A strong asymptotic local-global principle for integral Kleinian sphere packings

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Soddy sphere packings: The construction

**Figure:** Four mutually tangent spheres.

**Figure:** More tangent spheres.

**Figure:** A Soddy sphere packing.
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Figure: Four tangent spheres with two additional tangent spheres.
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Soddy sphere packings and the integers

Figure: An integral Soddy sphere packing. Image by Nicolas Hannachi.

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Which integers appear as bends?

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A strong asymptotic local-global principle for sphere packings
Label on sphere: 
\[ \text{bend} = 1/\text{radius} \]

All of the bends of this Soddy sphere packing are integers.

Which integers appear as bends?

Are there any congruence or local obstructions?

Figure: An integral Soddy sphere packing. Image by Nicolas Hannachi.
Definition (Admissible integers for Soddy sphere packings)

Let $\mathcal{P}$ be an integral Soddy sphere packing. An integer $m$ is **admissible (or locally represented)** if for every $q \geq 1$

$$m \equiv \text{bend of some sphere in } \mathcal{P} \pmod{q}.$$

Equivalently, $m$ is admissible if $m$ has no local obstructions.
**Admissible integers**

Theorem (Kontorovich, 2019)

$m$ is admissible in a primitive integral Soddy sphere packing $\mathcal{P}$ if and only if

$$m \equiv 0 \text{ or } \varepsilon(\mathcal{P}) \pmod{3},$$

where $\varepsilon(\mathcal{P}) \in \{\pm 1\}$ depends only on the packing.

**Example**

$m$ is admissible $\iff$

$$m \equiv 0 \text{ or } 1 \pmod{3}.$$
The bends of a fixed primitive integral Soddy sphere packing $\mathcal{P}$ satisfy a strong asymptotic local-global principle. That is, there is an $N_0 = N_0(\mathcal{P})$ so that, if $m > N_0$ and $m$ is admissible, then $m$ is the bend of a sphere in the packing.

Example

If $m \equiv 0$ or $1 \pmod{3}$ and $m$ is sufficiently large, then $m$ is the bend of a sphere in the packing.
1. Show that the automorphism/symmetry group of the Soddy sphere packing contains a congruence subgroup of $\text{PSL}_2(\mathbb{Z}[e^{\pi i/3}])$, and this congruence subgroup maps a particular sphere to itself. This implies that the set of bends contains “primitive” values of a shifted quaternary quadratic form.
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2. The shifted quaternary quadratic form gives you enough to work with so that you can quote the result of the circle method to say that every sufficiently large admissible number is represented by the quadratic form without the primitivity restriction.
1. Show that the automorphism/symmetry group of the Soddy sphere packing contains a congruence subgroup of $\text{PSL}_2(\mathbb{Z}[e^{\pi i/3}])$, and this congruence subgroup maps a particular sphere to itself. This implies that the set of bends contains “primitive” values of a shifted quaternary quadratic form.

2. The shifted quaternary quadratic form gives you enough to work with so that you can quote the result of the circle method to say that every sufficiently large admissible number is represented by the quadratic form without the primitivity restriction.

3. Show that the singular series (with the primitivity restriction) is bounded away from zero when $m$ is admissible.
Congruence subgroup of $\text{PSL}_2(\mathcal{O}_K)$

**Definition (Principal congruence subgroup of $\text{PSL}_2(\mathcal{O}_K)$)**

For an imaginary quadratic field $K$, a **principal congruence subgroup** of $\text{PSL}_2(\mathcal{O}_K)$ is a subgroup of $\text{PSL}_2(\mathcal{O}_K)$ of the form

$$\Lambda(\varrho) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{PSL}_2(\mathcal{O}_K) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{\varrho} \right\}$$

for a fixed element $\varrho$ of $\mathcal{O}_K$.

