Apollonian circle packings, integers, and higher-dimensional sphere packings

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Colloquium
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Given three pairwise tangent circles with disjoint points of tangency, there are exactly two circles tangent to the given ones. (Proved by Apollonius of Perga.)
Apollonian circle packings: The construction
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Figure: An Apollonian circle packing.
Apollonian circle packings: The construction

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Figure: An Apollonian circle packing.
Apollonian circle packings

Label on circle: bend = 1/radius

Figure: An Apollonian circle packing.
Apollonian circle packings

Figure: An Apollonian circle packing.

Label on circle:
\[ \text{bend} = \frac{1}{\text{radius}} \]

What do you notice about the bends that you can see in this Apollonian circle packing?

Figure: An Apollonian circle packing.
Apollonian circle packings

Label on circle: $\text{bend} = \frac{1}{\text{radius}}$

What do you notice about the bends that you can see in this Apollonian circle packing?

They are all integers.

Figure: An Apollonian circle packing.
Label on circle: \( \text{bend} = \frac{1}{\text{radius}} \)

What do you notice about the bends that you can see in this Apollonian circle packing?

They are all integers.

Why?
Four circles to the kissing come.
The smaller are the benter.
The bend is just the inverse of
The distance from the centre.
Though their intrigue left Euclid dumb
There’s now no need for rule of thumb.

Since zero bend’s a dead straight line
And concave bends have minus sign,
The sum of the squares of all four bends
Is half the square of their sum.

Figure: An excerpt of “The Kiss Precise” by F. Soddy in *Nature*, 1936.
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If $b_1, b_2, b_3, b_4$ are bends of four pairwise tangent circles, then

$$b_1^2 + b_2^2 + b_3^2 + b_4^2 = \frac{1}{2} (b_1 + b_2 + b_3 + b_4)^2.$$
Descartes circle theorem

Theorem (Descartes circle theorem, 1643)

If $b_1, b_2, b_3, b_4$ are bends of four pairwise tangent circles, then

$$(b_1 + b_2 + b_3 + b_4)^2 = 2(b_1^2 + b_2^2 + b_3^2 + b_4^2).$$
Theorem (Descartes circle theorem, 1643)

If $b_1, b_2, b_3, b_4$ are bends of four pairwise tangent circles, then

$$(b_1 + b_2 + b_3 + b_4)^2 = 2(b_1^2 + b_2^2 + b_3^2 + b_4^2).$$

Example

$b_1 = 0, b_2 = b_3 = 1, b_4 = 4$
Theorem (Descartes circle theorem, 1643)

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$$(b_1 + b_2 + b_3 + b_4)^2 = 2(b_1^2 + b_2^2 + b_3^2 + b_4^2).$$

Example

$b_1 = 0$, $b_2 = b_3 = 1$, $b_4 = 4$

$(0 + 1 + 1 + 4)^2 = 6^2 = 36$
Descartes circle theorem

**Theorem (Descartes circle theorem, 1643)**

*If* $b_1, b_2, b_3, b_4$ *are bends of four pairwise tangent circles, then*

$$(b_1 + b_2 + b_3 + b_4)^2 = 2(b_1^2 + b_2^2 + b_3^2 + b_4^2).$$

**Example**

$b_1 = 0, b_2 = b_3 = 1, b_4 = 4$

$$(0 + 1 + 1 + 4)^2 = 6^2 = 36$$

$2(0^2 + 1^2 + 1^2 + 4^2) = 2(18) = 36$$
Descartes circle theorem

**Theorem (Descartes circle theorem, 1643)**

If $b_1, b_2, b_3, b_4$ are bends of four pairwise tangent circles, then

$$(b_1 + b_2 + b_3 + b_4)^2 = 2(b_1^2 + b_2^2 + b_3^2 + b_4^2).$$

**Example**

- $b_1 = -11$, $b_2 = 21$, $b_3 = 24$, $b_4 = 28$
Descartes circle theorem

Theorem (Descartes circle theorem, 1643)

