What is the circle method?

Typically refers to the Hardy-Littlewood circle method. It may also refer to other methods that are used to provide an asymptotic formula for the number of ways an integer is represented by an integer-valued function on \( \mathbb{Z} \). Examples include the Kloosterman circle method and the delta method.

**Definition (Representation number)**

\[ R_F(n) = \# \{ m \in \mathbb{Z} : F(m) = n \} \]
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**Definition (Representation number)**

$$R_F(n) = \# \{ m \in \mathbb{Z}^s : F(m) = n \}$$
Originally developed by Hardy and Ramanujan (1918) to provide asymptotic formula for the partition function $p(n)$, the number of partitions of $n$

Proved that

$$p(n) \sim \frac{1}{4n\sqrt{3}} \exp \left( \pi \sqrt{\frac{2n}{3}} \right)$$
Partition function & modularity

\[ f(z) = \sum_{n=0}^{\infty} p(n)e(nz), \]

where \( e(z) = e^{2\pi iz} \) and \( \text{Im}(z) > 0. \)
Partition function & modularity

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where \( e(z) = e^{2\pi iz} \) and \( \text{Im}(z) > 0 \).

\[ f(z) = \frac{e(z/24)}{\eta(z)}, \]

where \( \eta(z) \) is the Dedekind eta function

\[ \eta(z) = e\left(\frac{z}{24}\right) \prod_{m=1}^{\infty} (1 - e(mz)). \]

Hardy and Ramanujan used the modularity of \( \eta \) to obtain the asymptotic formula.
Using the Cauchy integral formula, we find that
\[ p(n) = \frac{1}{2\pi i} \int_{|q|=r} F(q) q^n dq, \]
where \( 0 < r < 1 \).

Changing \( q \) into \( e^{x+iy} \), we obtain
\[ p(n) = \int_0^1 f(x+iy) e^{-n(x+iy)} dx, \]
where \( \text{Im}(y) > 0 \).
Hardy-Littlewood circle method & partition function

\[ f(z) = \sum_{n=0}^{\infty} p(n)e(nz) \]

\[ F(q) = \sum_{n=0}^{\infty} p(n)q^n \]

where \( q = e(z) \).

Using the Cauchy integral formula, we find that

\[ p(n) = \frac{1}{2\pi i} \int_{|q|=r} \frac{F(q)}{q^{n+1}} \, dq, \]

where \( 0 < r < 1 \).
Hardy-Littlewood circle method & partition function

\[ f(z) = \sum_{n=0}^{\infty} p(n) e(nz) \quad F(q) = \sum_{n=0}^{\infty} p(n) q^n \]

where \( q = e(z) \).

Using the Cauchy integral formula, we find that

\[ p(n) = \frac{1}{2\pi i} \int_{|q|=r} \frac{F(q)}{q^{n+1}} \, dq, \]

where \( 0 < r < 1 \).

Changing \( q \) into \( e(x + iy) \), we obtain

\[ p(n) = \int_{0}^{1} f(x + iy)e(-n(x + iy)) \, dx, \]

where \( \text{Im}(y) > 0 \).
Figure: Modulus of $\prod_{m=1}^{\infty}(1 - q^m)$ with $|q| < 1$. From Wikipedia.

Main contribution to integral from points near $e(a/q)$ where $q$ is small.
Split $[0, 1]$ into major arcs $\mathcal{M}$ and minor arcs $\mathcal{m}$. 

How close depends on the application of the method.
Major arcs and minor arcs

Split $[0, 1]$ into major arcs $\mathcal{M}$ and minor arcs $\mathcal{m}$.

$$\mathcal{M} = \left\{ x \in [0, 1] : x \text{ is “close to” } \frac{a}{q}, a, q \in \mathbb{Z}, 0 < q \leq Q \right\}.$$ 

How close depends on the application of the method.
Split $[0, 1]$ into major arcs $\mathcal{M}$ and minor arcs $m$.