Example (Soddy sphere packing, Kontorovich, 2019)

There exists a sphere $S_0 \in \mathcal{P}$ such that the stabilizer of $S_0$ in $\Gamma$ contains (up to conjugacy) the congruence subgroup

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{PSL}_2(\mathcal{O}_K) : b, c \equiv 0 \pmod{\varrho} \right\},$$

where $\mathcal{O} = \mathbb{Z}[\frac{\pi i}{3}]$ and $\varrho = 1 + e^{\frac{\pi i}{3}}$. 

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Definition (Principal congruence subgroup of PSL$_2(\mathcal{O}_K)$)

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where $\mathcal{O} = \mathbb{Z}[e^{\pi i/3}]$ and $\varrho = 1 + e^{\pi i/3}$. 
Examples of integral Kleinian sphere packings

**Figure:** An integral Soddy sphere packing. Image by Nicolas Hannachi.

**Figure:** An integral Kleinian (more specifically, an orthoplicial) sphere packing. Image by Kei Nakamura.

**Figure:** A fundamental domain of an integral Kleinian sphere packing. Image by Arseniy (Senia) Sheydvasser.
Definition (Kleinian sphere packing)

An \((n - 1)\)-sphere packing \(P\) is **Kleinian** if its limit set is that of a geometrically finite group \(\Gamma < \text{Isom}(\mathcal{H}^{n+1})\).

**Figure:** Apollonian circle packing as the limit set of \(\Gamma\). Image by Alex Kontorovich.
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- Action of \(\text{Isom}(\mathcal{H}^{n+1})\) extends continuously to \(\hat{\mathbb{R}}^n = \mathbb{R}^n \cup \{\infty\}\), the boundary of \(\mathcal{H}^{n+1}\).
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- \(\Gamma\) stabilizes \(P\) (i.e., \(\Gamma\) maps \(P\) to itself).
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- \(\Gamma\) stabilizes \(\mathcal{P}\) (i.e., \(\Gamma\) maps \(\mathcal{P}\) to itself).
- \(\Gamma\) is a thin group.

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A strong asymptotic local-global principle for sphere packings
**Goal:** Prove strong asymptotic local-global principles for certain integral Kleinian sphere packings, that is, prove:

*If \( m \) is admissible and sufficiently large, then \( m \) is the bend of an \((n - 1)\)-sphere in the packing.*

**Definition (Admissible integers)**

Let \( \mathcal{P} \) be an integral Kleinian sphere packing. An integer \( m \) is **admissible (or locally represented)** if for every \( q \geq 1 \)

\[
m \equiv \text{bend of some \((n - 1)\)-sphere in } \mathcal{P} \pmod{q}.
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Theorem (J., 2021+, in progress)

Let \( \mathcal{P} \) be a primitive integral Kleinian \((n - 1)\)-sphere packing in \( \hat{\mathbb{R}}^n \) with an orientation-preserving automorphism group \( \Gamma \) of Möbius transformations.

1. Suppose that there exists an \((n - 1)\)-sphere \( S_0 \) such that the stabilizer of \( S_0 \) in \( \Gamma \) contains (up to conjugacy) a congruence subgroup of \( \text{PSL}_2(\mathcal{O}_K) \), where \( K \) is an imaginary quadratic field and \( \mathcal{O}_K \) is the ring of integers of \( K \). This condition implies that \( n \geq 3 \).

2. Suppose that there is an \((n - 1)\)-sphere \( S_1 \in \mathcal{P} \) that is tangent to \( S_0 \).

3. Suppose that \( \mathcal{O}_K \) is a principal ideal domain.

Then every sufficiently large admissible integer is the bend of an \((n - 1)\)-sphere in \( \Gamma \cdot S_1 \subseteq \mathcal{P} \).
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A strong asymptotic local-global principle for sphere packings
Why should we have a strong asymptotic local-global principle?

