The circle method

Edna Jones

Duke University

35th Automorphic Forms Workshop
May 22, 2023

What is the circle method?

What is the circle method?

- Typically refers to the Hardy-Littlewood circle method

What is the circle method?

- Typically refers to the Hardy-Littlewood circle method
- May also refer to other methods that are used to provide an asymptotic formula for the number of ways an integer is represented by an integer-valued function on $\mathbb{Z}^s$
  - Kloosterman circle method
  - The delta method

What is the circle method?

- Typically refers to the Hardy-Littlewood circle method
- May also refer to other methods that are used to provide an asymptotic formula for the number of ways an integer is represented by an integer-valued function on $\mathbb{Z}^s$
  - Kloosterman circle method
  - The delta method

Definition (Representation number)

$$R_F(n) = \#\{\mathbf{m} \in \mathbb{Z}^s : F(\mathbf{m}) = n\}$$

Hardy-Littlewood circle method

- Originally developed by Hardy and Ramanujan (1918) to provide asymptotic formula for the partition function $p(n)$, the number of partitions of n
- Proved that

$$p(n) \sim \frac{1}{4n\sqrt{3}} \exp\left(\pi\sqrt{\frac{2n}{3}}\right)$$

Partition function & modularity

$$f(z) = \sum_{n=0}^{\infty} p(n)\mathrm{e}(nz)\,,$$

where $\mathrm{e}(z) = e^{2\pi iz}$ and $\mathrm{Im}(z) > 0$.

Partition function & modularity

$$f(z) = \sum_{n=0}^{\infty} p(n)\mathrm{e}(nz)\,,$$

where $\mathrm{e}(z) = e^{2\pi i z}$ and $\mathrm{Im}(z) > 0$.

$$f(z) = \frac{\mathrm{e}(z/24)}{\eta(z)},$$

where $\eta(z)$ is the Dedekind eta function

$$\eta(z) = \mathrm{e}\left(\frac{z}{24}\right) \prod_{m=1}^{\infty} (1 - \mathrm{e}(mz)).$$

Hardy and Ramanujan used the modularity of η to obtain the asymptotic formula.

$$f(z) = \sum_{n=0}^{\infty} p(n)\mathrm{e}(nz) \qquad\qquad F(q) = \sum_{n=0}^{\infty} p(n)q^n$$

where $q = \mathrm{e}(z)$.

$$f(z) = \sum_{n=0}^{\infty} p(n)\mathrm{e}(nz) \qquad\qquad F(q) = \sum_{n=0}^{\infty} p(n)q^n$$

where $q = \mathrm{e}(z)$.
Using the Cauchy integral formula, we find that

$$p(n) = \frac{1}{2\pi i} \int_{|q|=r} \frac{F(q)}{q^{n+1}}\, dq,$$

where $0 < r < 1$.

$$f(z) = \sum_{n=0}^{\infty} p(n)\mathrm{e}(nz) \qquad F(q) = \sum_{n=0}^{\infty} p(n)q^n$$

where $q = \mathrm{e}(z)$.
Using the Cauchy integral formula, we find that

$$p(n) = \frac{1}{2\pi i} \int_{|q|=r} \frac{F(q)}{q^{n+1}}\, dq,$$

where $0 < r < 1$.
Changing q into $\mathrm{e}(x+iy)$, we obtain

$$p(n) = \int_0^1 f(x+iy)\mathrm{e}(-n(x+iy))\, dx,$$

where $\mathrm{Im}(y) > 0$.

Partition function

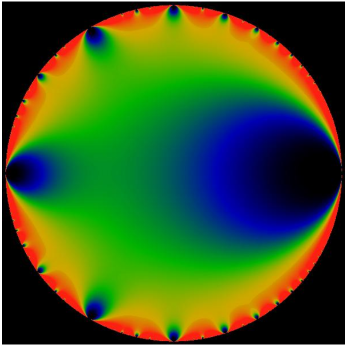


Figure: Modulus of $\prod_{m=1}^{\infty}(1-q^m)$ with $|q|<1$. From Wikipedia.