If $b_1, b_2, b_3, b_4$ are bends of four pairwise tangent circles, then

$$(b_1 + b_2 + b_3 + b_4)^2 = 2(b_1^2 + b_2^2 + b_3^2 + b_4^2).$$

Example

$\begin{align*}
&b_1 = -11, \quad b_2 = 21, \quad b_3 = 24, \quad b_4 = 28 \\
&(-11 + 21 + 24 + 28)^2 = 62^2 = 3844
\end{align*}$
Descartes circle theorem

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If $b_1, b_2, b_3, b_4$ are bends of four pairwise tangent circles, then

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Example

$b_1 = -11, b_2 = 21, b_3 = 24, b_4 = 28$

$$(−11 + 21 + 24 + 28)^2 = 62^2 = 3844$$

$2((-11)^2 + 21^2 + 24^2 + 28^2) = 2(1922) = 3844$$
Theorem (Descartes Circle Theorem, 1643)

If $b_1, b_2, b_3, b_4$ are bends of four pairwise tangent circles, then

$$(b_1 + b_2 + b_3 + b_4)^2 = 2(b_1^2 + b_2^2 + b_3^2 + b_4^2).$$

Fix $b_1, b_2, b_3$. What do I know about the solutions to $b_4$?
Descartes circle theorem

Theorem (Descartes Circle Theorem, 1643)

If $b_1, b_2, b_3, b_4$ are bends of four pairwise tangent circles, then

$$(b_1 + b_2 + b_3 + b_4)^2 = 2(b_1^2 + b_2^2 + b_3^2 + b_4^2).$$

Fix $b_1, b_2, b_3$. What do I know about the solutions to $b_4$?

If $b_4$ and $b'_4$ are solutions for fixed $b_1, b_2, b_3$, then, by the quadratic formula,

$$b_4 + b'_4 = 2(b_1 + b_2 + b_3).$$
Matrices and geometry

\[ b'_4 = 2b_1 + 2b_2 + 2b_3 - b_4 \]

Matrix form:

\[
\begin{pmatrix}
  b_1 \\
  b_2 \\
  b_3 \\
  b'_4 \\
\end{pmatrix} =
\begin{pmatrix}
  1 & 1 & \\
  1 & 1 & 1 \\
  2 & 2 & 2 \\
  2 & 2 & -1 \\
\end{pmatrix}
\begin{pmatrix}
  b_1 \\
  b_2 \\
  b_3 \\
  b_4 \\
\end{pmatrix}
\]

\[ M_4 \]
Matrices and geometry

\[ b'_4 = 2b_1 + 2b_2 + 2b_3 - b_4 \]

Matrix form:

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\begin{pmatrix}
  b_1 \\
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  b_3 \\
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\end{pmatrix} =
\begin{pmatrix}
  1 & 1 \\
  2 & 2 & 2 & -1 \\
\end{pmatrix}
\begin{pmatrix}
  b_1 \\
  b_2 \\
  b_3 \\
  b_4 \\
\end{pmatrix}
\]

\( M_4 \)

Figure: Four tangent circles and a reflection to a fifth circle.
Matrices and the Apollonian group

$M_1 = \begin{pmatrix} -1 & 2 & 2 & 2 \\ 1 & 1 & 1 \\ 2 & 2 & -1 & 2 \\ 1 & 2 & 2 & 1 \end{pmatrix}$,  

$M_2 = \begin{pmatrix} 1 & 2 & -1 & 2 \\ 2 & 1 & 1 \end{pmatrix}$,  

$M_3 = \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}$,  

$M_4 = \begin{pmatrix} 1 & 1 \\ 2 & 2 & 2 & -1 \end{pmatrix}$.