**Theorem (Kim, 2015)**

Let $\mathcal{P}$ be a Kleinian $(n - 1)$-sphere packing with $n \geq 2$. The number of spheres in $\mathcal{P}$ with bend at most $N$ (counted with multiplicity) is asymptotically equal to a constant times $N^\delta$, where $\delta = \text{the Hausdorff dimension of the closure of } \mathcal{P}$.

For us,

$$\delta > n - 1 \geq 2.$$  

Thus, we would expect that the multiplicity of a given admissible bend up to $N$ is roughly $N^{\delta - 1} \gg N$, so we should expect that every sufficiently large admissible number to be represented.
The assumptions that the stabilizer of $S_0$ in $\Gamma$ contains (up to conjugacy) a congruence subgroup of $\text{PSL}_2(\mathcal{O}_K)$ and that $S_1 \in \mathcal{P}$ is tangent to $S_0$ imply that the set of bends of $\mathcal{P}$ contains “primitive” values of a quadratic polynomial in 4 variables.
The assumptions that the stabilizer of $S_0$ in $\Gamma$ contains (up to conjugacy) a congruence subgroup of $\text{PSL}_2(\mathcal{O}_K)$ and that $S_1 \in \mathcal{P}$ is tangent to $S_0$ imply that the set of bends of $\mathcal{P}$ contains “primitive” values of a quadratic polynomial in 4 variables.

This quadratic polynomial in 4 variables should give you enough to work with so that you can apply the circle method to show that every sufficiently large admissible integer is represented as a bend.
Isometries of $\mathcal{H}^{n+1}$

- $\text{Isom}^0(\mathcal{H}^{n+1})$: group of orientation-preserving isometries of $\mathcal{H}^{n+1}$
- $\text{Möb}^0(\hat{\mathbb{R}}^n)$: group of orientation-preserving Möbius transformations acting on $\hat{\mathbb{R}}^n$
- $\text{Möb}^0(\hat{\mathbb{R}}^n) \cong \text{Isom}^0(\mathcal{H}^{n+1})$

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$$\text{Isom}^0(\mathcal{H}^{n+1}) : \mathcal{H}^{n+1} \to \mathcal{H}^{n+1}$$

$$\text{M"ob}^0(\mathbb{R}^n) : \mathbb{R}^n \to \mathbb{R}^n$$

$$z \mapsto g(z) = (az + b)(cz + d)^{-1},$$

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{PSL}(2, C_{n-1})$$

$a, b, c, d$ in a Clifford algebra $C_{n-1}$ with some restrictions.
Clifford algebras

Definition (Clifford algebra)

The **Clifford algebra** $C_m$ is the real associative algebra generated by $m$ elements $i_1, i_2, \ldots, i_m$ subject to the relations:

- $i_\ell^2 = -1$ ($1 \leq \ell \leq m$)
- $i_h i_\ell = -i_\ell i_h$ ($1 \leq h, \ell \leq m$, $h \neq \ell$)

Examples ($C_m$ for some $m$)

- $C_0 = \mathbb{R}$
- $C_1 \cong \mathbb{C}$, $z_0 + z_1 i_1 \leftrightarrow z_0 + z_1 i$
- $C_2 \cong \mathbb{H}$, $z_0 + z_1 i_1 + z_2 i_2 + z_{12} i_1 i_2 \leftrightarrow z_0 + z_1 i + z_2 j + z_{12} k$
- $C_3 \cong \mathbb{H} \oplus \mathbb{H}$
\[ V_{n-1} := \{ v_0 + v_1 i_1 + \cdots + v_{n-1} i_{n-1} \} \cong \mathbb{R}^n \]
\[ v_0 + v_1 i_1 + \cdots + v_{n-1} i_{n-1} \leftrightarrow (v_0, v_1, \ldots, v_{n-1}) \]

\[ \widehat{V}_{n-1} := V_{n-1} \cup \{ \infty \} \cong \mathbb{R}^n \cup \{ \infty \} = \mathbb{R}^n \]
Example \((n = 2)\)

