2.

Def: The <u>span</u> of $v_1, \dots, v_k \in \mathbb{R}^n$ is

"the set of all linear combinations of $v_1, \dots, v_k$"

$$\operatorname{span}(v_1, \dots, v_k) = \{\underbrace{c_1 v_1 + \dots + c_k v_k}_{\text{a linear combination of } v_1, \dots, v_k} \mid c_1, \dots, c_k \in \mathbb{R}\}$$

adding k vectors is like adding two because $+$ is associative!
$(\dots((v_1+v_2)+v_3)+\dots)+v_k$
why?

The <u>dimension</u> of this span is the minimum number of these vectors needed to span. e.g. dim(line) = ? dim(plane) = ? dim(point) = ?

<u>E.g.</u> Is $x = (1,3,-1,-2)$ a linear combination of $u = (1,1,0,-1)$ and $v = (2,0,1,1)$? $\iff$ Does x lie in the plane spanned by u and v?

Can we find $\overset{s}{\cancel{c_1}}, \overset{t}{\cancel{c_2}}$ so that

$$\overset{s}{\cancel{c_1}} u + \overset{t}{\cancel{c_2}} v = x \ ?$$

$$\| $$

$$(s,s,0,-s) + (2t,0,t,t) = (s+2t, s, t, -s+t) \overset{?}{=} (1,3,-1,-2)$$

$$\Rightarrow \begin{array}{l} s + 2t = 1 \\ \boxed{\begin{array}{l} s \quad\quad = 3 \\ \quad\quad t = -1 \end{array}} \\ -s + t = -2 \end{array}$$

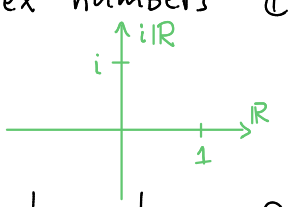
but $3u - v = (1,3,-1,-4) \neq x$, so: No.
This linear system is <u>inconsistent</u>.

systematic (algorithmic) way to solve: next week

Note about $\mathbb{R}$:

Nothing so far used $\mathbb{R}$! Could have used

- complex numbers $\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R}\}$, where $i^2 = -1$, so $i = \sqrt{-1}$
  = the set of linear combinations of 1 and i
  = the plane spanned by 1 and i



- rational numbers $\mathbb{Q} = \{\frac{a}{b} \mid a, b \text{ integers and } b \neq 0\}$ where $\frac{a}{b} = \frac{a'}{b'}$ if $ab' = a'b$
- binary field $\mathbb{F}_2 = \{0, 1\}$ where $0+0=0$, $0+1=1$, $1+1=0$; $0 \cdot 0 = 0$, $0 \cdot 1 = 0$, $1 \cdot 1 = 1$

but <u>not</u> integers $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$.

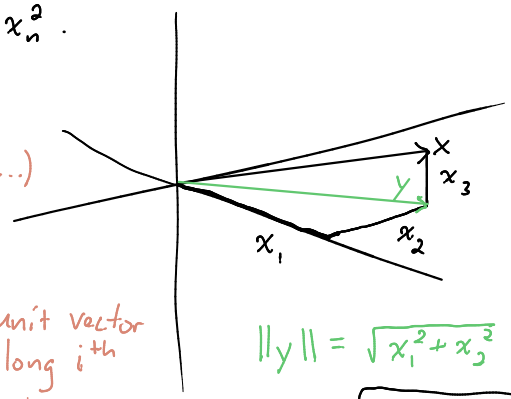
why? Because you might have to divide by 3 to solve a linear system

Q. What <u>is</u> special about $\mathbb{R}$?

A. For $x \in \mathbb{R}$, $x^2 \geq 0$. that's the special thing: $\mathbb{R}$ is ordered.

$\mathbb{C}$?
$\mathbb{F}_2$?
$\mathbb{Q}$?

Def: length of $x \in \mathbb{R}^n$ is $\|x\| = \sqrt{x_1^2 + \cdots + x_n^2}$.
(or magnitude)



Why? $x = x_1 e_1 + \cdots + x_n e_n$ $\quad e_i = (\ldots, 0, 1, 0, \ldots)$

$= \underbrace{x_1 e_1 + \cdots + x_{n-1} e_{n-1}}_{x_{n-1}} + x_n e_n$

i^{th} slot

$\Rightarrow e_i$ = unit vector along i^{th} axis

$\|y\| = \sqrt{x_1^2 + x_2^2}$

$\|x\| = \sqrt{\underbrace{\|y\|^2}_{x_1^2 + x_2^2} + x_3^2}$

$\Rightarrow \|x\|^2 = \underbrace{\|x_{n-1}\|^2}_{x_1^2 + \cdots + x_{n-1}^2 \text{ by induction on } n} + x_n^2$ by Pythagoras

$x = 0 \Leftrightarrow \|x\| = 0$

$x \neq 0 \Rightarrow \dfrac{x}{\|x\|}$ is a unit vector (length = 1) in the "same direction" as x.