The Apollonian group $\Gamma := \langle M_1, M_2, M_3, M_4 \rangle$ (set of products of $M_1, M_2, M_3, M_4$) maps bends of an Apollonian circle packing to more bends of the packing, "generates" all bends of the packing from four bends, and sends integer vectors to integer vectors.
Matrices and the Apollonian group

\[ M_1 = \begin{pmatrix} -1 & 2 & 2 & 2 \\ 1 & 1 & 1 \\ \end{pmatrix}, \quad M_2 = \begin{pmatrix} 1 & -1 & 2 & 2 \\ 2 & 1 & 1 \\ \end{pmatrix}, \]
\[ M_3 = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & -1 \\ \end{pmatrix}, \quad M_4 = \begin{pmatrix} 1 \\ 2 & 2 & 2 \\ 1 \end{pmatrix}. \]

The **Apollonian group** \( \Gamma := \langle M_1, M_2, M_3, M_4 \rangle \) (set of products of \( M_1, M_2, M_3, M_4 \))

- maps bends of an Apollonian circle packing to more bends of the packing,
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- maps bends of an Apollonian circle packing to more bends of the packing,
- “generates” all bends of the packing from four bends, and
- sends integer vectors to integer vectors.

Since we started with an integer vector of bends (namely, $(-11, 21, 24, 28)^\top$), **all of our bends are integers!**
The **Apollonian group** $\Gamma := \langle M_1, M_2, M_3, M_4 \rangle$

- maps bends of an Apollonian circle packing to more bends of the packing,
- "generates" all bends of the packing from four bends, and
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Since we started with an integer vector of bends (namely, $(-11, 21, 24, 28)^T$), all of our bends are integers!

Which integers appear as bends?
The **Apollonian group** \( \Gamma := \langle M_1, M_2, M_3, M_4 \rangle \)

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Since we started with an integer vector of bends (namely, \((-11, 21, 24, 28)^\top\)), all of our bends are integers!

Which integers appear as bends?

Are there any congruence or local obstructions?
Theorem (Fuchs, 2011)

For an integral, primitive Apollonian circle packing, there are local obstructions modulo 24 for the bends of the packing.
(The local obstructions depend on the packing.)

Example

Each bend is given by
\[\equiv 0, 4, 12, 13, 16, \text{ or } 21 \pmod{24}.\]
Admissible integers

Definition (Admissible integers for Apollonian circle packings)

Let $\mathcal{P}$ be an integral Apollonian circle packing.

An integer $m$ is **admissible** (or **locally represented**) if for every $q \geq 1$

$$m \equiv \text{bend of some circle in } \mathcal{P} \pmod{q}.$$

Equivalently, $m$ is admissible if $m$ has no local obstructions to being the bend of a circle in the packing.
Theorem (Fuchs, 2011)

$m$ is admissible if and only if $m$ is in certain congruence classes modulo 24.
(The congruence classes depend on the packing.)

Example

$m$ is admissible $\iff m \equiv 0, 4, 12, 13, 16, \text{ or } 21 \pmod{24}.$
Strong asymptotic local-global conjecture

Conjecture (Graham–Lagarias–Mallows–Wilks–Yan, 2003)

The bends of a fixed primitive, integral Apollonian circle packing $\mathcal{P}$ satisfy a strong asymptotic local-global principle. That is, there is an $N_0 = N_0(\mathcal{P})$ so that, if $m > N_0$ and $m$ is admissible, then $m$ is the bend of a circle in the packing.

Example

We think that if $m \equiv 0, 4, 12, 13, 16, \text{ or } 21 \pmod{24}$ and $m$ is sufficiently large, then $m$ is the bend of a circle in the packing.

We do not have a proof of this!
Why do we have a strong asymptotic local-global conjecture?

**Theorem (Kontorovich–Oh, 2011)**

The number of circles in an Apollonian circle packing $\mathcal{P}$ with bend at most $N$ (counted with multiplicity) is asymptotically equal to a constant times $N^\delta$, where $\delta = \text{the Hausdorff dimension of the closure of } \mathcal{P}$.

