A strong asymptotic local-global principle for integral Kleinian sphere packings

Edna Jones

Rutgers University

Number Theory Seminar
Louisiana State University
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Soddy sphere packings: The construction

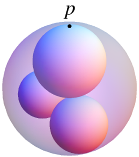

Figure: Four mutually tangent spheres.

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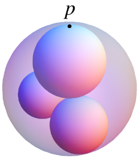

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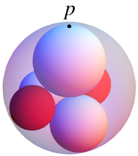

Figure: Four tangent spheres with two additional tangent spheres.

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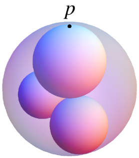

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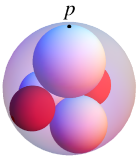

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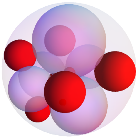

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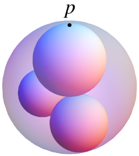

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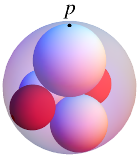

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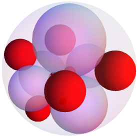

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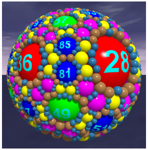

Figure: A Soddy sphere packing.

Soddy sphere packings and the integers

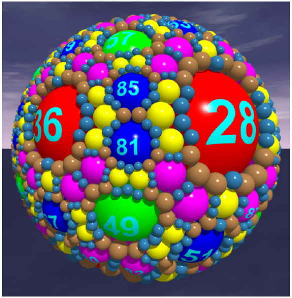

Figure: An integral Soddy sphere packing. Image by Nicolas Hannachi.

Label on sphere:

bend $= 1/\text{radius}$

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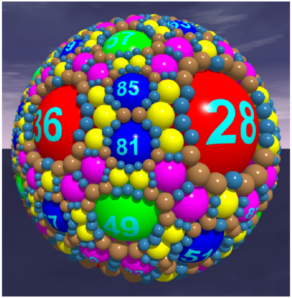

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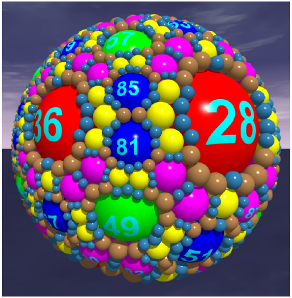

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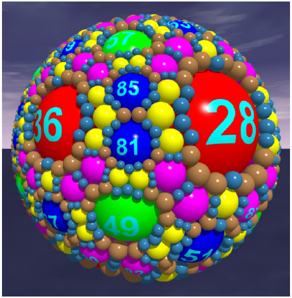

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Which integers appear as bends?

Are there any congruence or local obstructions?

Admissible integers

Definition (Admissible integers for Soddy sphere packings)

Let $\mathcal{P}$ be an integral Soddy sphere packing.
An integer m is **admissible (or locally represented)** if for every $q \geq 1$

$$m \equiv \text{bend of some sphere in } \mathcal{P} \pmod{q}.$$

Equivalently, m is admissible if m has no local obstructions.

Admissible integers

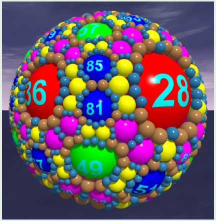
Theorem (Kontorovich, 2019)

m is admissible in a primitive integral Soddy sphere packing $\mathcal{P}$ if and only if

$$m \equiv 0 \text{ or } \varepsilon(\mathcal{P}) \pmod 3,$$

where $\varepsilon(\mathcal{P}) \in \{\pm 1\}$ depends only on the packing.

Example



m is admissible $\iff$
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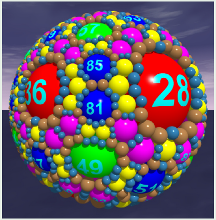
A strong asymptotic local-global theorem

Theorem (Kontorovich, 2019)

The bends of a fixed primitive integral Soddy sphere packing $\mathcal{P}$ satisfy a strong asymptotic local-global principle.
That is, there is an $N_0 = N_0(\mathcal{P})$ so that, if $m > N_0$ and m is admissible, then m is the bend of a sphere in the packing.

Example



If $m \equiv 0$ or $1 \pmod 3$ and m is sufficiently large, then m is the bend of a sphere in the packing.