Main contribution to integral from points near $\mathrm{e}(a/q)$ where q is small.

Major arcs and minor arcs

Split $[0,1]$ into major arcs $\mathfrak{M}$ and minor arcs $\mathfrak{m}$.

Major arcs and minor arcs

Split $[0, 1]$ into major arcs $\mathfrak{M}$ and minor arcs $\mathfrak{m}$.

$$\mathfrak{M} = \left\{ x \in [0,1] : x \text{ is "close to" } \frac{a}{q}, a, q \in \mathbb{Z}, 0 < q \leq Q \right\}.$$

How close depends on the application of the method.

Major arcs and minor arcs

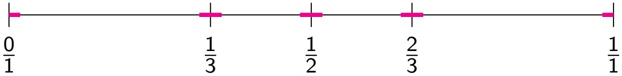
Split $[0,1]$ into major arcs $\mathfrak{M}$ and minor arcs $\mathfrak{m}$.

$$\mathfrak{M} = \left\{ x \in [0,1] : x \text{ is ``close to''} \frac{a}{q}, a, q \in \mathbb{Z}, 0 < q \leq Q \right\}.$$

How close depends on the application of the method.

$$\mathfrak{m} = [0,1] \setminus \mathfrak{M}.$$

Example of major arcs $\mathfrak{M}$ when $Q = 3$ for the Hardy-Littlewood circle method:



$$
\begin{aligned}
p(n) &= \int_0^1 f(x+iy)\mathrm{e}(-n(x+iy)) \, dx \\
&= \underbrace{\int_{\mathfrak{M}} f(x+iy)\mathrm{e}(-n(x+iy)) \, dx}_{\text{main term}} + \underbrace{\int_{\mathfrak{m}} f(x+iy)\mathrm{e}(-n(x+iy)) \, dx}_{\text{error term}}
\end{aligned}
$$

Real quadratic forms

F is a real quadratic form in s variables $\iff$
For all $\mathbf{m} \in \mathbb{R}^s$,

$$F(\mathbf{m}) = \frac{1}{2}\mathbf{m}^\top A\mathbf{m},$$

where A is a real symmetric $s \times s$ matrix and is the Hessian matrix of F.

Example (Example of a quadratic form in 2 variables)

$$\begin{aligned} F(\mathbf{m}) &= m_1^2 + m_1 m_2 + m_2^2 \\ &= \frac{1}{2}\mathbf{m}^\top \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \mathbf{m} \end{aligned}$$

Quadratic form definitions

Definition (Integral quadratic form)

A quadratic form F is **integral** if $F(\mathbf{m}) \in \mathbb{Z}$ for all $\mathbf{m} \in \mathbb{Z}^s$.

Definition (Positive definite quadratic form)

A quadratic form F is **positive definite** if $F(\mathbf{m}) > 0$ for all $\mathbf{m} \in \mathbb{R}^s \setminus \{\mathbf{0}\}$.

Examples (Examples of integral positive definite quadratic forms)

- $f_4(\mathbf{m}) = m_1^2 + m_2^2 + m_3^2 + m_4^2$
- $x^2 + xy + y^2$

Hardy-Littlewood circle method & quadratic forms

Want an asymptotic formula for $R_F(n)$ when F is a positive definite quadratic form.

Hardy-Littlewood circle method & quadratic forms

Want an asymptotic formula for $R_F(n)$ when F is a positive definite quadratic form.

Use same overall method for obtaining an asymptotic formula for the partition function.

Note that the theta function

$$\Theta(z) = \sum_{n=0}^{\infty} R_F(n)\mathrm{e}(nz)$$

is an automorphic form.

Singular series $\mathfrak{S}_F(n)$

$$\mathfrak{S}_F(n) = \sum_{q=1}^{\infty} \frac{1}{q^s} \sum_{d \in (\mathbb{Z}/q\mathbb{Z})^\times} \sum_{\mathbf{h} \in (\mathbb{Z}/q\mathbb{Z})^s} \mathrm{e}\left(\frac{d}{q}\left(F(\mathbf{h}) - n\right)\right)$$

The singular series $\mathfrak{S}_F(n)$ contains information about $F(\mathbf{m}) \equiv n \pmod{q}$ for all positive integers q.

