

# The circle method

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  - Kloosterman circle method
  - The delta method

## Definition (Representation number)

$$R_F(n) = \#\{\mathbf{m} \in \mathbb{Z}^s : F(\mathbf{m}) = n\}$$

# Hardy-Littlewood circle method

- Originally developed by Hardy and Ramanujan (1918) to provide asymptotic formula for the partition function  $p(n)$ , the number of partitions of  $n$
- Proved that

$$p(n) \sim \frac{1}{4n\sqrt{3}} \exp\left(\pi\sqrt{\frac{2n}{3}}\right)$$

# Partition function & modularity

$$f(z) = \sum_{n=0}^{\infty} p(n)e(nz),$$

where  $e(z) = e^{2\pi iz}$  and  $\text{Im}(z) > 0$ .

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$$f(z) = \frac{e(z/24)}{\eta(z)},$$

where  $\eta(z)$  is the Dedekind eta function

$$\eta(z) = e\left(\frac{z}{24}\right) \prod_{m=1}^{\infty} (1 - e(mz)).$$

Hardy and Ramanujan used the modularity of  $\eta$  to obtain the asymptotic formula.



# Hardy-Littlewood circle method & partition function

$$f(z) = \sum_{n=0}^{\infty} p(n)e(nz)$$

$$F(q) = \sum_{n=0}^{\infty} p(n)q^n$$

where  $q = e(z)$ .

# Hardy-Littlewood circle method & partition function

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where  $q = e(z)$ .

Using the Cauchy integral formula, we find that

$$p(n) = \frac{1}{2\pi i} \int_{|q|=r} \frac{F(q)}{q^{n+1}} dq,$$

where  $0 < r < 1$ .

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Changing  $q$  into  $e(x + iy)$ , we obtain

$$p(n) = \int_0^1 f(x + iy)e(-n(x + iy)) dx,$$

where  $\text{Im}(y) > 0$ .

# Partition function

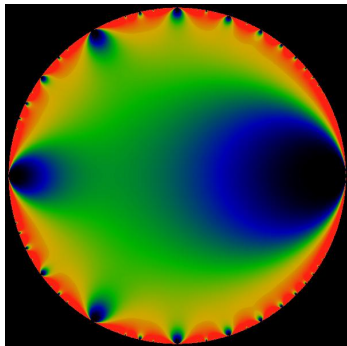


Figure: Modulus of  $\prod_{m=1}^{\infty} (1 - q^m)$  with  $|q| < 1$ . From Wikipedia.

Main contribution to integral from points near  $e(a/q)$  where  $q$  is small.

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Split  $[0, 1]$  into major arcs  $\mathfrak{M}$  and minor arcs  $\mathfrak{m}$ .

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$$\mathfrak{M} = \left\{ x \in [0, 1] : x \text{ is "close to"} \frac{a}{q}, a, q \in \mathbb{Z}, 0 < q \leq Q \right\}.$$

How close depends on the application of the method.

# Major arcs and minor arcs

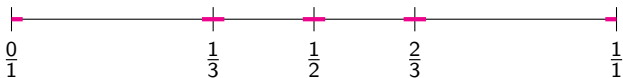
Split  $[0, 1]$  into major arcs  $\mathfrak{M}$  and minor arcs  $\mathfrak{m}$ .

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How close depends on the application of the method.

$$\mathfrak{m} = [0, 1] \setminus \mathfrak{M}.$$

Example of major arcs  $\mathfrak{M}$  when  $Q = 3$  for the Hardy-Littlewood circle method:



$$\begin{aligned}
 p(n) &= \int_0^1 f(x + iy)e(-n(x + iy)) \, dx \\
 &= \int_{\mathfrak{M}} f(x + iy)e(-n(x + iy)) \, dx + \int_{\mathfrak{m}} f(x + iy)e(-n(x + iy)) \, dx
 \end{aligned}$$



# Real quadratic forms

$F$  is a real quadratic form in  $s$  variables  $\iff$

For all  $\mathbf{m} \in \mathbb{R}^s$ ,

$$F(\mathbf{m}) = \frac{1}{2} \mathbf{m}^\top A \mathbf{m},$$

where  $A$  is a real symmetric  $s \times s$  matrix and is the Hessian matrix of  $F$ .

Example (Example of a quadratic form in 2 variables)

$$\begin{aligned} F(\mathbf{m}) &= m_1^2 + m_1 m_2 + m_2^2 \\ &= \frac{1}{2} \mathbf{m}^\top \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \mathbf{m} \end{aligned}$$

# Quadratic form definitions

## Definition (Integral quadratic form)

A quadratic form  $F$  is **integral** if  $F(\mathbf{m}) \in \mathbb{Z}$  for all  $\mathbf{m} \in \mathbb{Z}^s$ .

