Local Densities of Diagonal Integral Ternary Quadratic Forms at Odd Primes

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Quadratic Forms $Q(\mathbf{v}) = ax^2 + by^2 + cz^2$

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$a, b, c \in \mathbb{Z}$
$\gcd(a, b, c) = 1$

$$\mathbf{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Examples

- $Q(\mathbf{v}) = x^2 + 3y^2 + 5z^2$
- $Q(\mathbf{v}) = x^2 + 4y^2 + 4z^2$
- $Q(\mathbf{v}) = 3x^2 + 4y^2 + 5z^2$
- $Q(\mathbf{v}) = x^2 + 5y^2 + 7z^2$

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Let m be an integer. We would like to know when

$$Q(\mathbf{v}) = m$$

has an integer solution.

Easier Problem: Look (mod n)

Definition (Local representation number)

$$r_n(m, Q) = \#\left\{\mathbf{v} \in (\mathbb{Z}/n\mathbb{Z})^3 : Q(\mathbf{v}) \equiv m \pmod{n}\right\}.$$

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Because of Chinese Remainder Theorem, only need to look at $r_{p^k}(m, Q)$, p prime.

Local (representation) density or p-adic density

Let p be a prime. Let $\mathbb{Z}_p$ denote the set of p-adic integers with the usual Haar measure.

Definition (Local (representation) density or p-adic density)

$$\alpha_p(m, Q) = \lim_{U \to \{m\}} \frac{\mathrm{Vol}_{\mathbb{Z}_p^3}(Q^{-1}(U))}{\mathrm{Vol}_{\mathbb{Z}_p}(U)},$$

where U is an open set in $\mathbb{Z}_p$ containing m, $\mathrm{Vol}_{\mathbb{Z}_p^3}(Q^{-1}(U))$ is the volume of $Q^{-1}(U)$ in $\mathbb{Z}_p^3$, and $\mathrm{Vol}_{\mathbb{Z}_p}(U)$ is the volume of U in $\mathbb{Z}_p$.

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It can be shown that

$$\alpha_p(m, Q) = \lim_{k \to \infty} \frac{r_{p^k}(m, Q)}{p^{2k}}.$$

Why do we care about local densities?

Definition (Representation number)

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Definition (Representation number)

$$r(m, Q) = \# \left\{ \mathbf{v} \in \mathbb{Z}^3 : Q(\mathbf{v}) = m \right\}$$

The $\alpha_p(m, Q)$'s give us local information.

If $m \neq 0$, Hensel's lemma shows that

$$\alpha_p(m, Q) = 0 \iff r_{p^k}(m, Q) = 0 \text{ for some } k.$$

This implies that $r(m, Q) = 0$ if $\alpha_p(m, Q) = 0$ for some prime p. (Converse does not hold.)

Theorem (Siegel, 1935)

Let m be an integer and Q be a positive definite quadratic form of rank 3. Let $\{Q_j\}$ be a complete set representatives for classes in the same genus as Q. Then

$$\frac{\sum_j \frac{r(m,Q_j)}{\#O(Q_j)}}{\sum_j \frac{1}{\#O(Q_j)}} = \alpha_{\mathbb{R}}(m,Q) \prod_{p \text{ prime}} \alpha_p(m,Q),$$

where $O(Q_j)$ is the orthogonal group of Q_j over $\mathbb{Z}$,

$\alpha_{\mathbb{R}}(m,Q) = \lim_{U \to \{m\}} \frac{\mathrm{Vol}_{\mathbb{R}^3}(Q^{-1}(U))}{\mathrm{Vol}_{\mathbb{R}}(U)}$*, U is an open set in $\mathbb{R}$ containing m, $\mathrm{Vol}_{\mathbb{R}^3}(Q^{-1}(U))$ is the volume of $Q^{-1}(U)$ in $\mathbb{R}^3$, and $\mathrm{Vol}_{\mathbb{R}}(U)$ is the volume of U in $\mathbb{R}$.*

Corollary (Specialized Version of Siegel's Mass Formula)

Let m be an integer and Q be a positive definite quadratic form of rank 3. If Q is in a genus containing only one class, then

$$r(m, Q) = \alpha_{\mathbb{R}}(m, Q) \prod_{p \text{ prime}} \alpha_p(m, Q).$$

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- Lomadze (1971) computed the representation numbers for these 82 quadratic forms.