\(\text{Möb}(\hat{\mathbb{R}}^2) \cong \text{PSL}_2(\mathbb{C})\) acts on \(\hat{\mathbb{R}}^2 \cong \hat{\mathbb{C}}\) via

\[ z \mapsto g(z) = (az + b)(cz + d)^{-1}, \]

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**Example (n = 2)**

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z \mapsto g(z) = (az + b)(cz + d)^{-1},
\]

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g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{PSL}_2(\mathbb{C})
\]

Restrictions make explicitly stating what M"{o}b(\hat{\mathbb{R}}^n) is isomorphic to trickier for \( n > 2. \)

(For example, M"{o}b(\hat{\mathbb{R}}^3) \nless \not= \text{PSL}_2(\mathbb{H}), \text{ even though } \mathbb{C}_2 \cong \mathbb{H}.\)
Definition (Inversive coordinates)

Given an oriented generalized \((n - 1)\)-sphere \(S\), we define the **inversive coordinates** of \(S\) to be the ordered triple \((\beta(S), \hat{\beta}(S), \xi(S))\), where they are defined as follows:

1. **Bend** \(\beta(S)\) of \(S\) is \(1 / \text{radius of } S\), taken to be positive if \(S\) is positively oriented and negative otherwise.
2. **Co-bend** \(\hat{\beta}(S)\) of \(S\) is the bend of the reflection of \(S\) in the unit \((n - 1)\)-sphere.
3. **Bend-center** \(\xi(S)\) of \(S\) is \(\beta(S) \times \text{center of } S\).
4. If \(S\) is a hyperplane, then its bend-center is the unique unit normal vector to \(S\) pointing in the direction of the interior of \(S\).
Inversive coordinates

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- If $S$ is an oriented $(n - 1)$-sphere, then the **bend** $\beta(S)$ of $S$ is $1/(\text{radius of } S)$, taken to be positive if $S$ is positively oriented and negative otherwise.
- If $S$ is a hyperplane, then its bend is $\beta(S) = 0$. 

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  and negative otherwise.
  If \(S\) is a hyperplane, then its bend is \(\beta(S) = 0\).
- The **co-bend** \(\hat{\beta}(S)\) of \(S\) is the bend of the reflection of \(S\) in
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  If \(S\) is a hyperplane, then its bend is \(\beta(S) = 0\).

- The \textbf{co-bend} \(\hat{\beta}(S)\) of \(S\) is the bend of the reflection of \(S\) in the unit \((n - 1)\)-sphere.

- If \(S\) is an oriented \((n - 1)\)-sphere, then the \textbf{bend-center} \(\xi(S) \in \mathbb{R}^n\) of \(S\) is \(\beta(S) \times (\text{center of } S)\).
  
  If \(S\) is a hyperplane, then its bend-center is the unique unit normal vector to \(S\) pointing in the direction of the interior of \(S\).
Definition (Inversive-coordinate matrix)

Given an oriented generalized \((n - 1)\)-sphere \(S\), the **inversive-coordinate matrix** of \(S\) is the \(2 \times 2\) matrix

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M_S := \begin{pmatrix}
\hat{\beta}(S) & \xi(S) \\
\xi(S) & \beta(S)
\end{pmatrix}.
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**Example**

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\beta(S) = \frac{1}{2}
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Inversive-coordinate matrix

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**Example**

- \(\beta(S) = \frac{1}{2}\)
- \(\hat{\beta}(S) = 6\)
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**Example**