$\mathfrak{S}_F(n) = 0 \iff$
there exists a positive integer q such that $F(\mathbf{m}) \equiv n \pmod{q}$ has no solutions

An asymptotic for representation numbers from Hardy-Littlewood circle method

Theorem (Kloosterman, 1924)

Suppose that n is a positive integer.
Suppose that F is a positive definite integral quadratic form in $s \geq 5$ variables.
Let $A \in M_s(\mathbb{Z})$ be the Hessian matrix of F.
Then the number of integral solutions to $F(\mathbf{m}) = n$ is

$$R_F(n) = \mathfrak{S}_F(n) \frac{(2\pi)^{s/2}}{\Gamma(s/2)\sqrt{\det(A)}} n^{\frac{s}{2}-1} + O_{F,\varepsilon}\left(n^{\frac{s}{4}+\varepsilon} + n^{\frac{s}{2}-\frac{5}{4}+\varepsilon}\right)$$

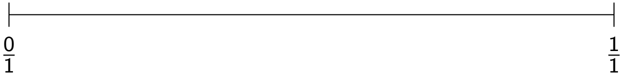
for any $\varepsilon > 0$.

- Want a better error term in asymptotic formula for $R_F(n)$ when F is a positive definite quadratic form.
- Split $[0, 1]$ differently.

Definition

For $Q \geq 1$, the **Farey sequence $\mathfrak{F}_Q$ of order Q** is the increasing sequence of all reduced fractions $\frac{a}{q}$ with $1 \leq q \leq Q$ and $\gcd(a, q) = 1$.

$Q = 1$

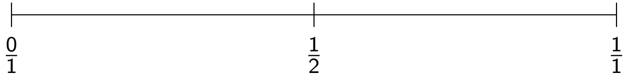


Farey sequence $\mathfrak{F}_Q$ of order Q

Definition

For $Q \geq 1$, the **Farey sequence** $\mathfrak{F}_Q$ **of order** Q is the increasing sequence of all reduced fractions $\frac{a}{q}$ with $1 \leq q \leq Q$ and $\gcd(a, q) = 1$.

$Q = 2$

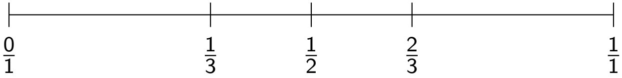


Farey sequence $\mathfrak{F}_Q$ of order Q

Definition

For $Q \geq 1$, the **Farey sequence $\mathfrak{F}_Q$ of order Q** is the increasing sequence of all reduced fractions $\frac{a}{q}$ with $1 \leq q \leq Q$ and $\gcd(a, q) = 1$.

$Q = 3$

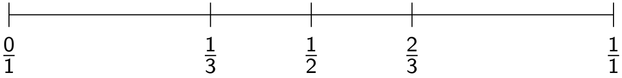


Farey sequence $\mathfrak{F}_Q$ of order Q

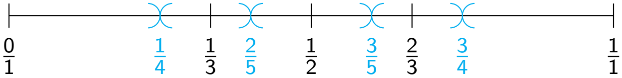
Definition

For $Q \geq 1$, the **Farey sequence $\mathfrak{F}_Q$ of order Q** is the increasing sequence of all reduced fractions $\frac{a}{q}$ with $1 \leq q \leq Q$ and $\gcd(a, q) = 1$.

$Q = 3$



Example of Farey dissection when $Q = 3$:



An asymptotic for representation numbers from the Kloosterman method

Theorem

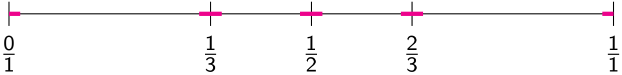
Suppose that n is a positive integer.
Suppose that F is a positive definite integral quadratic form in $s \geq 4$ variables.
Let $A \in M_s(\mathbb{Z})$ be the Hessian matrix of F.
Then the number of integral solutions to $F(\mathbf{m}) = n$ is

$$R_F(n) = \mathfrak{S}_F(n) \frac{(2\pi)^{s/2}}{\Gamma(s/2)\sqrt{\det(A)}} n^{\frac{s}{2}-1} + O_{F,\varepsilon}\left(n^{\frac{s-1}{4}+\varepsilon}\right)$$

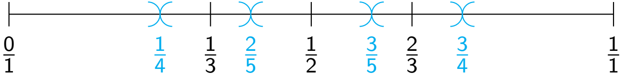
for any $\varepsilon > 0$.