Past Results on Local Densities

Complicated formulas (hard to tell when $\alpha_p(m, Q)$ is equal to zero):

- Yang (1998)
- Hanke (2004)

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Not in full generality:

- Siegel (1935): If $p \nmid 2abcm$, then

$$\alpha_p(m, Q) = 1 + \frac{1}{p}\left(\frac{-abcm}{p}\right),$$

  where $\left(\frac{\cdot}{p}\right)$ is the Legendre symbol.
- Berkovich and Jagy (2012)

Theorem (Berkovich and Jagy, 2012)

Let p be an odd prime and u be any integer with $\left(\frac{-u}{p}\right) = -1$. Let $Q(\mathbf{v}) = ux^2 + py^2 + upz^2$. Suppose m is a nonzero integer and $m = m_0 p^{m_1}$, where $\gcd(m_0, p) = 1$. Then

$$\alpha_p(m, Q) = \begin{cases} p^{-m_1/2}\left(1 - \left(\frac{-m_0}{p}\right)\right), & \text{if } m_1 \text{ is even,} \\ p^{(-m_1+1)/2}\left(1 + \frac{1}{p}\right), & \text{if } m_1 \text{ is odd.} \end{cases}$$

Formulas for Local Densities at Odd Primes

Theorem (J., 2020)

Let p be an odd prime. Suppose $p \nmid a$, $b = b_0 p^{b_1}$, and $c = c_0 p^{c_1}$, where $b_1 \leq c_1$, $\gcd(b_0, p) = 1$, and $\gcd(c_0, p) = 1$.
Suppose m is a nonzero integer and $m = m_0 p^{m_1}$, where $\gcd(m_0, p) = 1$.
$\alpha_p(m, Q)$ is easily computable using rational functions and Legendre symbols. Depends on a, b_0, b_1, c_0, c_1, m_0, m_1, and p.

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$\alpha_p(m, Q)$ is easily computable using rational functions and Legendre symbols. Depends on a, b_0, b_1, c_0, c_1, m_0, m_1, and p.

Multiple cases:

- $m_1 < b_1$ *and depends on parity of* m_1
- $b_1 \leq m_1 < c_1$ *and depends on parity of* b_1
- $m_1 \geq c_1$ *and depends on parities of* b_1, c_1, *and* m_1

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Suppose m is a nonzero integer and $m = m_0 p^{m_1}$, where $\gcd(m_0, p) = 1$.
$\alpha_p(m, Q)$ is easily computable using rational functions and Legendre symbols. Depends on a, b_0, b_1, c_0, c_1, m_0, m_1, and p.

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- $b_1 \leq m_1 < c_1$ *and depends on parity of* b_1
- $m_1 \geq c_1$ *and depends on parities of* b_1, c_1, *and* m_1

Also $\alpha_p(0, Q)$ is computable. Multiple cases dependent on parities b_1 and c_1.

Main Theorem when $m_1 < c_1$

Theorem (J., 2020)

If $m_1 < b_1$, then

$$\alpha_p(m,Q) = \begin{cases} p^{m_1/2}\left(1 + \left(\frac{am_0}{p}\right)\right), & \text{if } m_1 \text{ is even,} \\ 0, & \text{if } m_1 \text{ is odd.} \end{cases}$$

If $b_1 \leq m_1 < c_1$, then $\alpha_p(m,Q) =$

$$\begin{cases} p^{b_1/2}\left(1 - \frac{1}{p}\left(\frac{-ab_0}{p}\right)^{m_1+1} + \left(1 - \frac{1}{p}\right)\left(\frac{m_1 - b_1}{2} \right.\right. \\ \qquad \left.\left. + \frac{(-1)^{m_1} - 1}{4} + \left(\frac{-ab_0}{p}\right)\left(\frac{m_1 - b_1}{2} + \frac{1 - (-1)^{m_1}}{4}\right)\right)\right), \\ \qquad\qquad \text{if } b_1 \text{ is even,} \\ p^{(b_1-1)/2}\left(1 + \left(\frac{a}{p}\right)^{m_1+1}\left(\frac{b_0}{p}\right)^{m_1}\left(\frac{m_0}{p}\right)\right), \text{if } b_1 \text{ is odd.} \end{cases}$$