For Apollonian circle packings, we have

$$\delta \approx 1.30568 \ldots$$

Thus, we would expect the multiplicity of a given admissible bend up to $N$ to be roughly $N^{\delta-1} \approx N^{0.30568} \geq 1$, so we should expect every sufficiently large admissible number to be represented.
Theorem (Bourgain–Kontorovich, 2014)

Almost every admissible number is the bend of a circle in the Apollonian circle packing $\mathcal{P}$. Quantitatively, the number of exceptions up to $N$ is bounded above by $cN^{1-\eta}$, where $c, \eta > 0$ are constants only dependent on the packing.
The best we can do right now

Theorem (Bourgain–Kontorovich, 2014)

Almost every admissible number is the bend of a circle in the Apollonian circle packing \( \mathcal{P} \). Quantitatively, the number of exceptions up to \( N \) is bounded above by \( cN^{1-\eta} \), where \( c, \eta > 0 \) are constants only dependent on the packing.

Extended by Fuchs, Stange, and Zhang to certain other circle packings.
Given four pairwise tangent spheres with disjoint points of tangency, there are exactly two spheres tangent to the given ones.

**Figure:** Four tangent spheres.

**Figure:** Four tangent spheres with two additional tangent spheres.
Soddy sphere packings: The construction

**Figure:** Four tangent spheres.

**Figure:** Four tangent spheres with two more spheres.
Soddy sphere packings: The construction

**Figure:** Four tangent spheres.

**Figure:** Four tangent spheres with two more spheres.

**Figure:** More spheres.
Soddy sphere packings: The construction

Figure: Four tangent spheres.

Figure: Four tangent spheres with two more spheres.

Figure: More spheres.

Figure: A Soddy sphere packing.
Soddy sphere packings

Figure: A Soddy sphere packing.

Label on sphere:
\[
\text{bend} = \frac{1}{\text{radius}}
\]

What do you notice about the bends that you can see in this Soddy sphere packing?

Figure: A Soddy sphere packing.
Soddy sphere packings

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Label on sphere:
\[ \text{bend} = \frac{1}{\text{radius}} \]

What do you notice about the bends that you can see in this Soddy sphere packing?

They are all integers.

Why?
To spy out spherical affairs
An oscular surveyor
Might find the task laborious,
The sphere is much the gayer,
And now besides the pair of pairs
A fifth sphere in the kissing shares.
Yet, signs and zero as before,
For each to kiss the other four
The square of the sum of all five bends
Is thrice the sum of their squares.

F. SODDY.

Figure: The last stanza of “The Kiss Precise” by F. Soddy in *Nature*, 1936.
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F. Soddy.

If \( b_1, b_2, b_3, b_4, b_5 \) are bends of five pairwise tangent spheres, then

\[
(b_1 + b_2 + b_3 + b_4 + b_5)^2 = 3(b_1^2 + b_2^2 + b_3^2 + b_4^2 + b_5^2).
\]

Figure: The last stanza of “The Kiss Precise” by F. Soddy in *Nature*, 1936.
If $b_1, b_2, b_3, b_4, b_5$ are bends of five pairwise tangent spheres, then

$$(b_1 + b_2 + b_3 + b_4 + b_5)^2 = 3(b_1^2 + b_2^2 + b_3^2 + b_4^2 + b_5^2).$$

Fix $b_1, b_2, b_3, b_4$. What do I know about the solutions to $b_5$?
If $b_1, b_2, b_3, b_4, b_5$ are bends of five pairwise tangent spheres, then

$$(b_1 + b_2 + b_3 + b_4 + b_5)^2 = 3(b_1^2 + b_2^2 + b_3^2 + b_4^2 + b_5^2).$$

Fix $b_1, b_2, b_3, b_4$. What do I know about the solutions to $b_5$?