- \(\beta(S) = \frac{1}{2}\)
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- \(\xi(S) = \frac{1}{2}(4, 0) = (2, 0) \sim 2 + 0i = 2\)
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- \(\hat{\beta}(S) = 6\)
- \(\xi(S) = \frac{1}{2}(4, 0) = (2, 0)\) \(\sim 2 + 0i = 2\)
- \(M_S = \begin{pmatrix} 6 & 2 \\ 2 & \frac{1}{2} \end{pmatrix}\)
The group \( \text{SL}(2, C_{n-1}) \) acts on the set of inversive-coordinate matrices by

\[ g \cdot M := gM\overline{g}^T \]

for an inversive-coordinate matrix \( M \) and \( g \in \text{SL}(2, C_{n-1}) \). The group action of \( \text{SL}(2, C_{n-1}) \) on the set of inversive-coordinate matrices is equivalent to the group action of \( \text{SL}(2, C_{n-1}) \) on the set of oriented generalized \((n-1)\)-spheres. That is, if \( S \) is an oriented generalized \((n-1)\)-sphere and \( g \in \text{SL}(2, C_{n-1}) \), then

\[ M_{gS} = g \cdot M_S. \]

Extends work of Stange \((n = 2)\), Sheydvasser \((n = 3)\), and Litman & Sheydvasser \((n = 4)\).
Corollary (J., 2021+, proved)

Let

\[ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, C_{n-1}), \]

and let \( S_0 \) be an oriented generalized \((n - 1)\)-sphere with the inversive coordinates \((\beta, \hat{\beta}, \xi)\).

Then \( gS_0 \) has the following inversive coordinates:

- **bend** \( \beta(gS_0) = \hat{\beta}|c|^2 + d\overline{\xi}\overline{c} + c\xi\overline{d} + \beta|d|^2 \)
- **co-bend** \( \hat{\beta}(gS_0) = \hat{\beta}|a|^2 + b\overline{\xi}\overline{a} + a\xi\overline{b} + \beta|b|^2 \)
- **bend-center** \( \xi(gS_0) = a\overline{\beta}\overline{c} + b\overline{\xi}\overline{c} + a\xi\overline{d} + b\beta\overline{d} \)
The assumptions that the stabilizer of $S_0$ in $\Gamma$ contains (up to conjugacy) a congruence subgroup of $\text{PSL}_2(\mathcal{O}_K)$ and that $S_1 \in \mathcal{P}$ is tangent to $S_0$ imply that the set of bends of $\mathcal{P}$ contains “primitive” values of a quadratic polynomial in 4 variables.

Assume $S_0$ is the hyperplane with inversive coordinates $(0, 0, -i_{n-1})$ and $S_1$ is a hyperplane with inversive coordinates $(0, \hat{\beta}_1, i_{n-1})$, $\hat{\beta}_1 > 0$. 
The assumptions that the stabilizer of $S_0$ in $\Gamma$ contains (up to conjugacy) a congruence subgroup of $\text{PSL}_2(\mathcal{O}_K)$ and that $S_1 \in \mathcal{P}$ is tangent to $S_0$ imply that the set of bends of $\mathcal{P}$ contains “primitive” values of a quadratic polynomial in 4 variables.

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\begin{align*}
\mathcal{O}_K &= \mathbb{Z}[\omega], \\
\varrho &\in \mathcal{O}_K, \\
\gamma &= \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma, \\
g &= \begin{pmatrix} 1 + \varrho(a_0 + a_1\omega) & \varrho(b_0 + b_1\omega) \\ \varrho(c_0 + c_1\omega) & 1 + \varrho(d_0 + d_1\omega) \end{pmatrix} \in \Lambda[\varrho] < \text{PSL}_2(\mathcal{O}_K) \cap \Gamma
\end{align*}
The assumptions that the stabilizer of \( S_0 \) in \( \Gamma \) contains (up to conjugacy) a congruence subgroup of \( \text{PSL}_2(\mathcal{O}_K) \) and that \( S_1 \in \mathcal{P} \) is tangent to \( S_0 \) imply that the set of bends of \( \mathcal{P} \) contains “primitive” values of a quadratic polynomial in 4 variables.