Proof outline for Soddy sphere packing result

1. Show that the automorphism/symmetry group of the Soddy sphere packing contains a congruence subgroup of $\mathrm{PSL}_2(\mathbb{Z}[e^{\pi i/3}])$, and this congruence subgroup maps a particular sphere to itself. This implies that the set of bends contains "primitive" values of a shifted quaternary quadratic form.

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2. The shifted quaternary quadratic form gives you enough to work with so that you can quote the result of the circle method to say that every sufficiently large admissible number is represented by the quadratic form without the primitivity restriction.

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2. The shifted quaternary quadratic form gives you enough to work with so that you can quote the result of the circle method to say that every sufficiently large admissible number is represented by the quadratic form without the primitivity restriction.
3. Show that the singular series (with the primitivity restriction) is bounded away from zero when m is admissible.

Definition (Principal congruence subgroup of $\mathrm{PSL}_2(\mathcal{O}_K)$)

For an imaginary quadratic field K, a **principal congruence subgroup** of $\mathrm{PSL}_2(\mathcal{O}_K)$ is a subgroup of $\mathrm{PSL}_2(\mathcal{O}_K)$ of the form

$$\Lambda(\varrho) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2(\mathcal{O}_K) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{\varrho} \right\}$$

for a fixed element ϱ of $\mathcal{O}_K$.

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Example (Soddy sphere packing, Kontorovich, 2019)

There exists a sphere $S_0 \in \mathcal{P}$ such that the stabilizer of S_0 in Γ contains (up to conjugacy) the congruence subgroup

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2(\mathcal{O}) : b, c \equiv 0 \pmod{\varrho} \right\},$$

where $\mathcal{O} = \mathbb{Z}[e^{\pi i/3}]$ and $\varrho = 1 + e^{\pi i/3}$.

Examples of integral Kleinian sphere packings

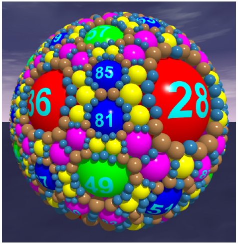

Figure: An integral Soddy sphere packing. Image by Nicolas Hannachi.

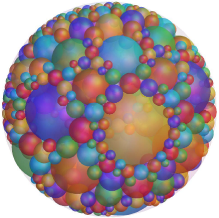

Figure: An integral Kleinian (more specifically, an orthoplicial) sphere packing. Image by Kei Nakamura.

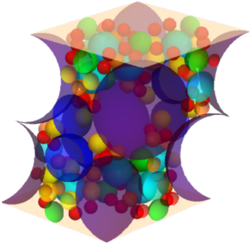

Figure: A fundamental domain of an integral Kleinian sphere packing. Image by Arseniy (Senia) Sheydvasser.

Kleinian sphere packings

Definition (Kleinian sphere packing)

An $(n-1)$-sphere packing $\mathcal{P}$ is **Kleinian** if its limit set is that of a geometrically finite group $\Gamma < \mathrm{Isom}(\mathcal{H}^{n+1})$.

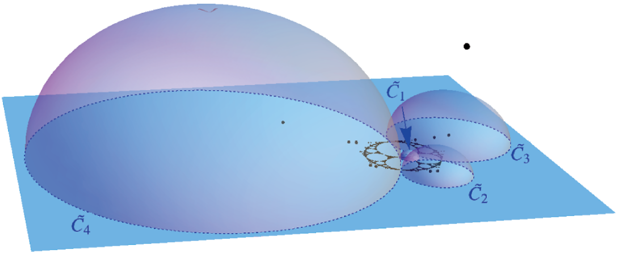

Figure: Apollonian circle packing as the limit set of Γ. Image by Alex Kontorovich.

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- Γ stabilizes $\mathcal{P}$ (i.e., Γ maps $\mathcal{P}$ to itself).
- Γ is a thin group.

Strong asymptotic local-global principles

Goal: Prove strong asymptotic local-global principles for certain integral Kleinian sphere packings, that is, prove:
If m is admissible and sufficiently large, then m is the bend of an $(n-1)$-sphere in the packing.

Definition (Admissible integers)

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Theorem in progress

Theorem (J., 2021+, in progress)

Let $\mathcal{P}$ be a primitive integral Kleinian $(n-1)$-sphere packing in $\widehat{\mathbb{R}^n}$ with an orientation-preserving automorphism group Γ of Möbius transformations.

Then every sufficiently large admissible integer is the bend of an $(n-1)$-sphere in $\Gamma \cdot S_1 \subseteq \mathcal{P}$.

