottom.

The echelon form is reduced if, in addition,

3. every leading entry is 1; and
4. in each column containing a leading entry, all other entries are 0.
e.g. just did one!

\[
\begin{bmatrix}
1 & 4 & 13 & -2 \\
0 & 10 & -10 & -30 \\
0 & 1 & -1 & 2 \\
\end{bmatrix}
\]

echelon form

\[
\begin{bmatrix}
1 & 0 & 0 & 1 \\
0 & 1 & -1 & 2 \\
\end{bmatrix}
\]

reduced echelon form

Thm 4.3: Each matrix has unique reduced echelon form.

Pf: Exercise 16 which you aren't asked to do, but you could. □

How to find it?

Algorithm (Gaussian elimination).

Echelon form

Init: i = 1

While: there is a nonzero entry in some row ≥ i

Do: 1. pick row ≥ i with a leftmost such entry

2. swap that row with row i

3. add multiples of row i to rows > i to cancel entry in pivot column

4. i \rightarrow i + 1

Output: the resulting matrix

Reduced echelon form given any echelon form

Init: i = 1

While: there is a nonzero entry in some row ≥ i

Do: 1. rescale row i so pivot is 1

2. add multiples of row i to rows < i to cancel entries in pivot column

3. i \rightarrow i + 1

Output: the resulting matrix