Another way to express it:

Def: For $x, y \in \mathbb{R}^n$ (or $\mathbb{C}^n, \mathbb{Q}^n, \mathbb{F}_2^n, \ldots$)
their dot product is $x \cdot y = x_1 y_1 + \cdots + x_n y_n$.

Thus $\|x\|^2 = x \cdot x$ in $\mathbb{R}^n$.

Note: $x \cdot x < 0$ is possible in $\mathbb{C}^n$: $(1, 2i) \cdot (1, 2i) = 1 - 4 = -3$.

Proposition:

commutative	1. $x \cdot y = y \cdot x$	for all $x, y \in \mathbb{R}^n$		works over any field
associative	2. $(cx) \cdot y = c(x \cdot y)$	"	and $c \in \mathbb{R}$	
distributive	3. $x \cdot (y + z) = x \cdot y + x \cdot z$	"	and $z \in \mathbb{R}^n$	
positive...	4. $x \cdot x = \|x\|^2 \geq 0$	for all $x \in \mathbb{R}^n$		needs $\mathbb{R}$
...definite	5. $x \cdot x = 0 \Leftrightarrow x = 0$.			

Pf: 1. $x_i y_i = y_i x_i$. $\longleftarrow$ still a sentence!

2. $(c x_i) y_i = c(x_i y_i)$.

3. $x_i (y_i + z_i) = x_i y_i + x_i z_i$.

4. $\|x\|$ is a sum of squares...

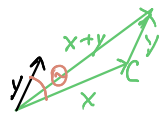
5. ... that is nonzero if $x \neq 0$. $\square$

E.g. $\|x+y\|^2 = (x+y)\cdot(x+y)$

$= (x+y)\cdot x + (x+y)\cdot y$

$= \|x\|^2 + y\cdot x + x\cdot y + \|y\|^2$

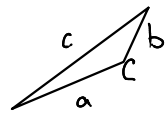
$= \|x\|^2 + 2x\cdot y + \|y\|^2 \Rightarrow \|x\|^2 + \|y\|^2 = \|x+y\|^2 - 2x\cdot y$



Over $\mathbb{R}$: "recall" Law of cosines: $a^2 + b^2 = c^2 + 2ab\cos C$.

$2ab\cos C = -2\|x\|\|y\|\cos\theta$

since $C = \pi - \theta$



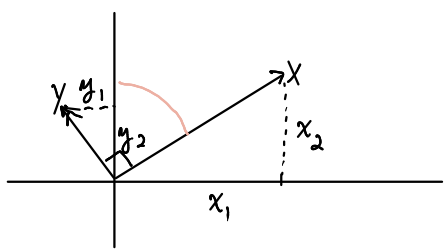
Thus $-2x\cdot y$ "should be" $-2\|x\|\|y\|\cos\theta$.

Def: The angle θ between x and y is defined by

$$\cos\theta = \frac{x\cdot y}{\|x\|\,\|y\|} = \frac{x}{\|x\|}\cdot\frac{y}{\|y\|}.$$

x and y are orthogonal ($x \perp y$) if $x\cdot y = 0$.

E.g. in $\mathbb{R}^2$:



similar Δs why? Both have an angle $\pi/2 - \triangledown$

$\Rightarrow \frac{x_1}{y_2} = \frac{x_2}{-y_1}$

since $y_1 < 0$

$\Rightarrow x_1 y_1 + x_2 y_2 = 0$.

For higher dim check by induction, or use invariance of $x\cdot y$ under rotation later in the course: put x, y in the plane. Actually, put x, y on axes!

Def of θ needs

Prop (Cauchy-Schwarz inequality):

$|x\cdot y| \le \|x\|\,\|y\|$. "=" holds $\Leftrightarrow$ one is a scalar times the other.

Pf: Easy if $x = 0$ or $y = 0$ so assume not.

First do case of unit vectors. Need $-1 \le x\cdot y \le 1$.

$\|\overset{\ge 0}{x+y}\|^2 = \overset{1}{\|x\|^2} + 2x\cdot y + \overset{1}{\|y\|^2} = 2(x\cdot y + 1) \ge 0 \Rightarrow x\cdot y \ge -1$.

$\|x-y\|^2 = \quad - \quad = \quad - \quad \Rightarrow x\cdot y \le 1$.

General: $\left|\frac{x}{\|x\|}\cdot\frac{y}{\|y\|}\right| \le 1 \Rightarrow |x\cdot y| \le \|x\|\,\|y\|$. $\square$