Proof Sketch

1. Use exponential sums and quadratic Gauss sums to compute $r_{p^k}(m, Q)$.
2. Divide by p^{2k} and take a limit.

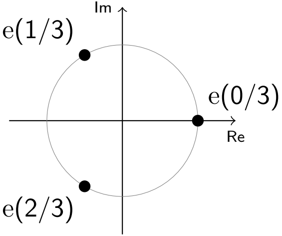
Abbreviate $\mathrm{e}(w) = e^{2\pi i w}$.

Definition

The *quadratic Gauss sum* $g(n;q)$ *over* $\mathbb{Z}/q\mathbb{Z}$ is defined by

$$g(n;q) = \sum_{j=0}^{q-1} \mathrm{e}\left(\frac{nj^2}{q}\right).$$

$$\sum_{t=0}^{q-1} \mathrm{e}\left(\frac{nt}{q}\right) = \begin{cases} q, & \text{if } n \equiv 0 \pmod{q}, \\ 0, & \text{otherwise.} \end{cases}$$



A Sum Containing $\mathrm{e}(w)$

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$$\sum_{t=0}^{p^k-1} \mathrm{e}\left(\frac{(Q(\mathbf{v})-m)t}{p^k}\right) = \begin{cases} p^k, & \text{if } Q(\mathbf{v}) \equiv m \pmod{p^k}, \\ 0, & \text{otherwise.} \end{cases}$$

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$$\frac{1}{p^k} \sum_{t=0}^{p^k-1} \mathrm{e}\left(\frac{(Q(\mathbf{v}) - m)t}{p^k}\right) = \begin{cases} 1, & \text{if } Q(\mathbf{v}) \equiv m \pmod{p^k}, \\ 0, & \text{otherwise.} \end{cases}$$

Counting Solutions (mod p^k)

$$\frac{1}{p^k}\sum_{t=0}^{p^k-1} \mathrm{e}\left(\frac{(Q(\mathbf{v})-m)t}{p^k}\right) = \begin{cases} 1, & \text{if } Q(\mathbf{v}) \equiv m \pmod{p^k}, \\ 0, & \text{otherwise.} \end{cases}$$

$$r_{p^k}(m,Q) = \#\left\{\mathbf{v} \in (\mathbb{Z}/p^k\mathbb{Z})^3 : Q(\mathbf{v}) \equiv m \pmod{p^k}\right\}.$$

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Counting Solutions (mod p^k)

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&r_{p^k}(m, Q) \\
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&= \sum_{x=0}^{p^k-1} \sum_{y=0}^{p^k-1} \sum_{z=0}^{p^k-1} \frac{1}{p^k} \sum_{t=0}^{p^k-1} \mathrm{e}\left(\frac{(ax^2+by^2+cz^2-m)t}{p^k}\right) \\
&= \frac{1}{p^k} \sum_{t=0}^{p^k-1} \mathrm{e}\left(\frac{-mt}{p^k}\right) \sum_{x=0}^{p^k-1} \mathrm{e}\left(\frac{atx^2}{p^k}\right) \sum_{y=0}^{p^k-1} \mathrm{e}\left(\frac{bty^2}{p^k}\right) \sum_{z=0}^{p^k-1} \mathrm{e}\left(\frac{ctz^2}{p^k}\right)
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&= \frac{1}{p^k} \sum_{t=0}^{p^k-1} \mathrm{e}\left(\frac{-mt}{p^k}\right) \sum_{x=0}^{p^k-1} \mathrm{e}\left(\frac{atx^2}{p^k}\right) \sum_{y=0}^{p^k-1} \mathrm{e}\left(\frac{bty^2}{p^k}\right) \sum_{z=0}^{p^k-1} \mathrm{e}\left(\frac{ctz^2}{p^k}\right) \\
&= \frac{1}{p^k} \sum_{t=0}^{p^k-1} \mathrm{e}\left(\frac{-mt}{p^k}\right) g\left(at; p^k\right) g\left(bt; p^k\right) g\left(ct; p^k\right).
\end{aligned}
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Counting Solutions (mod p^k)

