Def: The span of \( v_1, \ldots, v_k \in \mathbb{R}^n \) is
\[
\text{span}(v_1, \ldots, v_k) = \{c_1v_1 + \cdots + c_kv_k \mid c_1, \ldots, c_k \in \mathbb{R}\}
\]
(a linear combination of \( v_1, \ldots, v_k \))

"the set of all linear combinations of \( v_1, \ldots, v_k \)")

The dimension of this span is the minimum number of these vectors needed to span. E.g. \( \dim(\text{line}) = ? \), \( \dim(\text{plane}) = ? \), \( \dim(\text{point}) = ? \)

E.g. Is \( x = (1, 3, -1, -2) \) a linear combination of \( u = (1, 1, 0, -1) \) and \( v = (2, 0, 1, 1) \)? \( \iff \) u and v?

Can we find \( s, t \) so that \( s u + t v = x \)?

\[
su + tv = x \\
(5, 5, 0, -5) + (2t, 0, t, -5t) = (5, 5, 0, -5) = (1, 3, -1, -2)
\]

\[
\Rightarrow \begin{align*}
    s + 2t &= 1 \\
    5s + 3t &= 3 \\
    t &= -1 \\
    s + t &= -2
\end{align*}
\]

but \( 3u - v = (1, 3, -1, -4) \neq x \), so: No. 

This linear system is inconsistent.

Note about \( \mathbb{R} \):

Nothing so far used \( \mathbb{R} \)! Could have used:

- complex numbers \( \mathbb{C} = \{a + bi \mid a, b \in \mathbb{R}\} \), where \( i^2 = -1 \)
  - the set of linear combinations of 1 and \( i \)
  - the plane spanned by 1 and \( i \)

- rational numbers \( \mathbb{Q} = \{\frac{a}{b} \mid a, b \text{ integers and } b \neq 0\} \) where \( \frac{a}{b} = \frac{a'}{b'} \) if \( ab' = a'b \)

- binary field \( \mathbb{F}_2 = \{0, 1\} \) where \( 0 + 0 = 0 \), \( 0 \cdot 0 = 0 \), \( 0 + 1 = 1 \), \( 0 \cdot 1 = 0 \), \( 1 + 1 = 0 \), \( 1 \cdot 1 = 1 \).

but not integers \( \mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\} \).

\( \checkmark \) why? Because you might have to divide by 3 to solve a linear system.

Q. What is special about \( \mathbb{R} \)?
A. For \( x \in \mathbb{R}, x^2 \geq 0 \). 

- that's the special thing: \( \mathbb{R} \) is ordered. 

Q?
**Def:** Length of \( x \in \mathbb{R}^n \) is \( \| x \| = \sqrt{x_1^2 + \ldots + x_n^2} \).

(or magnitude)

Why? \( x = x_1 e_1 + \ldots + x_n e_n \) where \( e_i = (\ldots, 0, 1, 0, \ldots) \) with a 1 in the \( i \)th position.

\[
\| x \| = \sqrt{x_1^2 + \ldots + x_n^2} \]

by induction on \( n \)

\[ \Rightarrow \| x \| = \sqrt{\sum_{i=1}^{n} x_i^2} \]

by Pythagoras

**Another way to express it:**

**Def:** For \( x, y \in \mathbb{R}^n \) (or \( \mathbb{C}^n, \mathbb{Q}^n, \mathbb{R}_2^n, \ldots \))

their dot product is \( x \cdot y = x_1 y_1 + \ldots + x_n y_n \).

Thus \( \| x \|^2 = x \cdot x \) in \( \mathbb{R}^n \).

Note: \( x \cdot x < 0 \) is possible in \( \mathbb{C}^n \): \( (1, 2i) \cdot (1, 2i) = 1 - 4 = -3 \).

**Proposition:**

1. Commutative: \( x \cdot y = y \cdot x \) for all \( x, y \in \mathbb{R}^n \) and \( c \in \mathbb{R} \)

2. Associative: \( (c x) \cdot y = c (x \cdot y) \)

3. Distributive: \( x \cdot (y + z) = x \cdot y + x \cdot z \)

4. Positive: \( x \cdot x = \| x \|^2 \geq 0 \) for all \( x \in \mathbb{R}^n \)

...Definite: 5. \( x \cdot x = 0 \Leftrightarrow x = 0 \).

**Pf:**

1. \( x_i y_i = y_i x_i \) — still a sentence!

2. \( (c x_i) y_i = c (x_i y_i) \)

3. \( x_i (y_i + z_i) = x_i y_i + x_i z_i \)

4. \( \| x \| \) is a sum of squares...

5. ...that is nonzero if \( x \neq 0 \). \( \Box \)
\[ \|x + y\|^2 = (x + y) \cdot (x + y) \]
\[ = (x + y) \cdot x + (x + y) \cdot y \]
\[ = \|x\|^2 + y \cdot x + x \cdot y + \|y\|^2 \]
\[ = \|x\|^2 + 2x \cdot y + \|y\|^2 \Rightarrow \|x\|^2 + \|y\|^2 = \|x + y\|^2 - 2x \cdot y \]

Over \( \mathbb{R} \): "recall" Law of cosines:
\[ a^2 + b^2 = c^2 + 2ab \cos \theta. \]

Since \( C = \pi - \theta \)
Thus \(-2x \cdot y\) "should be" \(-2\|x\|\|y\|\cos \theta\).

**Def:** The angle \( \theta \) between \( x \) and \( y \) is defined by
\[ \cos \theta = \frac{x \cdot y}{\|x\| \|y\|} = \frac{x}{\|x\|} \cdot \frac{y}{\|y\|}. \]

\( x \) and \( y \) are orthogonal \((x \perp y)\) if \( x \cdot y = 0 \).

**e.g. in \( \mathbb{R}^2 \):**

\[ \text{similar } \Delta s \quad \text{why? Both have an angle } \pi/2 - \Delta \]
\[ \Rightarrow x_1 = \frac{x_2}{y_2}, \quad y_1 = 0. \quad \text{since } y_1 < 0 \]
\[ \Rightarrow x_1y_1 + x_2y_2 = 0. \]

For higher dim check by induction, or use invariance of \( x \cdot y \) under rotation later in the course: put \( x, y \) in the plane. Actually, put \( x, y \) on axes!

**Def of \( \theta \) needs**

**Prop (Cauchy-Schwarz inequality):**
\[ |x \cdot y| \leq \|x\| \|y\|. \quad "\theta" \ holds \Rightarrow \ one \ is \ a \ scalar \ times \ the \ other. \]

**Pf:** Easy if \( x = 0 \) or \( y = 0 \) so assume not.

First do case of unit vectors. Need \(-1 \leq x \cdot y \leq 1\).

\[ \|x + y\|^2 = \|x\|^2 + 2x \cdot y + \|y\|^2 = 2(x \cdot y + 1) \geq 0 \Rightarrow x \cdot y \geq -1. \]
\[ \|x - y\|^2 = \|x\|^2 \quad \|y\|^2 \Rightarrow x \cdot y \leq 1. \]

**General:**
\[ \left| \frac{x}{\|x\|} \cdot \frac{y}{\|y\|} \right| \leq 1 \Rightarrow |x \cdot y| \leq \|x\| \|y\|. \]