## Definition (Positive definite quadratic form)

A quadratic form  $F$  is **positive definite** if  $F(\mathbf{m}) > 0$  for all  $\mathbf{m} \in \mathbb{R}^s \setminus \{\mathbf{0}\}$ .

## Examples (Examples of integral positive definite quadratic forms)

- $f_4(\mathbf{m}) = m_1^2 + m_2^2 + m_3^2 + m_4^2$
- $x^2 + xy + y^2$

# Hardy-Littlewood circle method & quadratic forms

Want an asymptotic formula for  $R_F(n)$  when  $F$  is a positive definite quadratic form.

# Hardy-Littlewood circle method & quadratic forms

Want an asymptotic formula for  $R_F(n)$  when  $F$  is a positive definite quadratic form.

Use same overall method for obtaining an asymptotic formula for the partition function.

Note that the theta function

$$\Theta(z) = \sum_{n=0}^{\infty} R_F(n)e(nz)$$

is an automorphic form.

# Singular series $\mathfrak{S}_F(n)$

$$\mathfrak{S}_F(n) = \sum_{q=1}^{\infty} \frac{1}{q^s} \sum_{d \in (\mathbb{Z}/q\mathbb{Z})^\times} \sum_{\mathbf{h} \in (\mathbb{Z}/q\mathbb{Z})^s} e\left(\frac{d}{q} (F(\mathbf{h}) - n)\right)$$

The singular series  $\mathfrak{S}_F(n)$  contains information about  $F(\mathbf{m}) \equiv n \pmod{q}$  for all positive integers  $q$ .

$$\mathfrak{S}_F(n) = 0 \iff$$

there exists a positive integer  $q$  such that  $F(\mathbf{m}) \equiv n \pmod{q}$  has no solutions

# An asymptotic for representation numbers from Hardy-Littlewood circle method

## Theorem (Kloosterman, 1924)

*Suppose that  $n$  is a positive integer.*

*Suppose that  $F$  is a positive definite integral quadratic form in  $s \geq 5$  variables.*

*Let  $A \in M_s(\mathbb{Z})$  be the Hessian matrix of  $F$ .*

*Then the number of integral solutions to  $F(\mathbf{m}) = n$  is*

$$R_F(n) = \mathfrak{S}_F(n) \frac{(2\pi)^{s/2}}{\Gamma(s/2) \sqrt{|\det(A)|}} n^{\frac{s}{2}-1} + O_{F,\varepsilon} \left( n^{\frac{s}{4}+\varepsilon} + n^{\frac{s}{2}-\frac{5}{4}+\varepsilon} \right)$$

*for any  $\varepsilon > 0$ .*

# Motivation for the Kloosterman circle method

- Want a better error term in asymptotic formula for  $R_F(n)$  when  $F$  is a positive definite quadratic form.
- Split  $[0, 1]$  differently.

# Farey sequence $\mathfrak{F}_Q$ of order $Q$

## Definition

For  $Q \geq 1$ , the **Farey sequence  $\mathfrak{F}_Q$  of order  $Q$**  is the increasing sequence of all reduced fractions  $\frac{a}{q}$  with  $1 \leq q \leq Q$  and  $\gcd(a, q) = 1$ .

$$Q = 1$$



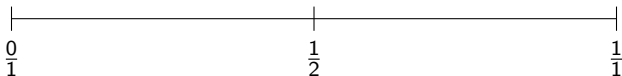


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$$Q = 2$$

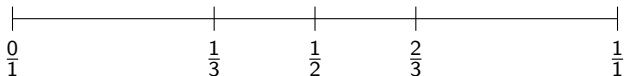


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$$Q = 3$$

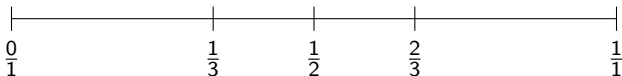


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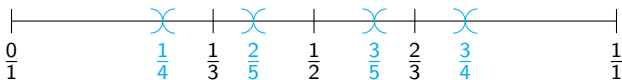
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$$Q = 3$$



Example of Farey dissection when  $Q = 3$ :



# An asymptotic for representation numbers from the Kloosterman method

## Theorem

*Suppose that  $n$  is a positive integer.*

*Suppose that  $F$  is a positive definite integral quadratic form in  $s \geq 4$  variables.*

*Let  $A \in M_s(\mathbb{Z})$  be the Hessian matrix of  $F$ .*

*Then the number of integral solutions to  $F(\mathbf{m}) = n$  is*

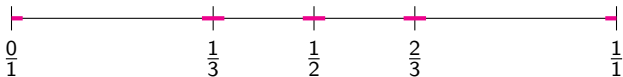
$$R_F(n) = \mathfrak{S}_F(n) \frac{(2\pi)^{s/2}}{\Gamma(s/2) \sqrt{|\det(A)|}} n^{\frac{s}{2}-1} + O_{F,\varepsilon} \left( n^{\frac{s-1}{4} + \varepsilon} \right)$$

*for any  $\varepsilon > 0$ .*

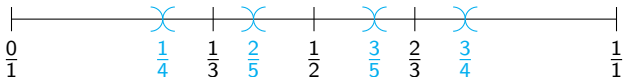
Kloosterman proved this (with a worse error term) in 1926 for diagonal quadratic forms ( $F(\mathbf{m}) = a_1 m_1^2 + \cdots + a_s m_s^2$ ), using what is now called the Kloosterman circle method.