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1. *Suppose that there exists an $(n-1)$-sphere S_0 such that the stabilizer of S_0 in Γ contains (up to conjugacy) a congruence subgroup of $\mathrm{PSL}_2(\mathcal{O}_K)$, where K is an imaginary quadratic field and $\mathcal{O}_K$ is the ring of integers of K. This condition implies that $n \geq 3$.*

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2. *Suppose that there is an $(n-1)$-sphere $S_1 \in \mathcal{P}$ that is tangent to S_0.*

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2. *Suppose that there is an $(n-1)$-sphere $S_1 \in \mathcal{P}$ that is tangent to S_0.*
3. *Suppose that $\mathcal{O}_K$ is a principal ideal domain.*

Then every sufficiently large admissible integer is the bend of an $(n-1)$-sphere in $\Gamma \cdot S_1 \subseteq \mathcal{P}$.

Why should we have a strong asymptotic local-global principle?

Theorem (Kim, 2015)

Let $\mathcal{P}$ be a Kleinian $(n-1)$-sphere packing with $n \geq 2$.
The number of spheres in $\mathcal{P}$ with bend at most N (counted with multiplicity) is asymptotically equal to a constant times N^{δ}, where $\delta =$ the Hausdorff dimension of the closure of $\mathcal{P}$.

For us,

$$\delta > n - 1 \geq 2.$$

Thus, we would would expect that the multiplicity of a given admissible bend up to N is roughly $N^{\delta-1} \gg N$, so we should expect that every sufficiently large admissible number to be represented.

Proof outline of my theorem in progress

1. The assumptions that the stabilizer of S_0 in Γ contains (up to conjugacy) a congruence subgroup of $\mathrm{PSL}_2(\mathcal{O}_K)$ and that $S_1 \in \mathcal{P}$ is tangent to S_0 imply that the set of bends of $\mathcal{P}$ contains "primitive" values of a quadratic polynomial in 4 variables.

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2. This quadratic polynomial in 4 variables should give you enough to work with so that you can apply the circle method to show that every sufficiently large admissible integer is represented as a bend.

Isometries of $\mathcal{H}^{n+1}$

- $\text{Isom}^0(\mathcal{H}^{n+1})$: group of orientation-preserving isometries of $\mathcal{H}^{n+1}$
- $\text{Möb}^0(\widehat{\mathbb{R}^n})$: group of orientation-preserving Möbius transformations acting on $\widehat{\mathbb{R}^n}$
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$$\begin{aligned} \mathrm{Isom}^0(\mathcal{H}^{n+1}) &: \mathcal{H}^{n+1} \to \mathcal{H}^{n+1} \\ \mathrm{Möb}^0(\widehat{\mathbb{R}^n}) &: \widehat{\mathbb{R}^n} \to \widehat{\mathbb{R}^n} \\ & z \mapsto g(z) = (az+b)(cz+d)^{-1}, \\ & g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}(2, C_{n-1}) \end{aligned}$$

a, b, c, d in a Clifford algebra C_{n-1} with some restrictions.

Clifford algebras

Definition (Clifford algebra)

The **Clifford algebra** C_m is the real associative algebra generated by m elements $i_1, i_2, \ldots, i_m$ subject to the relations:

- $i_\ell^2 = -1$ $(1 \leq \ell \leq m)$
- $i_h i_\ell = -i_\ell i_h$ $(1 \leq h, \ell \leq m,\ h \neq \ell)$

Examples (C_m for some m)

- $C_0 = \mathbb{R}$
- $C_1 \cong \mathbb{C}$, $z_0 + z_1 i_1 \leftrightarrow z_0 + z_1 i$
- $C_2 \cong \mathbb{H}$, $z_0 + z_1 i_1 + z_2 i_2 + z_{12} i_1 i_2 \leftrightarrow z_0 + z_1 i + z_2 j + z_{12} k$
- $C_3 \cong \mathbb{H} \oplus \mathbb{H}$

Clifford vectors

$$V_{n-1} := \{v_0 + v_1 i_1 + \cdots + v_{n-1} i_{n-1}\} \cong \mathbb{R}^n$$

$$v_0 + v_1 i_1 + \cdots + v_{n-1} i_{n-1} \leftrightarrow (v_0, v_1, \ldots, v_{n-1})$$

$$\widehat{V_{n-1}} := V_{n-1} \cup \{\infty\} \cong \mathbb{R}^n \cup \{\infty\} = \widehat{\mathbb{R}^n}$$