Assume \( S_0 \) is the hyperplane with inversive coordinates \((0, 0, -i_{n-1})\) and \( S_1 \) is a hyperplane with inversive coordinates \((0, \hat{\beta}_1, i_{n-1})\), \( \hat{\beta}_1 > 0 \).

\[
\mathcal{O}_K = \mathbb{Z}[\omega], \quad \rho \in \mathcal{O}_K, \quad \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma,
\]

\[
g = \begin{pmatrix} 1 + \rho(a_0 + a_1\omega) & \rho(b_0 + b_1\omega) \\ \rho(c_0 + c_1\omega) & 1 + \rho(d_0 + d_1\omega) \end{pmatrix} \in \Lambda[\rho] < \text{PSL}_2(\mathcal{O}_K) \cap \Gamma
\]

\[
\beta(\gamma g S_1) = \hat{\beta}_1 |C(1 + \rho(a_0 + a_1\omega)) + D\rho(c_0 + c_1\omega)|^2 - Di_{n-1} \bar{C} + Ci_{n-1} \bar{D}
\]
Quadratic form for Kleinian sphere packings

\[ \beta(\gamma g S_1) = \hat{\beta}_1 |C(1 + \varrho(a_0 + a_1 \omega)) + D \varrho(c_0 + c_1 \omega)|^2 \\
- Di_{n-1} \bar{C} + Ci_{n-1} \bar{D} \]

\[ \sim \]

\[ f_\gamma(a_0, a_1, c_0, c_1) = \hat{\beta}_1 |C(1 + \varrho(a_0 + a_1 \omega)) + D \varrho(c_0 + c_1 \omega)|^2 \\
- Di_{n-1} \bar{C} + Ci_{n-1} \bar{D} \]

with \( a_0, a_1, c_0, c_1 \in \mathbb{Z} \) and

\[ (1 + \varrho(a_0 + a_1 \omega)) \mathcal{O}_K + \varrho(c_0 + c_1 \omega) \mathcal{O}_K = \mathcal{O}_K. \quad (*) \]
Quadratic form for Kleinian sphere packings

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$$\sim$$

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$$(1 + \varrho(a_0 + a_1 \omega))\mathcal{O}_{K} + \varrho(c_0 + c_1 \omega)\mathcal{O}_{K} = \mathcal{O}_{K}. \quad (*)$$

Want to know which integers are represented by $f_\gamma(a_0, a_1, c_0, c_1)$ as $\gamma, a_0, a_1, c_0, c_1$ vary subject to coprimality condition $(\ast)$
$R_N(m) = \sum_{\gamma \in \Gamma} \sum_{\|\gamma\| \leq T} 1_{\{m = f_\gamma(a_0, a_1, c_0, c_1)\}} \gamma_X(a_0, a_1, c_0, c_1),$ 

where $N = T^2X^2$, $T$ is very small compared to $X$, $\gamma_X$ is a nonnegative bump function so that $a_0, a_1, c_0, c_1$ are of size $X$, and 

$$1_{\{m = f_\gamma(a_0, a_1, c_0, c_1)\}} = \begin{cases} 
1 & \text{if } m = f_\gamma(a_0, a_1, c_0, c_1), \\
0 & \text{otherwise}.
\end{cases}$$
$R_N(m) = \sum_{\gamma \in \Gamma} \sum_{\|\gamma\| \leq T} 1_{\{m = f_\gamma(a_0, a_1, c_0, c_1)\}} \gamma X(a_0, a_1, c_0, c_1),$ 

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Want to know when $R_N(m) > 0$
$$R_N(m) = \sum_{\gamma \in \Gamma} \sum \mathbb{1}_{\{m=f_\gamma(a_0,a_1,c_0,c_1)\}} \chi(a_0, a_1, c_0, c_1)$$

- Addresses admissibility conditions (make sure singular series isn’t too small when \(m\) is admissible)
- Uses spectral theory and expander graphs
\[ R_N(m) = \sum_{\gamma \in \Gamma} \sum_{\|\gamma\| \leq T} \sum_{a_0, a_1, c_0, c_1 \in \mathbb{Z}} 1\{m=f_{\gamma}(a_0, a_1, c_0, c_1)\} \gamma \chi(a_0, a_1, c_0, c_1) \]

