Math 230, Fall 2012: HW 9 Solutions

Problem 1 (p.345 #4)**.** Let X and Y be independent random variables each uniformly distributed on $(0,1)$. Find:

a) $P(|X-Y| \leq 0.25)$;

b) $P(|X/Y-1| \leq 0.25)$;

c) $P(Y \geq X \mid Y \geq 0.25)$.

SOLUTION. These problems are most easily solved by drawing pictures and computing the areas of the triangular regions that you see. For part (a) the answer is $1-2(\frac{1}{2})(\frac{3}{4})(\frac{3}{4}) = \frac{7}{16}$ (we did this as an example in class). For part (b) the answer is $1-\frac{1}{2}(1)(\frac{4}{5}) - \frac{1}{2}(1)(\frac{3}{4}) = \frac{9}{40}$. For part (c),

$$P(Y \geq X, Y \geq 0.25) = P(Y \geq X) - P(Y \geq X, Y < 0.25) = \frac{1}{2} - \frac{1}{2}(\frac{1}{4})(\frac{1}{4}) = \frac{15}{32}$$

so $P(Y \geq X \mid Y \geq 0.25) = \frac{15/32}{3/4} = \frac{15}{24}$.

Problem 2 (p.345 #7)**.** Let X and Y be two independent Uniform$(0,1)$ random variables. Let $M = \min\{X, Y\}$. Let $0 < x < 1$.

a) Represent the event $\{M \geq x\}$ as a region in the plane, and find $P(M \geq x)$ as the area of this region.

b) Use your result in (a) to find the c.d.f. and density of M. Sketch the graph of these functions.

SOLUTION. For the event $\{M \geq x\}$ to occur, both X and Y must be at least x. The region in the plane that corresponds to this event is the square with corners at (x,x), $(x,1)$, $(1,x)$, and $(1,1)$. The area of this region is $(1-x)^2$, so this is the probability $P(M \geq x)$. If F is the c.d.f. of M, then $F(x) = P(M \leq x) = 1-(1-x)^2$. The density of M is $f(x) = F'(x) = 2(1-x)$ for $x \in (0,1)$.

Problem 3 (p.345 #9)**.** Suppose a straight stick is broken in three at two points chosen independently at random along its length. What is the chance that the three sticks so formed can be made into the sides of a triangle?

SOLUTION. Let X and Y be the locations of the breaks, so $X, Y \sim$Uniform(0,1) are independent. The three sticks to form a triangle if and only if their lengths satisfy the triangle inequalities. That is, the sum of the lengths of any two sticks must be larger than the third. First suppose $X < Y$, so the lengths of the sticks are X, $Y-X$ and $1-Y$. In this case, we want to satisfy the inequalities:

$$\begin{aligned} X + (Y-X) &> 1-Y \\ X + (1-Y) &> Y-X \\ (Y-X)+(1-Y) &> X \end{aligned}$$

simplifying these expressions gives

$$\begin{aligned} Y &> \frac{1}{2} \\ Y &< X + \frac{1}{2} \\ X &< \frac{1}{2}. \end{aligned}$$

When $Y < X$, we get the same three inequalities except with the rolls of X and Y reversed, and the probabilities are the same. Therefore, if Δ is the event that a triangle can be formed from the three sticks

$$P(\Delta) = 2 \cdot P\left(X < \frac{1}{2} < Y < X + \frac{1}{2}\right) = 2\left(\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2}\right) = \frac{1}{4}$$

since the region enclosed by these inequalities is an isosceles right triangle with side $1/2$.

Problem 4 (p.354 #2)**.** Suppose (X, Y) is uniformly distributed over the region $\{(x, y)\ :\ 0 < |x| + |y| < 1\}$. Find:

a) the joint density of (X, Y);

b) the marginal densities $f_X(x)$ and $f_Y(y)$.

c) Are X and Y independent?

d) Find EX and EY.

SOLUTION. The region is a square rotated by 45 degrees with corners at $(\pm 1, 0)$ and $(0, \pm 1)$. The area of this square region is 2, so the density is

$$f(x, y) = \begin{cases} \frac{1}{2} & 0 < |x| + |y| < 1 \\ 0 & \text{else.} \end{cases}$$

To find the marginal of X, we compute

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f(x, y) dy \\ &= \int_{|x|-1}^{1-|x|} \frac{1}{2} dy \\ &= 1 - |x| \end{aligned}$$

for $x = \in [-1, 1]$ and 0 otherwise. Likewise, $f_Y(y) = 1 - |y|$ for $y \in [-1, 1]$ and 0 otherwise. Clearly X and Y are not independent. For example, $f(0, 0) = \frac{1}{2}$ but $f_X(0) f_Y(0) = 1$.