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&= \frac{1}{p^k} \left(g\left(0; p^k\right)\right)^3 + \frac{1}{p^k} \sum_{t=1}^{p^k-1} \mathrm{e}\left(\frac{-mt}{p^k}\right) g\left(at; p^k\right) g\left(bt; p^k\right) g\left(ct; p^k\right).
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\end{aligned}
$$

Since $g\left(0; p^k\right) = p^k$,

$$
r_{p^k}(m, Q) = p^{2k} + \frac{1}{p^k} \sum_{t=1}^{p^k-1} \mathrm{e}\left(\frac{-mt}{p^k}\right) g\left(at; p^k\right) g\left(bt; p^k\right) g\left(ct; p^k\right).
$$

Formulas for Quadratic Gauss Sums

Lemma

Suppose k is a positive integer, p is an odd prime, and $n \neq 0$. Let $n = n_0 p^\ell$ so that $\gcd(n_0, p) = 1$. Then

$$g\left(n; p^k\right) = \begin{cases} p^k, & \text{if } \ell \geq k, \\ p^{(k+\ell)/2} \left(\frac{n_0}{p^{k-\ell}}\right) \varepsilon_{p^{k-\ell}}, & \text{if } \ell < k, \end{cases}$$

where

$$\varepsilon_{p^{k-\ell}} = \begin{cases} 1, & \text{if } p^{k-\ell} \equiv 1 \pmod 4, \\ i, & \text{if } p^{k-\ell} \equiv 3 \pmod 4, \end{cases}$$

and $\left(\frac{\cdot}{p^{k-\ell}}\right)$ is the Jacobi symbol.

Formulas for Quadratic Gauss Sums

Lemma

Suppose p is an odd prime and $a \in \mathbb{Z}$. Then

$$g(a;p) = \sum_{t=0}^{p-1} \left(1 + \left(\frac{t}{p}\right)\right) \mathrm{e}\left(\frac{at}{p}\right).$$

If $a \not\equiv 0 \pmod{p}$, then

$$g(a;p) = \sum_{t=0}^{p-1} \left(\frac{t}{p}\right) \mathrm{e}\left(\frac{at}{p}\right).$$

Proof for the previous lemma.

Let t be an integer. The number of solutions modulo p of the congruence

$$j^2 \equiv t \pmod{p}$$

is $1 + \left(\frac{t}{p}\right)$. Therefore,

$$g(a;p) = \sum_{j=0}^{p-1} \mathrm{e}\left(\frac{aj^2}{p}\right) = \sum_{t=0}^{p-1} \left(1 + \left(\frac{t}{p}\right)\right) \mathrm{e}\left(\frac{at}{p}\right).$$

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When $a \not\equiv 0 \pmod{p}$,

$$g(a;p) = \sum_{t=0}^{p-1} \left(\frac{t}{p}\right) \mathrm{e}\left(\frac{at}{p}\right)$$

since $\sum_{t=0}^{p-1} \mathrm{e}\left(\frac{at}{p}\right) = 0$.

Counting Solutions (mod p^k)

$$
\begin{aligned}
&r_{p^k}(m, Q) \\
&= p^{2k} + \frac{1}{p^k} \sum_{t=1}^{p^k-1} \mathrm{e}\left(\frac{-mt}{p^k}\right) g\left(at; p^k\right) g\left(bt; p^k\right) g\left(ct; p^k\right) \\
&= p^{2k} + \frac{1}{p^k} \sum_{t=1}^{p^k-1} \mathrm{e}\left(\frac{-m_0 p^{m_1} t}{p^k}\right) g\left(at; p^k\right) g\left(b_0 p^{b_1} t; p^k\right) g\left(c_0 p^{c_1} t; p^k\right).
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\end{aligned}
$$