- Can be removed using the Möbius function on ideals
- Removal currently uses the fact that \( \mathcal{O}_K \) is a PID since \( f_{\gamma} \) is not invariant over elements of an ideal.
$$R_N(m) = \sum_{\gamma \in \Gamma} \sum_{\|\gamma\| \leq T} 1 \{ m = f_\gamma(a_0, a_1, c_0, c_1) \} \gamma \chi(a_0, a_1, c_0, c_1)$$

- Circle method with a Kloosterman refinement obtains the bulk of the main term and the error term.
Future directions

- Remove the condition that $\mathcal{O}_K$ is a PID.
Future directions

- Remove the condition that $O_K$ is a PID.
- Remove condition about $S_1 \in \mathcal{P}$ is tangent to $S_0$. 
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  - Have quadratic polynomial with 8 variables instead of 4.
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Future directions

- Remove the condition that $\mathcal{O}_K$ is a PID.
- Remove condition about $S_1 \in \mathcal{P}$ is tangent to $S_0$.
  - Have quadratic polynomial with 8 variables instead of 4.
  - The coprimality condition becomes a determinant condition.
- How large is sufficiently large?
Besides the illustrations previously credited and a few circle illustrations created by the presenter, the illustrations for this talk came from the following paper:

Thank you for listening!
Conjecture (Graham–Lagarias–Mallows–Wilks–Yan, 2003)

The bends of a fixed primitive integral Apollonian circle packing $\mathcal{P}$ satisfy a strong asymptotic local-global principle.

That is, there is an $N_0 = N_0(\mathcal{P})$ so that, if $m > N_0$ and $m$ is admissible, then $m$ is the bend of a circle in the packing.

Example

We think that if $m \equiv 0, 4, 12, 13, 16, \text{ or } 21 \pmod{24}$ and $m$ is sufficiently large, then $m$ is the bend of a circle in the packing.

We do not have a proof of this!
Why do we have a strong asymptotic local-global conjecture?

Theorem (Kontorovich–Oh, 2011)

The number of circles in an Apollonian circle packing \( \mathcal{P} \) with bend at most \( N \) (counted with multiplicity) is asymptotically equal to a constant times \( N^{\delta} \), where \( \delta = \) the Hausdorff dimension of the closure of \( \mathcal{P} \).

For Apollonian circle packings, we have

\[
\delta \approx 1.30568 \ldots
\]

Thus, we would expect that the multiplicity of a given admissible bend up to \( N \) is roughly \( N^{\delta - 1} \approx N^{0.30568} \geq 1 \), so we should expect that every sufficiently large admissible number to be represented.
Theorem (Bourgain–Kontorovich, 2014)

Almost every admissible number is the bend of a circle in a fixed primitive integral Apollonian circle packing $\mathcal{P}$. Quantitatively, the number of exceptions up to $N$ is bounded by $O(N^{1-\eta})$, where $\eta > 0$ is effectively computable.
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Proof outline:

1. Show that the automorphism group of the Apollonian circle packing contains the congruence subgroup $\Gamma(2)$ of $\text{PSL}_2(\mathbb{Z})$, and $\Gamma(2)$ is the stabilizer of a particular circle. This implies that the set of bends contains primitive values of a shifted binary quadratic form. (Sarnak, 2007)
The best we can do right now

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2. The shifted binary quadratic form gives you enough to work with so that you can apply the circle method (with some other tools, including spectral theory, used in the major arc and minor arc analyses) to obtain an “almost all” statement.
Result for Kleinian 1-sphere (circle) packings

Theorem (Fuchs–Stange–Zhang, 2019)

Suppose that $P$ is a primitive integral Kleinian $1$-sphere packing in $\hat{\mathbb{R}}^2$ with an automorphism group $\Gamma < \text{PSL}_2(\mathbb{Z})$, where $\mathbb{Z}$ is a fractional ideal of an imaginary quadratic field $K$, there exist circles $S_0$ and $S_1$ such that $S_1$ is in $P$ and is tangent to $S_0$, and the stabilizer of $S_0$ in $\Gamma$ (up to conjugacy) contains a congruence subgroup of $\text{PSL}_2(\mathbb{Z})$.