The expected values of X and Y can be easily computed, but since both marginal densities are even functions (symmetric about the vertical axis through 0), $EX = EY = 0$.

Problem 5 (p. 354 #9)**.** Let $X = \min(S, T)$ and $Y = \max(S, T)$ where S and T are independent $\text{Exp}(\lambda)$ random variables. Let $Z = Y - X$.

a) Find the joint density of X and Y. Are X and Y independent? [*Hint: Consider the probability $P(s < X < Y < t)$ expressed as a double integral of the joint density. What happens when you differentiate this by s then by t?*]

b) Find the joint density of X and Z. Are X and Z independent?

c) Identify the marginal distributions of X and Z.

SOLUTION. Using the hint, we find that $P(s < X < Y < t) = (e^{-\lambda s} - e^{-\lambda t})^2$ for any $0 < s < t$. If $f(x, y)$ is the joint density of X and Y, then

$$P(s < X < Y < t) = \int_s^t \int_s^y f(x, y)dxdy.$$

Differentiating first by t then by s gives

$$\begin{aligned}
\frac{d}{ds}\left[\frac{d}{dt}\int_s^t \int_s^y f(x, y)dxdy\right] &= \frac{d}{ds}\left[\frac{d}{dt}(e^{-\lambda s} - e^{-\lambda t})^2\right] \\
\frac{d}{ds}\int_s^t f(x, t)dx &= \frac{d}{ds}\left[2\lambda e^{-\lambda t}(e^{-\lambda s} - e^{-\lambda t})\right] \\
-f(s, t) &= -2\lambda^2 e^{-\lambda t} e^{-\lambda s} \\
f(s, t) &= 2\lambda^2 e^{-\lambda(t+s)}.
\end{aligned}$$

X and Y are not independent since $f_X(s)f_Y(t) = \lambda^2 e^{-\lambda(t+s)}$ when $s > 0$ and $t > 0$.

There are several ways to go about part (b). For instance, for $s, t > 0$ we can use part (a) to compute

$$\begin{aligned}
P(X > s,\ Z > t) &= P(X > s,\ Y > X + t) \\
&= \int_s^\infty \int_{x+t}^\infty 2\lambda^2 e^{-\lambda(x+y)} dydx \\
&= \int_s^\infty \int_t^\infty g(x, z)dzdx
\end{aligned}$$

where $g(x, z)$ is the joint density of X and Z. Differentiating twice gives $g(x, z) = 2\lambda^2 e^{-\lambda(2x+z)}$. The marginal of X is $f_X(x) = 2\lambda e^{-2\lambda x}$ and the marginal of Z is $f_Z(z) = \lambda e^{-\lambda z}$ so we see that X and Z are independent.

Problem 6 (p. 355 #11)**.** Suppose X and Y are independent random variables such that $X \sim \text{Uniform}(0, 1)$ and $Y \sim \text{Exp}(1)$. Calculate:

a) $E(X + Y)$;

b) $E(XY)$;

c) $E[(X - Y)^2]$;

d) $E(X^2 e^{2Y})$.

SOLUTION. Using independence and the facts that $EX = \frac{1}{2}$, $E(X^2) = \frac{1}{3}$, $EY = 1$ and $E(Y^2) = 2$ gives the answers for (a)-(c) as: $E(X+Y) = \frac{3}{2}$, $E(XY) = \frac{1}{2}$, and $E[(X-Y)^2] = E(X^2) - 2EXEY + E(Y^2) = \frac{4}{3}$. For part (d) we must also compute

$$E(e^{2Y}) = \int_0^\infty e^{2y} e^{-y} dy = \int_0^\infty e^y dy = \infty.$$

Therefore, $E(X^2 e^{2Y}) = \infty$.