Group of Möbius transformations

Example ($n = 2$)

$\text{Möb}(\widehat{\mathbb{R}^2}) \cong \text{PSL}_2(\mathbb{C})$ acts on $\widehat{\mathbb{R}^2} \cong \widehat{\mathbb{C}}$ via

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Restrictions make explicitly stating what $\text{Möb}(\widehat{\mathbb{R}^n})$ is isomorphic to trickier for $n > 2$.
(For example, $\text{Möb}(\widehat{\mathbb{R}^3}) \ncong \text{PSL}_2(\mathbb{H})$, even though $C_2 \cong \mathbb{H}$.)

Inversive coordinates

Definition (Inversive coordinates)

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- The **co-bend** $\hat{\beta}(S)$ of S is the bend of the reflection of S in the unit $(n-1)$-sphere.
- If S is an oriented $(n-1)$-sphere, then the **bend-center** $\xi(S) \in \mathbb{R}^n$ of S is $\beta(S) \times (\text{center of } S)$.
  If S is a hyperplane, then its bend-center is the unique unit normal vector to S pointing in the direction of the interior of S.

Inversive-coordinate matrix

Definition (Inversive-coordinate matrix)

Given an oriented generalized $(n-1)$-sphere S, the **inversive-coordinate matrix** of S is the 2×2 matrix

$$M_S := \begin{pmatrix} \hat{\beta}(S) & \xi(S) \\ \overline{\xi(S)} & \beta(S) \end{pmatrix}.$$

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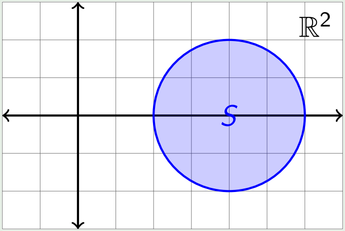

- $\beta(S) = \frac{1}{2}$

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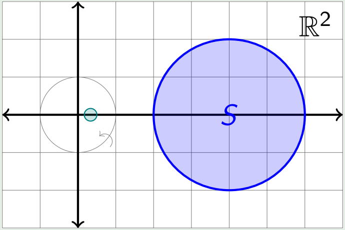

- $\beta(S) = \frac{1}{2}$
- $\hat{\beta}(S) = 6$

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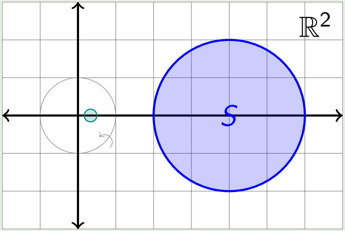

- $\beta(S) = \frac{1}{2}$
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  $\sim 2 + 0i = 2$

Inversive-coordinate matrix

Definition (Inversive-coordinate matrix)

Given an oriented generalized $(n-1)$-sphere S, the **inversive-coordinate matrix** of S is the 2×2 matrix

$$M_S := \begin{pmatrix} \hat{\beta}(S) & \xi(S) \\ \overline{\xi(S)} & \beta(S) \end{pmatrix}.$$

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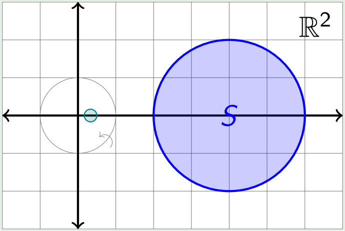

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  $\sim 2 + 0i = 2$
- $M_S = \begin{pmatrix} 6 & 2 \\ 2 & \frac{1}{2} \end{pmatrix}$

Lemma (J., 2021+, proved)

The group $\mathrm{SL}(2, C_{n-1})$ *acts on the set of inversive-coordinate matrices by*

$$g.M := gM\overline{g}^{\top}$$

for an inversive-coordinate matrix M *and* $g \in \mathrm{SL}(2, C_{n-1})$. *The group action of* $\mathrm{SL}(2, C_{n-1})$ *on the set of inversive-coordinate matrices is equivalent to the group action of* $\mathrm{SL}(2, C_{n-1})$ *on the set of oriented generalized* $(n-1)$*-spheres. That is, if* S *is an oriented generalized* $(n-1)$*-sphere and* $g \in \mathrm{SL}(2, C_{n-1})$, *then*

$$M_{gS} = g.M_S.$$

Extends work of Stange ($n = 2$), Sheydvasser ($n = 3$), and Litman & Sheydvasser ($n = 4$).