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How close depends on the application of the method.

$$m = [0, 1] \setminus \mathcal{M}.$$
Example of major arcs $\mathcal{M}$ when $Q = 3$ for the Hardy-Littlewood circle method:

$$p(n) = \int_{0}^{1} f(x + iy)e(-n(x + iy)) \, dx$$

$$= \int_{\mathcal{M}} f(x + iy)e(-n(x + iy)) \, dx + \int_{\mathcal{M}} f(x + iy)e(-n(x + iy)) \, dx$$

main term  error term
Real quadratic forms

$F$ is a real quadratic form in $s$ variables $\iff$ For all $m \in \mathbb{R}^s$,

$$F(m) = \frac{1}{2} m^\top A m,$$

where $A$ is a real symmetric $s \times s$ matrix and is the Hessian matrix of $F$.

**Example (Example of a quadratic form in 2 variables)**

$$F(m) = m_1^2 + m_1 m_2 + m_2^2$$

$$= \frac{1}{2} m^\top \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} m$$
Definition (Integral quadratic form)
A quadratic form $F$ is integral if $F(m) \in \mathbb{Z}$ for all $m \in \mathbb{Z}^s$.

Definition (Positive definite quadratic form)
A quadratic form $F$ is positive definite if $F(m) > 0$ for all $m \in \mathbb{R}^s \setminus \{0\}$.

Examples (Examples of integral positive definite quadratic forms)
- $f_4(m) = m_1^2 + m_2^2 + m_3^2 + m_4^2$
- $x^2 + xy + y^2$
Want an asymptotic formula for $R_F(n)$ when $F$ is a positive definite quadratic form.
Want an asymptotic formula for \( R_F(n) \) when \( F \) is a positive definite quadratic form.

Use same overall method for obtaining an asymptotic formula for the partition function.

Note that the theta function

\[
\Theta(z) = \sum_{n=0}^{\infty} R_F(n)e(nz)
\]

is an automorphic form.
The singular series $\mathcal{S}_F(n)$ contains information about $F(m) \equiv n \pmod{q}$ for all positive integers $q$.

$\mathcal{S}_F(n) = 0 \iff$ there exists a positive integer $q$ such that $F(m) \equiv n \pmod{q}$ has no solutions
Theorem (Kloosterman, 1924)

Suppose that \( n \) is a positive integer.
Suppose that \( F \) is a positive definite integral quadratic form in \( s \geq 5 \) variables.
Let \( A \in M_s(\mathbb{Z}) \) be the Hessian matrix of \( F \).
Then the number of integral solutions to \( F(m) = n \) is

\[
R_F(n) = \mathcal{S}_F(n) \frac{(2\pi)^{s/2}}{\Gamma(s/2)\sqrt{\det(A)}} n^{\frac{s}{2}-1} + O_F,\varepsilon \left( n^{\frac{s}{4}+\varepsilon} + n^{\frac{s}{2}-\frac{5}{4}+\varepsilon} \right)
\]

for any \( \varepsilon > 0 \).
Motivation for the Kloosterman circle method

- Want a better error term in asymptotic formula for $R_F(n)$ when $F$ is a positive definite quadratic form.
- Split $[0, 1]$ differently.
For $Q \geq 1$, the Farey sequence $\mathcal{F}_Q$ of order $Q$ is the increasing sequence of all reduced fractions $\frac{a}{q}$ with $1 \leq q \leq Q$ and $\gcd(a, q) = 1$. 

$Q = 1$

\[ \begin{array}{ccc}
0 & & 1 \\
\frac{1}{1} & & \frac{1}{1}
\end{array} \]
Definition

For $Q \geq 1$, the **Farey sequence** $\mathcal{F}_Q$ of order $Q$ is the increasing sequence of all reduced fractions $\frac{a}{q}$ with $1 \leq q \leq Q$ and $\gcd(a, q) = 1$.