Let $t = t_0 p^\tau$, where $0 \leq \tau \leq k-1$ and $t_0 \in (\mathbb{Z}/p^{k-\tau}\mathbb{Z})^*$. Then

$$
\begin{aligned}
r_{p^k}(m, Q) = p^{2k} + \frac{1}{p^k} \sum_{\tau=0}^{k-1} \sum_{t_0 \in (\mathbb{Z}/p^{k-\tau}\mathbb{Z})^*} \mathrm{e}\left(\frac{-m_0 t_0 p^{m_1+\tau}}{p^k}\right) g\left(a t_0 p^\tau; p^k\right) \\
\cdot g\left(b_1 t_0 p^{b_1+\tau}; p^k\right) g\left(c_0 t_0 p^{c_1+\tau}; p^k\right).
\end{aligned}
$$

Counting Solutions (mod p^k)

Let

$$s_{k,\tau} = \sum_{t_0 \in (\mathbb{Z}/p^{k-\tau}\mathbb{Z})^*} \mathrm{e}\left(\frac{-m_0 t_0 p^{m_1+\tau}}{p^k}\right) g\left(a t_0 p^{\tau}; p^k\right) \cdot g\left(b_1 t_0 p^{b_1+\tau}; p^k\right) g\left(c_0 t_0 p^{c_1+\tau}; p^k\right)$$

so that

$$r_{p^k}(m, Q) = p^{2k} + \frac{1}{p^k} \sum_{\tau=0}^{k-1} s_{k,\tau}.$$

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so that

$$r_{p^k}(m, Q) = p^{2k} + \frac{1}{p^k} \sum_{\tau=0}^{k-1} s_{k,\tau}.$$

Compute $s_{k,\tau}$ under different conditions depending on b_1, c_1, m_1, k, and τ. Then compute $r_{p^k}(m, Q)$ and $\alpha_p(m, Q)$.

Computing $s_{k,\tau}$ when $0 \leq \tau \leq k - m_1 - 2$

Lemma

For $0 \leq \tau \leq k - m_1 - 2$, $s_{k,\tau} = 0$.

Lemma

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Proof.

Suppose that $0 \leq \tau \leq k - m_1 - 2$. Then let $t_0 = t_1 + t_2 p$, where $1 \leq t_1 \leq p - 1$ and $0 \leq t_2 \leq p^{k-\tau-1} - 1$, so

$$s_{k,\tau} = \sum_{t_1=1}^{p-1} \sum_{t_2=0}^{p^{k-\tau-1}-1} \mathrm{e}\left(\frac{-m_0(t_1 + t_2 p)p^{m_1+\tau}}{p^k}\right) g\left(a(t_1 + t_2 p)p^{\tau}; p^k\right) \cdot g\left(b_1(t_1 + t_2 p)p^{b_1+\tau}; p^k\right) g\left(c_0(t_1 + t_2 p)p^{c_1+\tau}; p^k\right)$$

$$= \sum_{t_1=1}^{p-1} \sum_{t_2=0}^{p^{k-\tau-1}-1} \mathrm{e}\left(\frac{-m_0 t_1}{p^{k-m_1-\tau}}\right) \mathrm{e}\left(\frac{-m_0 t_2}{p^{k-m_1-1-\tau}}\right) g\left(a t_1 p^{\tau}; p^k\right) \cdot g\left(b_1 t_1 p^{b_1+\tau}; p^k\right) g\left(c_0 t_1 p^{c_1+\tau}; p^k\right)$$

Computing $s_{k,\tau}$ when $0 \leq \tau \leq k - m_1 - 2$

Lemma

For $0 \leq \tau \leq k - m_1 - 2$, $s_{k,\tau} = 0$.

Proof (continued).

$$s_{k,\tau} = \sum_{t_1=1}^{p-1} \mathrm{e}\left(\frac{-m_0 t_1}{p^{k-m_1-\tau}}\right) g\left(a t_1 p^{\tau}; p^k\right) g\left(b_1 t_1 p^{b_1+\tau}; p^k\right)$$

$$\cdot\, g\left(c_0 t_1 p^{c_1+\tau}; p^k\right) \sum_{t_2=0}^{p^{k-\tau-1}-1} \mathrm{e}\left(\frac{-m_0 t_2}{p^{k-m_1-1-\tau}}\right).$$

Computing $s_{k,\tau}$ when $0 \leq \tau \leq k - m_1 - 2$

Lemma

For $0 \leq \tau \leq k - m_1 - 2$, $s_{k,\tau} = 0$.