If $b_5$ and $b'_5$ are solutions for fixed $b_1, b_2, b_3, b_4$, then, by the quadratic formula,

$$b_5 + b'_5 = b_1 + b_2 + b_3 + b_4.$$
Matrices and geometry

\[ b'_5 = b_1 + b_2 + b_3 + b_4 - b_5 \]

Matrix form:

\[
\begin{pmatrix}
  b_1 \\
  b_2 \\
  b_3 \\
  b_4 \\
  b'_5
\end{pmatrix} = 
\begin{pmatrix}
  1 & & & & \\
  & 1 & & & \\
  & & 1 & & \\
  & & & 1 & \\
  & & & & -1
\end{pmatrix}
\begin{pmatrix}
  b_1 \\
  b_2 \\
  b_3 \\
  b_4 \\
  b_5
\end{pmatrix}
\]

\[ M_5 \]

Figure: Five tangent spheres and a reflection to a sixth sphere.
Matrices and geometry

\[ b'_5 = b_1 + b_2 + b_3 + b_4 - b_5 \]

Matrix form:

\[
\begin{pmatrix}
  b_1 \\
  b_2 \\
  b_3 \\
  b_4 \\
  b'_5
\end{pmatrix}
= \begin{pmatrix}
  1 & 1 & 1 & 1 & 1 \\
  1 & 1 & 1 & 1 & -1 \\
\end{pmatrix}
\begin{pmatrix}
  b_1 \\
  b_2 \\
  b_3 \\
  b_4 \\
  b_5
\end{pmatrix}
\]

\[ M_5 \]

Figure: Five tangent spheres and a reflection to a sixth sphere.
Matrices and the Soddy group

\[ M_1 = \begin{pmatrix} -1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} , \quad M_2 = \begin{pmatrix} 1 & -1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} , \quad M_3 = \begin{pmatrix} 1 & 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 & 1 \end{pmatrix} , \quad M_4 = \begin{pmatrix} 1 & 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 & 1 \end{pmatrix} , \quad M_5 = \begin{pmatrix} 1 & 1 & 1 & 1 & -1 \\ 1 & 1 & 1 & 1 & -1 \\ 1 & 1 & 1 & 1 & -1 \\ 1 & 1 & 1 & 1 & -1 \\ 1 & 1 & 1 & 1 & -1 \end{pmatrix} . \]
Matrices and the Soddy group

\[
M_1 = \begin{pmatrix}
-1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1
\end{pmatrix},
M_2 = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 \\
1 & -1 & 1 & 1 & 1 \\
1 & 1 & -1 & 1 & 1 \\
1 & 1 & 1 & -1 & 1 \\
1 & 1 & 1 & 1 & 1
\end{pmatrix},
M_3 = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1
\end{pmatrix},
M_4 = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1
\end{pmatrix},
M_5 = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1
\end{pmatrix},
\]

The **Soddy group** \( \Gamma := \langle M_1, M_2, M_3, M_4, M_5 \rangle \) (set of products of \( M_1, M_2, M_3, M_4, M_5 \))

- maps bends of a Soddy sphere packing to more bends of the packing,
- “generates” all bends of the packing from five bends, and
- sends integer vectors to integer vectors.
The Soddy group $\Gamma := \langle M_1, M_2, M_3, M_4, M_5 \rangle$
- maps bends of a Soddy sphere packing to more bends of the packing,
- “generates” all bends of the packing from five bends, and
- sends integer vectors to integer vectors.

Since we started with an integer vector of five bends (namely, $(-11, 21, 25, 27, 28)^T$), all of our bends are integers!
Integrality of bends

The **Soddy group** $\Gamma := \langle M_1, M_2, M_3, M_4, M_5 \rangle$

- maps bends of a Soddy sphere packing to more bends of the packing,
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Which integers appear as bends?
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Since we started with an integer vector of five bends (namely, $(-11, 21, 25, 27, 28)^T$), all of our bends are integers!

Which integers appear as bends?