# Hardy-Littlewood vs. Kloosterman

Example of major arcs when  $Q = 3$  for the Hardy-Littlewood circle method:



Example of Farey dissection when  $Q = 3$  for the Kloosterman circle method:



# Hardy-Littlewood vs. Kloosterman

Hardy-Littlewood:

$$R_F(n) = \mathfrak{S}_F(n) \frac{(2\pi)^{s/2}}{\Gamma(s/2)\sqrt{\det(A)}} n^{\frac{s}{2}-1} + O_{F,\varepsilon} \left( n^{\frac{s}{4}+\varepsilon} + n^{\frac{s}{2}-\frac{5}{4}+\varepsilon} \right)$$

Kloosterman:

$$R_F(n) = \mathfrak{S}_F(n) \frac{(2\pi)^{s/2}}{\Gamma(s/2)\sqrt{\det(A)}} n^{\frac{s}{2}-1} + O_{F,\varepsilon} \left( n^{\frac{s-1}{4}+\varepsilon} \right)$$

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# The delta method

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- Rewrite  $\delta(n)$ , the indicator function for zero, using bump functions
- More versatile than Kloosterman circle method
- Has been used for a variety of applications, including
  - Asymptotic formulas for the representation numbers of quadratic forms (Heath-Brown)
  - Subconvexity bounds for (twists of) automorphic forms (Munshi)

## Definition (Representation number)

$$R_F(n) = \#\{\mathbf{m} \in \mathbb{Z}^s : F(\mathbf{m}) = n\}$$

$$R_F(n) = \sum_{\mathbf{m} \in \mathbb{Z}^s} \mathbf{1}_{\{F(\mathbf{m})=n\}},$$

where  $\mathbf{1}_{\{F(\mathbf{m})=n\}}$  is the indicator function

$$\mathbf{1}_{\{F(\mathbf{m})=n\}} = \begin{cases} 1 & \text{if } F(\mathbf{m}) = n, \\ 0 & \text{otherwise.} \end{cases}$$

# Indicator function

$$\delta(n) = \mathbf{1}_{\{n=0\}} = \begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{otherwise.} \end{cases}$$

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$$R_F(n) = \sum_{\mathbf{m} \in \mathbb{Z}^s} \delta(F(\mathbf{m}) - n)$$

# The delta method & bump functions

## Definition (Bump function)

The space of real-valued, infinitely differentiable, and compactly supported functions on  $\mathbb{R}$  is denoted by  $C_c^\infty(\mathbb{R})$ . A function  $w \in C_c^\infty(\mathbb{R})$  is called a **bump function**.

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Require  $w(0) = 0$  and  $\sum_{q=1}^{\infty} w(q) \neq 0$  for the delta method.

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Require  $w(0) = 0$  and  $\sum_{q=1}^{\infty} w(q) \neq 0$  for the delta method.

If  $n$  is an integer, then

$$\delta(n) = \frac{1}{\sum_{q=1}^{\infty} w(q)} \sum_{q|n} \left( w(q) - w\left(\frac{|n|}{q}\right) \right),$$

where the sum over  $q | n$  is taken to be the sum over the positive divisors of  $n$ .



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Thank you for listening!

# Lemma for Kloosterman circle method

## Lemma

Let  $f: \mathbb{R} \rightarrow \mathbb{C}$  be a periodic function of period 1 and with real Fourier coefficients (so that  $\overline{f(x)} = f(-x)$  for all  $x \in \mathbb{R}$ ). Then

$$\int_0^1 f(x) dx = 2 \operatorname{Re} \left( \sum_{1 \leq q \leq Q} \int_0^{\frac{1}{qQ}} \sum_{\substack{Q < d \leq q+Q \\ qdx < 1 \\ \gcd(d,q)=1}} f\left(x - \frac{d^*}{q}\right) dx \right),$$

where  $d^*$  is the multiplicative inverse of  $d$  modulo  $q$ .

Use this for

$$f(x) = \sum_{\mathbf{m} \in \mathbb{Z}^s} e((x + iy)(F(\mathbf{m}) - n)),$$

where  $y > 0$ .