Proof (continued).

$$s_{k,\tau} = \sum_{t_1=1}^{p-1} \mathrm{e}\left(\frac{-m_0 t_1}{p^{k-m_1-\tau}}\right) g\left(a t_1 p^{\tau}; p^k\right) g\left(b_1 t_1 p^{b_1+\tau}; p^k\right) \cdot g\left(c_0 t_1 p^{c_1+\tau}; p^k\right) \sum_{t_2=0}^{p^{k-\tau-1}-1} \mathrm{e}\left(\frac{-m_0 t_2}{p^{k-m_1-1-\tau}}\right).$$

Now

$$\sum_{t_2=0}^{p^{k-\tau-1}-1} \mathrm{e}\left(\frac{-m_0 t_2}{p^{k-m_1-1-\tau}}\right) = p^{m_1} \sum_{t_2=0}^{p^{k-m_1-\tau-1}-1} \mathrm{e}\left(\frac{-m_0 t_2}{p^{k-m_1-1-\tau}}\right) = p^{m_1} \cdot 0 = 0.$$

□

Computing $s_{k,\tau}$ when $k - \min(m_1, b_1) \le \tau \le k - 1$

Lemma

For $k - \min(m_1, b_1) \le \tau \le k - 1$*,*

$$s_{k,\tau} = \begin{cases} p^{3k+(k-\tau)/2}\left(1 - \dfrac{1}{p}\right), & \text{if } k - \tau \text{ is even,} \\ 0, & \text{if } k - \tau \text{ is odd.} \end{cases}$$

Computing $s_{k,\tau}$ when $k - \min(m_1, b_1) \leq \tau \leq k - 1$

Lemma

For $k - \min(m_1, b_1) \leq \tau \leq k - 1$,

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Proof.

Suppose that $k - \min(m_1, b_1) \leq \tau \leq k - 1$. Then

$$s_{k,\tau} = \sum_{t_0 \in (\mathbb{Z}/p^{k-\tau}\mathbb{Z})^*} p^{(k+\tau)/2} \left(\frac{at_0}{p^{k-\tau}}\right) \varepsilon_{p^{k-\tau}} p^{2k}$$

$$= \varepsilon_{p^{k-\tau}} p^{5k/2+\tau/2} \left(\frac{a}{p}\right)^{k-\tau} \sum_{t_0 \in (\mathbb{Z}/p^{k-\tau}\mathbb{Z})^*} \left(\frac{t_0}{p}\right)^{k-\tau}.$$

Computing $s_{k,\tau}$ when $k - \min(m_1, b_1) \leq \tau \leq k-1$

Lemma

For $k - \min(m_1, b_1) \leq \tau \leq k-1$,

$$s_{k,\tau} = \begin{cases} p^{3k+(k-\tau)/2}\left(1 - \frac{1}{p}\right), & \text{if } k-\tau \text{ is even,} \\ 0, & \text{if } k-\tau \text{ is odd.} \end{cases}$$

Proof (continued).

$$s_{k,\tau} = \varepsilon_{p^{k-\tau}} p^{5k/2+\tau/2} \left(\frac{a}{p}\right)^{k-\tau} \sum_{t_0 \in (\mathbb{Z}/p^{k-\tau}\mathbb{Z})^*} \left(\frac{t_0}{p}\right)^{k-\tau}$$

$$= \begin{cases} p^{5k/2+\tau/2} p^{k-\tau}\left(1 - \frac{1}{p}\right), & \text{if } k-\tau \text{ is even,} \\ 0, & \text{if } k-\tau \text{ is odd.} \end{cases}$$

□

Computing $\sum_{\tau=k-\min(m_1,b_1)}^{k-1} s_{k,\tau}$

Lemma

Let $n_1 = \min(m_1, b_1)$. *Then*

$$\sum_{\tau=k-n_1}^{k-1} s_{k,\tau} = \sum_{\substack{\tau=k-n_1 \\ k-\tau \text{ is even}}}^{k-1} p^{3k+(k-\tau)/2}\left(1-\frac{1}{p}\right) = p^{3k}(p^{\lfloor n_1/2 \rfloor} - 1),$$

where $\lfloor x \rfloor$ *is the greatest integer less than or equal to* x.