Are there any congruence or local obstructions?
Lemma (Kontorovich, 2019)

For an integral, primitive Soddy sphere packing $\mathcal{P}$, there is an $\varepsilon(\mathcal{P}) \in \{1, 2\}$ such that each bend of the packing is

$$\equiv 0 \text{ or } \varepsilon(\mathcal{P}) \pmod{3}.$$
Definition (Admissible integers for Soddy sphere packings)

Let $\mathcal{P}$ be an integral Soddy sphere packing. An integer $m$ is **admissible (or locally represented)** if for every $q \geq 1$

$$m \equiv \text{bend of some sphere in } \mathcal{P} \pmod{q}.$$ 

Equivalently, $m$ is admissible if $m$ has no local obstructions to being the bend of a sphere in the packing.
Theorem (Kontorovich, 2019)

$m$ is admissible in a primitive integral Soddy sphere packing $\mathcal{P}$ if and only if

$$m \equiv 0 \text{ or } \varepsilon(\mathcal{P}) \pmod{3},$$

where $\varepsilon(\mathcal{P}) \in \{1, 2\}$ depends only on the packing.

Example

$m$ is admissible $\iff$

$$m \equiv 0 \text{ or } 1 \pmod{3}.$$
Theorem (Kontorovich, 2019)

The bends of a fixed primitive integral Soddy sphere packing $\mathcal{P}$ satisfy a strong asymptotic local-global principle. That is, there is an $N_0 = N_0(\mathcal{P})$ so that, if $m > N_0$ and $m$ is admissible, then $m$ is the bend of a sphere in the packing.

Example

If $m \equiv 0$ or $1 \pmod{3}$ and $m$ is sufficiently large, then $m$ is the bend of a sphere in the packing.
Why should we have a strong asymptotic local-global principle?

**Theorem (Kim, 2015)**

Let \( \mathcal{P} \) be a certain type of \((n-1)\)-sphere packing (called a Kleinian sphere packing) in dimension \( n \geq 2 \).

The number of spheres in \( \mathcal{P} \) with bend at most \( N \) (counted with multiplicity) is asymptotically equal to a constant times \( N^\delta \), where \( \delta = \) the Hausdorff dimension of the closure of \( \mathcal{P} \).

For Soddy sphere packings, we have

\[
\delta \approx 2.4739 \ldots
\]

Thus, we would expect the multiplicity of a given admissible bend up to \( N \) to be roughly \( N^{\delta-1} \approx N^{1.4739} \geq 1 \), so we should expect every sufficiently large admissible number to be represented.
**Goal:** Prove strong asymptotic local-global principles for certain integral Kleinian sphere packings, that is, prove:

If \( m \) is admissible and sufficiently large, then \( m \) is the bend of an \((n - 1)\)-sphere in the packing.

**Definition (Admissible integers)**

Let \( \mathcal{P} \) be an integral Kleinian sphere packing. An integer \( m \) is **admissible (or locally represented)** if for every \( q \geq 1 \)

\[
m \equiv \text{bend of some } (n - 1)\text{-sphere in } \mathcal{P} \pmod{q}.
\]
My original dissertation research problem

**Goal:** Prove strong asymptotic local-global principles for bends of certain integral Kleinian sphere packings in dimension at least 3.

**Figure:** An integral Soddy sphere packing. Image by Nicolas Hannachi.

**Figure:** An integral Kleinian (more specifically, an orthoplicial) sphere packing. Image by Kei Nakamura.

**Figure:** A fundamental domain of an integral Kleinian sphere packing. Image by Arseniy (Senia) Sheydvasser.
I got stuck with my original dissertation problem.

I developed a tool that could potentially be used to prove strong asymptotic local-global principles for bends of certain integral Kleinian sphere packings.