Kloosterman proved this (with a worse error term) in 1926 for diagonal quadratic forms ($F(\mathbf{m}) = a_1 m_1^2 + \cdots + a_s m_s^2$), using what is now called the Kloosterman circle method.

Example of major arcs when $Q = 3$ for the Hardy-Littlewood circle method:



Example of Farey dissection when $Q = 3$ for the Kloosterman circle method:



Hardy-Littlewood vs. Kloosterman

Hardy-Littlewood:

$$R_F(n) = \mathfrak{S}_F(n) \frac{(2\pi)^{s/2}}{\Gamma(s/2)\sqrt{\det(A)}} n^{\frac{s}{2}-1} + O_{F,\varepsilon}\left(n^{\frac{s}{4}+\varepsilon} + n^{\frac{s}{2}-\frac{5}{4}+\varepsilon}\right)$$

Kloosterman:

$$R_F(n) = \mathfrak{S}_F(n) \frac{(2\pi)^{s/2}}{\Gamma(s/2)\sqrt{\det(A)}} n^{\frac{s}{2}-1} + O_{F,\varepsilon}\left(n^{\frac{s-1}{4}+\varepsilon}\right)$$

The delta method

- Rewrite $\delta(n)$, the indicator function for zero, using bump functions

The delta method

- Rewrite $\delta(n)$, the indicator function for zero, using bump functions
- More versatile than Kloosterman circle method

The delta method

- Rewrite $\delta(n)$, the indicator function for zero, using bump functions
- More versatile than Kloosterman circle method
- Developed by Duke, Friedlander, and Iwaniec in 1993 to compute bounds for automorphic L-functions

The delta method

- Rewrite $\delta(n)$, the indicator function for zero, using bump functions
- More versatile than Kloosterman circle method
- Developed by Duke, Friedlander, and Iwaniec in 1993 to compute bounds for automorphic L-functions
- Has been used for a variety of applications, including
  - Asymptotic formulas for the representation numbers of quadratic forms (Heath-Brown)
  - Subconvexity bounds for (twists of) automorphic forms (Munshi)

Definition (Representation number)

$$R_F(n) = \#\{\mathbf{m} \in \mathbb{Z}^s : F(\mathbf{m}) = n\}$$

$$R_F(n) = \sum_{\mathbf{m} \in \mathbb{Z}^s} \mathbf{1}_{\{F(\mathbf{m})=n\}},$$

where $\mathbf{1}_{\{F(\mathbf{m})=n\}}$ is the indicator function

$$\mathbf{1}_{\{F(\mathbf{m})=n\}} = \begin{cases} 1 & \text{if } F(\mathbf{m}) = n, \\ 0 & \text{otherwise.} \end{cases}$$

Indicator function

$$\delta(n) = \mathbf{1}_{\{n=0\}} = \begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Indicator function

$$\delta(n) = \mathbf{1}_{\{n=0\}} = \begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{otherwise.} \end{cases}$$

$$\mathbf{1}_{\{F(\mathbf{m})=n\}} = \delta(F(\mathbf{m}) - n)$$

Indicator function

$$\delta(n) = \mathbf{1}_{\{n=0\}} = \begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{otherwise.} \end{cases}$$

$$\mathbf{1}_{\{F(\mathbf{m})=n\}} = \delta(F(\mathbf{m}) - n)$$

$$R_F(n) = \sum_{\mathbf{m} \in \mathbb{Z}^s} \delta(F(\mathbf{m}) - n)$$

The delta method & bump functions

Definition (Bump function)

The space of real-valued, infinitely differentiable, and compactly supported functions on $\mathbb{R}$ is denoted by $C_c^\infty(\mathbb{R})$. A function $w \in C_c^\infty(\mathbb{R})$ is called a **bump function**.

