Q. What operations on \([A|b]\) preserve sol set \(S\)?

A. (i) Swap any pair of rows.
(ii) Multiply a row by a scalar \(\neq 0\).
(iii) Replace any row by its sum with a multiple of another row.

Def (i), (ii), and (iii) are elementary row operations.

**Theorem 4.1** Applying any sequence of elementary row ops to \([A|b]\) results in a system with the same solution set.

**Pf:** (i) merely lists the equations \(A_i x = b_i\) in a different order.

For (ii), note that \(A_i x = b_i \iff c(A_i x) = c b_i\) if \(c, 0 \in \mathbb{R}\) \(\iff (cA_i) x = c b_i\).

For (iii), let \([A'|b']\) be the system obtained from \([A|b]\) by replacing the row \([A_i|b_i]\) with \([A_i + cA_j|b_i + cb_j]\).

Let \(S' = \text{sols of } A' x = b'\). Then

\[
S \subseteq S'
\]

because \((A_i + cA_j)x = A_i x + cA_j x = b_i + cb_j\) when \(x \in S\).

Aside: need \(S = S'\), not just \(S \subseteq S'\).

\[
S \subseteq S' \quad \text{and} \quad S' \subseteq S
\]

But \(A_i = A_i' - cA_j'\) (i) so also \(S' \subseteq S\).

Finally, since each of (i), (ii), (iii) preserves \(S\), any sequence of them does, as well. \(\square\)

**E.g.** \(Ax = b\) for \(A = \begin{bmatrix} 3 & -2 & 2 & 9 \\ 2 & 2 & -2 & -4 \end{bmatrix}\) and \(b = \begin{bmatrix} 4 \\ 6 \end{bmatrix}\):
\[
\begin{bmatrix}
A | b
\end{bmatrix} =
\begin{bmatrix}
3 & -2 & 2 & 9 & 4 \\
2 & 2 & -2 & -4 & 6
\end{bmatrix}
\]

\[
A_1 \Rightarrow A_1 - A_2
\]
\[
\begin{bmatrix}
1 & -4 & 4 & 13 & -2 \\
2 & 2 & -2 & -4 & 6
\end{bmatrix}
\]

\[
A_2 \Rightarrow A_2 - 2A_1
\]
\[
\begin{bmatrix}
1 & -4 & 4 & 13 & -2 \\
0 & 10 & -10 -30 & 10
\end{bmatrix}
\]

\[
A_3 \Rightarrow A_3 - 3A_1
\]
\[
\begin{bmatrix}
1 & -4 & 4 & 13 & -2 \\
0 & 1 & -1 & -3 & 1
\end{bmatrix}
\]

\[
A_4 \Rightarrow A_4 - 4A_1
\]
\[
\begin{bmatrix}
1 & 0 & 0 & 1 & 2 \\
0 & 1 & -1 & -3 & 1
\end{bmatrix} = \begin{bmatrix}
A' | b'
\end{bmatrix}
\]

\(\Rightarrow A'x = b'\) has same sols \(S\) as \(A'x = b'\).

\(\Rightarrow S = \{ x \in \mathbb{R}^4 \mid x_1 + x_4 = 2, \quad x_2 - x_3 - 3x_4 = 1 \}\)

In particular,\[x_1 = 2, \quad x_2 = 1, \quad x_3 = x_4 = 0\]so \(x \in S \iff x = 2 \begin{bmatrix}
1 \\
0 \\
0 \\
0
\end{bmatrix} \in S\).

A particular solution \(x_0\) is\[x_0 = 2 \begin{bmatrix}
1 \\
0 \\
0 \\
0
\end{bmatrix}\]

Free variables \(x_3, x_4\) can take on any values.

General solution:

\[\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix} = \begin{bmatrix}
2 \\
1 \\
1 \\
1
\end{bmatrix} + \begin{bmatrix}
x_3 \\
x_3 \\
x_4 \\
x_4
\end{bmatrix} + \begin{bmatrix}
-3x_4 \\
0 \\
0 \\
0
\end{bmatrix}
\]

Def: A matrix is in echelon form if

1. the leading (leftmost nonzero) entries progress to the right from each row to the next;

2. all 0 rows are at the bottom.

The echelon form is reduced if, in addition,

3. every leading entry is 1;

4. in each column containing a leading entry, all other entries are 0.
E.g. just did one!

\[
\begin{bmatrix}
1 & -4 & 4 & 13 & -2 \\
0 & 10 & -10 & -30 & 10 \\
0 & 1 & -1 & -3 & 2
\end{bmatrix}
\] echelon form

\[
\begin{bmatrix}
1 & 0 & 0 & 1 & 2
\end{bmatrix}
\] reduced echelon form

Thm 4.3: Each matrix has unique reduced echelon form.

Pf: Exercise 16 which you aren't asked to do, but you could. □

How to find it?

**Algorithm (Gaussian elimination).**

**Echelon form**

Init: \( i = 1 \)

While: there is a nonzero entry in some row \( \geq i \)

Do: 1. pick row \( \geq i \) with a leftmost such entry
   2. swap that row with row \( i \)
   3. add multiples of row \( i \) to rows \( > i \) to cancel entry in pivot column
   4. \( i \rightarrow i+1 \)

Output: the resulting matrix

**Reduced echelon form** given any echelon form

Init: \( i = 1 \)

While: there is a nonzero entry in some row \( \geq i \)

Do: 1. rescale row \( i \) so pivot is 1
   2. add multiples of row \( i \) to rows \( < i \) to cancel entries in pivot column
   3. \( i \rightarrow i+1 \)

Output: the resulting matrix