This tool is a technical version of the circle method. The circle method is used to provide an asymptotic formula for the number of ways an integer is represented by an integer-valued function on $\mathbb{Z}$ (such as an integral quadratic form).
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The circle method is used to provide an asymptotic formula for the number of ways an integer is represented by an integer-valued function on $\mathbb{Z}^r$ (such as an integral quadratic form).
Besides the illustrations previously credited and a few Apollonian circle packing construction illustrations created by the presenter, the illustrations for this talk came from the following papers:


Thank you for listening!
Show that the automorphism group of the Apollonian circle packing contains the congruence subgroup $\Gamma(2)$ of $\text{PSL}_2(\mathbb{Z})$, and $\Gamma(2)$ is the stabilizer of a particular circle. This implies that the set of bends contains primitive values of a shifted binary (2-variable) quadratic form. (Sarnak, 2007)
Proof outline for Bourgain’s and Kontorovich’s Apollonian circle packing result

1. Show that the automorphism group of the Apollonian circle packing contains the congruence subgroup $\Gamma(2)$ of $\text{PSL}_2(\mathbb{Z})$, and $\Gamma(2)$ is the stabilizer of a particular circle. This implies that the set of bends contains primitive values of a shifted binary (2-variable) quadratic form. (Sarnak, 2007)

2. The shifted binary quadratic form gives you enough to work with so that you can apply the circle method (with some other tools, including spectral theory, used in the major arc and minor arc analyses) to obtain an “almost all” statement.
Show that the Soddy group contains a congruence subgroup of $\text{PSL}_2(\mathbb{Z}[e^{\pi i/3}])$, and this congruence subgroup maps a particular sphere to itself. This implies that the set of bends contains “primitive” values of a shifted **quaternary (4-variable)** quadratic form.
Proof outline for Kontorovich’s Soddy sphere packing result

1. Show that the Soddy group contains a congruence subgroup of $\text{PSL}_2(\mathbb{Z}[e^{\pi i/3}])$, and this congruence subgroup maps a particular sphere to itself. This implies that the set of bends contains “primitive” values of a shifted quaternary (4-variable) quadratic form.

2. The shifted quaternary quadratic form gives you enough to work with so that you can quote the circle method to show that every sufficiently large admissible number is represented by the quadratic form without the primitivity restriction.
Proof outline for Kontorovich’s Soddy sphere packing result

1. Show that the Soddy group contains a congruence subgroup of $\text{PSL}_2(\mathbb{Z}[e^{\pi i/3}])$, and this congruence subgroup maps a particular sphere to itself. This implies that the set of bends contains “primitive” values of a shifted quaternary (4-variable) quadratic form.

2. The shifted quaternary quadratic form gives you enough to work with so that you can quote the circle method to show that every sufficiently large admissible number is represented by the quadratic form without the primitivity restriction.

3. Show that the singular series (with the primitivity restriction) is bounded away from zero when $m$ is admissible.
Definition (Kleinian sphere packing)

An \((n - 1)\)-sphere packing \(\mathcal{P}\) is **Kleinian** if its limit set is that of a geometrically finite group \(\Gamma < \text{Isom}(\mathcal{H}^{n+1})\).

**Figure:** Apollonian circle packing as the limit set of \(\Gamma\).
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- Action of \(\text{Isom}(\mathcal{H}^{n+1})\) extends continuously to \(\mathbb{R}^n = \mathbb{R}^n \cup \{\infty\}\), the boundary of \(\mathcal{H}^{n+1}\).
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An \((n-1)\)-sphere packing \(\mathcal{P}\) is **Kleinian** if its limit set is that of a geometrically finite group \(\Gamma < \text{Isom}(\mathcal{H}^{n+1})\).

- Action of \(\text{Isom}(\mathcal{H}^{n+1})\) extends continuously to \(\widetilde{\mathbb{R}^n} = \mathbb{R}^n \cup \{\infty\}\), the boundary of \(\mathcal{H}^{n+1}\).
- \(\Gamma\) stabilizes \(\mathcal{P}\) (i.e., \(\Gamma\) maps \(\mathcal{P}\) to itself).