Almost every admissible number is the bend of a circle in $P$. Quantitatively, the number of exceptions up to $N$ is bounded by $O(N^{1-\eta})$.

Generalizes Bourgain–Kontorovich result for Apollonian circle packings to other Kleinian 1-sphere packings.
Theorem (Fuchs–Stange–Zhang, 2019)

Suppose that

- $\mathcal{P}$ is a primitive integral Kleinian 1-sphere packing in $\hat{\mathbb{R}}^2$ with an automorphism group $\Gamma < \text{PSL}_2(\alpha)$, where $\alpha$ is a fractional ideal of an imaginary quadratic field $K$,

Almost every admissible number is the bend of a circle in $\mathcal{P}$. Quantitatively, the number of exceptions up to $N$ is bounded by $O(N^{1-\eta})$.

Generalizes Bourgain–Kontorovich result for Apollonian circle packings to other Kleinian 1-sphere packings.
Theorem (Fuchs–Stange–Zhang, 2019)

Suppose that

- $\mathcal{P}$ is a primitive integral Kleinian 1-sphere packing in $\hat{\mathbb{R}}^2$ with an automorphism group $\Gamma < \text{PSL}_2(\mathfrak{a})$, where $\mathfrak{a}$ is a fractional ideal of an imaginary quadratic field $K$,
- there exist circles $S_0$ and $S_1$ such that $S_1$ is in $\mathcal{P}$ and is tangent to $S_0$, and

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Theorem (Fuchs–Stange–Zhang, 2019)

Suppose that

- \( \mathcal{P} \) is a primitive integral Kleinian 1-sphere packing in \( \hat{\mathbb{R}}^2 \) with an automorphism group \( \Gamma < \text{PSL}_2(\mathfrak{a}) \), where \( \mathfrak{a} \) is a fractional ideal of an imaginary quadratic field \( K \),
- there exist circles \( S_0 \) and \( S_1 \) such that \( S_1 \) is in \( \mathcal{P} \) and is tangent to \( S_0 \), and
- the stabilizer of \( S_0 \) in \( \Gamma \) (up to conjugacy) contains a congruence subgroup of \( \text{PSL}_2(\mathbb{Z}) \).

Almost every admissible number is the bend of a circle in \( \Gamma \cdot S_1 \subseteq \mathcal{P} \). Quantitatively, the number of exceptions up to \( N \) is bounded by \( O(N^{1-\eta}) \).

Generalizes Bourgain–Kontorovich result for Apollonian circle packings to other Kleinian 1-sphere packings.
Theorem (Fuchs–Stange–Zhang, 2019)

If $\mathcal{P}$ is a primitive integral Kleinian 1-sphere packing in $\mathbb{R}^2$ satisfying certain conditions, almost every admissible number is the bend of a circle in $\mathcal{P}$.

Figure: An integral Kleinian (more specifically, cuboctahedral) 1-sphere packing that satisfies the conditions of the theorem. Figure taken from “Local-Global Principles in Circle Packings” by Fuchs, Stange, and Zhang.
The assumption that the stabilizer of $S_0$ in $\Gamma$ contains (up to conjugacy) a congruence subgroup of $\text{PSL}_2(\mathbb{Z})$ implies that the set of bends of $\mathcal{P}$ contains primitive values of a shifted binary quadratic form.
Proof outline

1. The assumption that the stabilizer of $S_0$ in $\Gamma$ contains (up to conjugacy) a congruence subgroup of $\text{PSL}_2(\mathbb{Z})$ implies that the set of bends of $\mathcal{P}$ contains primitive values of a shifted binary quadratic form.

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