$Q = 2$

\[\begin{array}{c|c|c}
0 & \frac{1}{2} & 1 \\
\frac{1}{1} & \frac{2}{2} & \frac{1}{1}
\end{array}\]
Farey sequence $\mathcal{F}_Q$ of order $Q$

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$Q = 3$

\[
\begin{array}{ccccccc}
& & & & & & \\
& 0 & & 1 & & 1 & & 2 & & 2 & & 3 & & 1 & \\
& 1 & & 3 & & 2 & & 3 & & 2 & & 5 & & 3 & & 5 & & 3 & & 4 & & 1 & & 1 & \\
\end{array}
\]
Farey sequence $\mathcal{F}_Q$ of order $Q$

**Definition**

For $Q \geq 1$, the **Farey sequence** $\mathcal{F}_Q$ of order $Q$ is the increasing sequence of all reduced fractions $\frac{a}{q}$ with $1 \leq q \leq Q$ and $\gcd(a, q) = 1$.

$Q = 3$

\[
\begin{array}{cccccccc}
0 & 1 & 1 & 2 & 3 & 1 \\
\frac{1}{1} & \frac{1}{3} & \frac{1}{2} & \frac{2}{3} & \frac{1}{1}
\end{array}
\]

Example of Farey dissection when $Q = 3$:

\[
\begin{array}{cccccccc}
0 & 1 & \frac{1}{4} & \frac{1}{3} & \frac{2}{5} & \frac{1}{2} & \frac{3}{5} & \frac{2}{3} & \frac{3}{4} & 1 \\
\frac{1}{1} & \frac{1}{4} & \frac{1}{3} & \frac{2}{5} & \frac{1}{2} & \frac{3}{5} & \frac{2}{3} & \frac{3}{4} & \frac{1}{1}
\end{array}
\]
Theorem

Suppose that \( n \) is a positive integer.
Suppose that \( F \) is a positive definite integral quadratic form in \( s \geq 4 \) variables.
Let \( A \in M_s(\mathbb{Z}) \) be the Hessian matrix of \( F \).
Then the number of integral solutions to \( F(m) = n \) is

\[
R_F(n) = \mathcal{G}_F(n) \frac{(2\pi)^{s/2}}{\Gamma(s/2) \sqrt{\det(A)}} n^{s-1} + O_{F,\varepsilon} \left( n^{s-1/4} + \varepsilon \right)
\]

for any \( \varepsilon > 0 \).

Kloosterman proved this (with a worse error term) in 1926 for diagonal quadratic forms \( (F(m) = a_1 m_1^2 + \cdots + a_s m_s^2) \), using what is now called the Kloosterman circle method.
Example of major arcs when $Q = 3$ for the Hardy-Littlewood circle method:

Example of Farey dissection when $Q = 3$ for the Kloosterman circle method:
Hardy-Littlewood:

\[ R_F(n) = \mathcal{S}_F(n) \frac{(2\pi)^{s/2}}{\Gamma(s/2) \sqrt{\det(A)}} n^{s-1} + O_{F,\varepsilon} \left( n^{s_4 + \varepsilon} + n^{s_2 - \frac{5}{4} + \varepsilon} \right) \]

Kloosterman:

\[ R_F(n) = \mathcal{S}_F(n) \frac{(2\pi)^{s/2}}{\Gamma(s/2) \sqrt{\det(A)}} n^{s-1} + O_{F,\varepsilon} \left( n^{s_4 - \frac{1}{4} + \varepsilon} \right) \]
Rewrite $\delta(n)$, the indicator function for zero, using bump functions.
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More versatile than Kloosterman circle method
The delta method

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- Developed by Duke, Friedlander, and Iwaniec in 1993 to compute bounds for automorphic $L$-functions
The delta method

- Rewrite $\delta(n)$, the indicator function for zero, using bump functions
- More versatile than Kloosterman circle method
- Developed by Duke, Friedlander, and Iwaniec in 1993 to compute bounds for automorphic $L$-functions
- Has been used for a variety of applications, including
  - Asymptotic formulas for the representation numbers of quadratic forms (Heath-Brown)
  - Subconvexity bounds for (twists of) automorphic forms (Munshi)
Definition (Representation number)

\[ R_F(n) = \# \{ m \in \mathbb{Z}^s : F(m) = n \} \]

\[ R_F(n) = \sum_{m \in \mathbb{Z}^s} 1_{\{F(m) = n\}}, \]

where \( 1_{\{F(m) = n\}} \) is the indicator function

\[ 1_{\{F(m) = n\}} = \begin{cases} 1 & \text{if } F(m) = n, \\ 0 & \text{otherwise.} \end{cases} \]
Indicator function

\[ \delta(n) = 1_{\{n=0\}} = \begin{cases} 
1 & \text{if } n = 0, \\
0 & \text{otherwise.} 
\end{cases} \]
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\[ 1_{\{F(m)=n\}} = \delta(F(m) - n) \]

\[ R_F(n) = \sum_{m \in \mathbb{Z}^s} \delta(F(m) - n) \]
Definition (Bump function)