The delta method & bump functions

Definition (Bump function)

The space of real-valued, infinitely differentiable, and compactly supported functions on $\mathbb{R}$ is denoted by $C_c^\infty(\mathbb{R})$. A function $w \in C_c^\infty(\mathbb{R})$ is called a **bump function**.

Require $w(0) = 0$ and $\sum_{q=1}^{\infty} w(q) \neq 0$ for the delta method.

Definition (Bump function)

The space of real-valued, infinitely differentiable, and compactly supported functions on $\mathbb{R}$ is denoted by $C_c^\infty(\mathbb{R})$. A function $w \in C_c^\infty(\mathbb{R})$ is called a **bump function**.

Require $w(0) = 0$ and $\sum_{q=1}^{\infty} w(q) \neq 0$ for the delta method.

If n is an integer, then

$$\delta(n) = \frac{1}{\sum_{q=1}^{\infty} w(q)} \sum_{q \mid n} \left(w(q) - w\left(\frac{|n|}{q}\right)\right),$$

where the sum over $q \mid n$ is taken to be the sum over the positive divisors of n.

The delta method & bump functions

Using the fact that

$$\frac{1}{q} \sum_{a \pmod q} \mathrm{e}\left(\frac{an}{q}\right) = \begin{cases} 1 & \text{if } q \mid n, \\ 0 & \text{otherwise,} \end{cases}$$

The delta method & bump functions

Using the fact that

$$\frac{1}{q} \sum_{a \pmod{q}} \mathrm{e}\left(\frac{an}{q}\right) = \begin{cases} 1 & \text{if } q \mid n, \\ 0 & \text{otherwise,} \end{cases}$$

we have

$$\begin{aligned} \delta(n) &= \frac{1}{\sum_{q=1}^{\infty} w(q)} \sum_{q \mid n} \left(w(q) - w\left(\frac{|n|}{q}\right)\right) \\ &= \frac{1}{\sum_{q=1}^{\infty} w(q)} \sum_{q=1}^{\infty} \frac{1}{q} \sum_{a \pmod{q}} \mathrm{e}\left(\frac{an}{q}\right) \left(w(q) - w\left(\frac{|n|}{q}\right)\right) \end{aligned}$$

if n is an integer.

Some applications of the circle method

- Asymptotic formula for the partition function (Hardy–Ramanujan)

- Asymptotic formula for the partition function (Hardy–Ramanujan)
- Waring's problem (Vaughan, Wooley)

Some applications of the circle method

- Asymptotic formula for the partition function (Hardy–Ramanujan)
- Waring's problem (Vaughan, Wooley)
- Asymptotic formulas for the representation numbers of quadratic forms (Kloosterman, Heath-Brown)

Some applications of the circle method

- Asymptotic formula for the partition function (Hardy–Ramanujan)
- Waring's problem (Vaughan, Wooley)
- Asymptotic formulas for the representation numbers of quadratic forms (Kloosterman, Heath-Brown)
- Subconvexity bounds for (twists of) automorphic forms (Duke–Friedlander–Iwaniec, Munshi)

Thank you for listening!

Lemma for Kloosterman circle method

Lemma

Let $f : \mathbb{R} \to \mathbb{C}$ be a periodic function of period 1 and with real Fourier coefficients (so that $\overline{f(x)} = f(-x)$ for all $x \in \mathbb{R}$). Then

$$\int_0^1 f(x)\, dx = 2\,\mathrm{Re}\left(\sum_{1 \le q \le Q} \int_0^{\frac{1}{qQ}} \sum_{\substack{Q < d \le q+Q \\ qdx < 1 \\ \gcd(d,q)=1}} f\left(x - \frac{d^*}{q}\right)\, dx \right),$$

where d^* is the multiplicative inverse of d modulo q.

Use this for

$$f(x) = \sum_{\mathbf{m} \in \mathbb{Z}^s} \mathrm{e}((x+iy)(F(\mathbf{m}) - n)),$$

where $y > 0$.