Lemma

Let $n_1 = \min(m_1, b_1)$. *Then*

$$\sum_{\tau=k-n_1}^{k-1} s_{k,\tau} = \sum_{\substack{\tau=k-n_1 \\ k-\tau \text{ is even}}}^{k-1} p^{3k+(k-\tau)/2}\left(1-\frac{1}{p}\right) = p^{3k}(p^{\lfloor n_1/2 \rfloor} - 1),$$

where $\lfloor x \rfloor$ *is the greatest integer less than or equal to* x.

Proof sketch:

1. Let $\tau_1 = \frac{k-\tau}{2}$.
2. Apply formulas for geometric sums.

Thank you for listening!

Formulas for Local Densities at Odd Primes

Theorem (J., 2020)

Let Q be the integral quadratic form $ax^2 + by^2 + cz^2$, where a, b, and c are integers. Let p be an odd prime. Suppose $p \nmid a$, $b = b_0 p^{b_1}$, and $c = c_0 p^{c_1}$, where $b_1 \leq c_1$, $\gcd(b_0, p) = 1$, and $\gcd(c_0, p) = 1$.
Suppose m is a nonzero integer and $m = m_0 p^{m_1}$, where $\gcd(m_0, p) = 1$.
If $m_1 < b_1$, then

$$\alpha_p(m, Q) = \begin{cases} p^{m_1/2}\left(1 + \left(\frac{am_0}{p}\right)\right), & \text{if } m_1 \text{ is even}, \\ 0, & \text{if } m_1 \text{ is odd}. \end{cases}$$

Formulas for Local Densities at Odd Primes

Theorem (J., 2020, continued)

If $b_1 \leq m_1 < c_1$, then

$$\alpha_p(m, Q) = \begin{cases} p^{b_1/2}\left(1 - \frac{1}{p}\left(\frac{-ab_0}{p}\right)^{m_1+1} \right. \\ \qquad + \left(1-\frac{1}{p}\right)\left(\frac{m_1-b_1}{2} + \frac{(-1)^{m_1}-1}{4}\right. \\ \qquad \left.\left. + \left(\frac{-ab_0}{p}\right)\left(\frac{m_1-b_1}{2} + \frac{1-(-1)^{m_1}}{4}\right)\right)\right), \\ \qquad \text{if } b_1 \text{ is even,} \\ p^{(b_1-1)/2}\left(1 + \left(\frac{a}{p}\right)^{m_1+1}\left(\frac{b_0}{p}\right)^{m_1}\left(\frac{m_0}{p}\right)\right), \\ \qquad \text{if } b_1 \text{ is odd.} \end{cases}$$

Formulas for Local Densities at Odd Primes

Theorem (J., 2020, continued)

If $m_1 \geq c_1$ and b_1 and c_1 are even, then

$$\alpha_p(m, Q) = \begin{cases} p^{b_1/2}\left(1 + \frac{1}{p} + p^{-m_1/2 + c_1/2 - 1}\left(\left(\left(\frac{-ab_0c_0m_0}{p}\right) - 1\right) \right.\right. \\ \qquad \left.\left. + \left(1 - \frac{1}{p}\right)\left(\frac{c_1 - b_1}{2} + \left(\frac{-ab_0}{p}\right)\frac{c_1 - b_1}{2}\right)\right)\right), \\ \qquad \text{if } m_1 \text{ is even}, \\ p^{b_1/2}\left(\left(1 + \frac{1}{p}\right)\left(1 - p^{-(m_1+1)/2 + c_1/2}\right) \right. \\ \qquad \left. + \left(1 - \frac{1}{p}\right)\left(\frac{c_1 - b_1}{2} + \left(\frac{-ab_0}{p}\right)\frac{c_1 - b_1}{2}\right)\right), \\ \qquad \text{if } m_1 \text{ is odd}. \end{cases}$$

Formulas for Local Densities at Odd Primes

Theorem (J., 2020, continued)