Corollary (J., 2021+, proved)

Let

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, C_{n-1}),$$

and let S_0 be an oriented generalized $(n-1)$-sphere with the inversive coordinates $(\beta, \hat{\beta}, \xi)$.
Then gS_0 has the following inversive coordinates:

- *bend* $\beta(gS_0) = \hat{\beta}|c|^2 + d\overline{\xi}\overline{c} + c\xi\overline{d} + \beta|d|^2$
- *co-bend* $\hat{\beta}(gS_0) = \hat{\beta}|a|^2 + b\overline{\xi}\overline{a} + a\xi\overline{b} + \beta|b|^2$
- *bend-center* $\xi(gS_0) = a\hat{\beta}\overline{c} + b\overline{\xi}\overline{c} + a\xi\overline{d} + b\beta\overline{d}$

Quadratic form for Kleinian sphere packings

The assumptions that the stabilizer of S_0 in Γ contains (up to conjugacy) a congruence subgroup of $\mathrm{PSL}_2(\mathcal{O}_K)$ and that $S_1 \in \mathcal{P}$ is tangent to S_0 imply that the set of bends of $\mathcal{P}$ contains "primitive" values of a quadratic polynomial in 4 variables.

Assume S_0 is the hyperplane with inversive coordinates $(0, 0, -i_{n-1})$ and S_1 is a hyperplane with inversive coordinates $(0, \hat{\beta}_1, i_{n-1})$, $\hat{\beta}_1 > 0$.

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$$\mathcal{O}_K = \mathbb{Z}[\omega], \qquad \varrho \in \mathcal{O}_K, \qquad \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma,$$

$$g = \begin{pmatrix} 1 + \varrho(a_0 + a_1\omega) & \varrho(b_0 + b_1\omega) \\ \varrho(c_0 + c_1\omega) & 1 + \varrho(d_0 + d_1\omega) \end{pmatrix} \in \Lambda[\varrho] < \mathrm{PSL}_2(\mathcal{O}_K) \cap \Gamma$$

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$$\beta(\gamma g S_1) = \hat{\beta}_1 |C(1 + \varrho(a_0 + a_1\omega)) + D\varrho(c_0 + c_1\omega)|^2 - D i_{n-1} \bar{C} + C i_{n-1} \bar{D}$$

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Want to know which integers are represented by $f_\gamma(a_0,a_1,c_0,c_1)$ as $\gamma, a_0, a_1, c_0, c_1$ vary subject to coprimality condition $(*)$

Representation number

$$R_N(m) = \sum_{\substack{\gamma\in\Gamma \\ \|\gamma\|\le T}} \sum_{\substack{a_0,a_1,c_0,c_1\in\mathbb{Z} \\ (1+\varrho(a_0+a_1\omega))\mathcal{O}_K+\varrho(c_0+c_1\omega)\mathcal{O}_K=\mathcal{O}_K}} \mathbf{1}_{\{m=f_\gamma(a_0,a_1,c_0,c_1)\}} \Upsilon_X(a_0,a_1,c_0,c_1),$$

where $N = T^2X^2$, T is very small compared to X, Υ_X is a nonnegative bump function so that a_0, a_1, c_0, c_1 are of size X, and

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Want to know when $R_N(m) > 0$

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- Addresses admissibility conditions (make sure singular series isn't too small when m is admissible)
- Uses spectral theory and expander graphs

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- Can be removed using the Möbius function on ideals
- Removal currently uses the fact that $\mathcal{O}_K$ is a PID since f_γ is not invariant over elements of an ideal.

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- Circle method with a Kloosterman refinement obtains the bulk of the main term and the error term.

Future directions

- Remove the condition that $\mathcal{O}_K$ is a PID.

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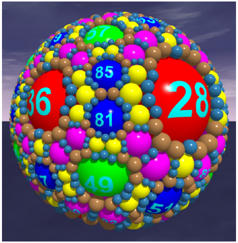
  - Have quadratic polynomial with 8 variables instead of 4.
  - The coprimality condition becomes a determinant condition.
- How large is sufficiently large?