The space of real-valued, infinitely differentiable, and compactly supported functions on $\mathbb{R}$ is denoted by $C_\infty^\infty(\mathbb{R})$. A function $w \in C_\infty^\infty(\mathbb{R})$ is called a bump function.
Definition (Bump function)

The space of real-valued, infinitely differentiable, and compactly supported functions on $\mathbb{R}$ is denoted by $C^\infty_c(\mathbb{R})$. A function $w \in C^\infty_c(\mathbb{R})$ is called a **bump function**.

Require $w(0) = 0$ and $\sum_{q=1}^{\infty} w(q) \neq 0$ for the delta method.
The delta method & bump functions

**Definition (Bump function)**

The space of real-valued, infinitely differentiable, and compactly supported functions on $\mathbb{R}$ is denoted by $C_c^\infty(\mathbb{R})$. A function $w \in C_c^\infty(\mathbb{R})$ is called a **bump function**.

Require $w(0) = 0$ and $\sum_{q=1}^\infty w(q) \neq 0$ for the delta method.

If $n$ is an integer, then

$$
\delta(n) = \frac{1}{\sum_{q=1}^\infty w(q)} \sum_{q|n} \left( w(q) - w \left( \frac{|n|}{q} \right) \right),
$$

where the sum over $q \mid n$ is taken to be the sum over the positive divisors of $n$.  

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The circle method
Using the fact that

\[ \frac{1}{q} \sum_{a \pmod{q}} e\left(\frac{an}{q}\right) = \begin{cases} 1 & \text{if } q \mid n, \\ 0 & \text{otherwise}, \end{cases} \]
Using the fact that

\[ \frac{1}{q} \sum_{a \equiv (mod \ q)} e\left(\frac{an}{q}\right) = \begin{cases} 1 & \text{if } q \mid n, \\ 0 & \text{otherwise}, \end{cases} \]

we have

\[ \delta(n) = \frac{1}{\sum_{q=1}^{\infty} w(q)} \sum_{q \mid n} \left( w(q) - w\left(\frac{|n|}{q}\right) \right) \]

\[ = \frac{1}{\sum_{q=1}^{\infty} w(q)} \sum_{q=1}^{\infty} \frac{1}{q} \sum_{a \equiv (mod \ q)} e\left(\frac{an}{q}\right) \left( w(q) - w\left(\frac{|n|}{q}\right) \right) \]

if \( n \) is an integer.
Some applications of the circle method

- Asymptotic formula for the partition function (Hardy–Ramanujan)
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- Asymptotic formula for the partition function (Hardy–Ramanujan)
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- Subconvexity bounds for (twists of) automorphic forms (Duke–Friedlander–Iwaniec, Munshi)
Thank you for listening!
Lemma for Kloosterman circle method

**Lemma**

Let \( f : \mathbb{R} \to \mathbb{C} \) be a periodic function of period 1 and with real Fourier coefficients (so that \( f(x) = f(-x) \) for all \( x \in \mathbb{R} \)). Then

\[
\int_0^1 f(x) \, dx = 2 \text{Re} \left( \sum_{1 \leq q \leq Q} \int_0^{\frac{1}{qQ}} \sum_{Q < d \leq q+Q \atop qd < 1} f \left( x - \frac{d^{*}}{q} \right) \, dx \right),
\]

where \( d^{*} \) is the multiplicative inverse of \( d \) modulo \( q \).

Use this for

\[
f(x) = \sum_{m \in \mathbb{Z}^s} e((x + iy)(F(m) - n)),
\]

where \( y > 0 \).