If $m_1 \geq c_1$, b_1 is even, and c_1 is odd, then

$$\alpha_p(m, Q) =$$

$$\begin{cases} p^{b_1/2}\left(1 - p^{-m_1/2+(c_1-1)/2}\left(\frac{-ab_0}{p}\right)\left(1+\frac{1}{p}\right) + \frac{1}{p}\left(\frac{-ab_0}{p}\right) \right. \\ \qquad \left. + \left(1-\frac{1}{p}\right)\left(\frac{c_1-b_1-1}{2} + \left(\frac{-ab_0}{p}\right)\frac{c_1-b_1+1}{2}\right)\right), \\ \qquad\qquad \text{if } m_1 \text{ is even,} \\ p^{b_1/2}\left(1 + p^{-(m_1+1)/2+(c_1-1)/2}\left(\left(\frac{c_0 m_0}{p}\right) - \left(\frac{-ab_0}{p}\right)\right) \right. \\ \qquad + \frac{1}{p}\left(\frac{-ab_0}{p}\right) \\ \qquad \left. + \left(1-\frac{1}{p}\right)\left(\frac{c_1-b_1-1}{2} + \left(\frac{-ab_0}{p}\right)\frac{c_1-b_1+1}{2}\right)\right), \\ \qquad\qquad \text{if } m_1 \text{ is odd.} \end{cases}$$

Formulas for Local Densities at Odd Primes

Theorem (J., 2020, continued)

If $m_1 \geq c_1$, b_1 is odd, and c_1 is even, then

$$\alpha_p(m, Q) = \begin{cases} p^{(b_1-1)/2} \left(1 + \left(\frac{-ac_0}{p}\right) \right. \\ \qquad \left. -p^{-m_1/2+c_1/2} \left(1 + \frac{1}{p}\right) \left(\frac{-ac_0}{p}\right)\right), \\ \qquad \text{if } m_1 \text{ is even,} \\ p^{(b_1-1)/2} \left(1 + \left(\frac{-ac_0}{p}\right) \right. \\ \qquad \left. +p^{-(m_1+1)/2+c_1/2} \left(\left(\frac{b_0 m_0}{p}\right) - \left(\frac{-ac_0}{p}\right)\right)\right), \\ \qquad \text{if } m_1 \text{ is odd.} \end{cases}$$

Theorem (J., 2020, continued)

If $m_1 \geq c_1$ and b_1 and c_1 are odd, then

$$\alpha_p(m, Q) = \begin{cases} p^{(b_1-1)/2}\left(1 + \left(\frac{-b_0 c_0}{p}\right)\right. \\ \qquad \left.+ p^{-m_1/2+(c_1-1)/2}\left(\left(\frac{a m_0}{p}\right) - \left(\frac{-b_0 c_0}{p}\right)\right)\right), \\ \qquad \text{if } m_1 \text{ is even,} \\ p^{(b_1-1)/2}\left(1 + \left(\frac{-b_0 c_0}{p}\right)\right. \\ \qquad \left.- p^{(-m_1+c_1)/2}\left(1 + \frac{1}{p}\right)\left(\frac{-b_0 c_0}{p}\right)\right), \\ \qquad \text{if } m_1 \text{ is odd.} \end{cases}$$

Formulas for Local Densities at Odd Primes

Theorem (J., 2020, continued)

Furthermore,

$$\alpha_p(0, Q) =$$

$$\begin{cases} p^{b_1/2}\left(1 + \frac{1}{p} + \left(1 - \frac{1}{p}\right)\left(\frac{c_1 - b_1}{2} + \left(\frac{-ab_0}{p}\right)\frac{c_1 - b_1}{2}\right)\right), & \text{if } b_1 \text{ and } c_1 \text{ are even,} \\ p^{b_1/2}\left(1 + \frac{1}{p}\left(\frac{-ab_0}{p}\right) + \left(1 - \frac{1}{p}\right)\left(\frac{c_1 - b_1 - 1}{2} + \left(\frac{-ab_0}{p}\right)\frac{c_1 - b_1 + 1}{2}\right)\right), & \text{if } b_1 \text{ is even and } c_1 \text{ is odd,} \\ p^{(b_1-1)/2}\left(1 + \left(\frac{-ac_0}{p}\right)\right), & \text{if } b_1 \text{ is odd and } c_1 \text{ is even,} \\ p^{(b_1-1)/2}\left(1 + \left(\frac{-b_0 c_0}{p}\right)\right), & \text{if } b_1 \text{ and } c_1 \text{ are odd.} \end{cases}$$