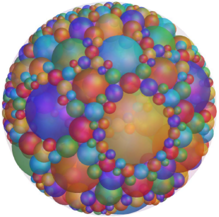
Illustrations credits

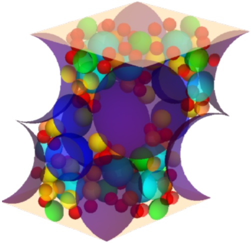
Besides the illustrations previously credited and a few circle illustrations created by the presenter, the illustrations for this talk came from the following paper:

Alex Kontorovich, "The Local-Global Principle for Integral Soddy Sphere Packings," *Journal of Modern Dynamics*, volume 15, pp. 209-236, 2019, `https://www.aimsciences.org/article/doi/10.3934/jmd.2019019`.

Thank you for listening!





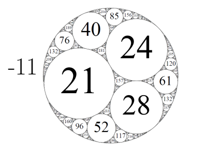


Strong asymptotic local-global conjecture

Conjecture (Graham–Lagarias–Mallows–Wilks–Yan, 2003)

The bends of a fixed primitive integral Apollonian circle packing $\mathcal{P}$ satisfy a strong asymptotic local-global principle.
That is, there is an $N_0 = N_0(\mathcal{P})$ so that, if $m > N_0$ and m is admissible, then m is the bend of a circle in the packing.

Example



We think that if
$m \equiv 0, 4, 12, 13, 16,$ or $21 \pmod{24}$
and m is sufficiently large,
then m is the bend of a circle in the packing.
We do not have a proof of this!

Why do we have a strong asymptotic local-global conjecture?

Theorem (Kontorovich–Oh, 2011)

The number of circles in an Apollonian circle packing $\mathcal{P}$ with bend at most N (counted with multiplicity) is asymptotically equal to a constant times N^{δ}, where $\delta =$ the Hausdorff dimension of the closure of $\mathcal{P}$.

For Apollonian circle packings, we have

$$\delta \approx 1.30568\ldots$$

Thus, we would would expect that the multiplicity of a given admissible bend up to N is roughly $N^{\delta-1} \approx N^{0.30568} \geq 1$, so we should expect that every sufficiently large admissible number to be represented.

The best we can do right now

Theorem (Bourgain–Kontorovich, 2014)

Almost every admissible number is the bend of a circle in a fixed primitive integral Apollonian circle packing $\mathcal{P}$. Quantitatively, the number of exceptions up to N is bounded by $O(N^{1-\eta})$, where $\eta > 0$ is effectively computable.

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Proof outline:

1. Show that the automorphism group of the Apollonian circle packing contains the congruence subgroup $\Gamma(2)$ of $\mathrm{PSL}_2(\mathbb{Z})$, and $\Gamma(2)$ is the stabilizer of a particular circle. This implies that the set of bends contains primitive values of a shifted binary quadratic form. (Sarnak, 2007)

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2. The shifted binary quadratic form gives you enough to work with so that you can apply the circle method (with some other tools, including spectral theory, used in the major arc and minor arc analyses) to obtain an "almost all" statement.

Result for Kleinian 1-sphere (circle) packings

Theorem (Fuchs–Stange–Zhang, 2019)

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Generalizes Bourgain–Kontorovich result for Apollonian circle packings to other Kleinian 1-sphere packings.

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Theorem (Fuchs–Stange–Zhang, 2019)

Suppose that

- $\mathcal{P}$ *is a primitive integral Kleinian* 1*-sphere packing in* $\widehat{\mathbb{R}^2}$ *with an automorphism group* $\Gamma < \mathrm{PSL}_2(\mathfrak{a})$*, where* $\mathfrak{a}$ *is a fractional ideal of an imaginary quadratic field* K*,*

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Result for Kleinian 1-sphere (circle) packings

Theorem (Fuchs–Stange–Zhang, 2019)

If $\mathcal{P}$ is a primitive integral Kleinian 1-sphere packing in $\widehat{\mathbb{R}^2}$ satisfying certain conditions, almost every admissible number is the bend of a circle in $\mathcal{P}$.

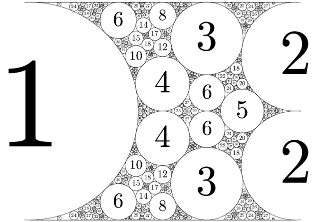

Figure: An integral Kleinian (more specifically, cuboctahedral) 1-sphere packing that satisfies the conditions of the theorem. Figure taken from "Local-Global Principles in Circle Packings" by Fuchs, Stange, and Zhang.

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