Problem 7 (p. 367 #6)**.** Let X and Y be independent standard normal variables. Find:

a) $P(3X + 2Y > 5)$;

SOLUTION: $V(3X+2Y) = 3^2V(X) + 2^2V(Y) = 13$ so $Z = 3X + 2Y \sim \mathcal{N}(0, 13)$. Therefore $P(3X + 2Y > 5) = P(Z > 5) = P(Z/\sqrt{13} > 5/\sqrt{13}) = 1 - \Phi(5/\sqrt{13})$.

b) $P(\min(X, Y) < 1)$;

SOLUTION:

$$\begin{aligned} P(\min(X,Y) < 1) &= 1 - P(\min(X,Y) > 1) \\ &= 1 - P(X > 1, Y > 1) \\ &= 1 - P(X > 1)P(Y > 1) \\ &= 1 - (1 - \Phi(1))^2. \end{aligned}$$

c) $P(|\min(X, Y)| < 1)$;

SOLUTION:

$$\begin{aligned} P(|\min(X,Y)| < 1) &= P(-1 < \min(X,Y) < 1) \\ &= P(\min(X,Y) > -1) - P(\min(X,Y) > 1) \\ &= \Phi(1)^2 - (1 - \Phi(1))^2 \end{aligned}$$

d) $P(\min(X, Y) > \max(X, Y) - 1)$.

SOLUTION:

$$\begin{aligned} P(\min(X,Y) > \max(X,Y) - 1) &= P(\max(X,Y) - \min(X,Y) < 1) \\ &= P(|X - Y| < 1) \\ &= P(-1 < X - Y < 1) \\ &= P(-1 < Z < 1) \end{aligned}$$

where $Z = X - Y \sim \mathcal{N}(0, 2)$ so

$$\begin{aligned} P(\min(X,Y) > \max(X,Y) - 1) &= P(-\frac{1}{\sqrt{2}} < Z/\sqrt{2} < \frac{1}{\sqrt{2}}) \\ &= 2\Phi(1/\sqrt{2}) - 1 \end{aligned}$$

Problem 8 (p.367 #7)**.** Suppose the DATA bus is scheduled to arrive at my corner at 8:10 AM, but its actual arrival time is a normal random variable with mean 8:10 AM, and standard deviation 40 seconds. Suppose I try to arrive at the corner at 8:09 AM, but my arrival time is actually normally distributed with mean 8:09 AM, and standard deviation 30 seconds.

a) What percentage of the time do I arrive at the corner before the bus is scheduled to arrive?

SOLUTION: Let X be the time that I arrive in minutes after 8:00 AM, so $X \sim \mathcal{N}(9, \frac{1}{4})$. The probability that we want is $P(X < 10) = P((X - 9)/(1/2) < 2) = \Phi(2)$

b) What percentage of the time do I arrive at the corner before the bus does?

SOLUTION: Let $Y \sim \mathcal{N}(10, \frac{4}{9})$ be the time that the bus arrives. We want to compute $P(X < Y) = P(X - Y < 0)$. The mean of $X - Y$ is -1 and the variance is $\frac{25}{36}$ so $X - Y \sim \mathcal{N}(-1, \frac{25}{36})$ (assuming that the time that I arrive and the bus arrives are independent). Therefore $P(X < Y) = P((X - Y + 1)/(5/6) < 6/5) = \Phi(6/5)$.

c) If I arrive at the stop at 8:09 AM and the bus still hasn't come by 8:12 AM, what is the probability that I have already missed it?

SOLUTION: We want

$$
\begin{aligned}
P(Y < 9 | \{Y < 9\} \cup \{Y > 12\}) &= \frac{P(Y < 9)}{P(Y < 9) + P(Y > 12)} \\
&= \frac{1 - \Phi(3/2)}{1 - \Phi(3/2) + 1 - \Phi(3)}
\end{aligned}
$$

(State your assumptions carefully.)

Problem 9 (p.369 #12)**.** Suppose two shots are fired at a target. Assume each shot hits with independent normally distributed coordinates, with the same means and equal unit variances.

a) Find the mean of the distance between the points where the two shots strike.

SOLUTION: Let (X_1, Y_1) and (X_2, Y_2) be the coordinates of the two shots, so X_1, X_2, Y_1, Y_2 are *i.i.d.* $\mathcal{N}(0, 1)$. The distance between the two points is $Z = \sqrt{(X_1 - X_2)^2 + (Y_1 - Y_2)^2}$. Observe that $X_1 - X_2 \sim \mathcal{N}(0, 2)$ and $Y_1 - Y_2 \sim \mathcal{N}(0, 2)$, so $X = X_1 - X_2/\sqrt{2} \sim \mathcal{N}(0, 1)$ and $Y = (Y_1 - Y_2)/\sqrt{2} \sim \mathcal{N}(0, 1)$. Now we can write:

$$
\begin{aligned}
Z &= \sqrt{(X_1 - X_2)^2 + (Y_1 - Y_2)^2} \\
&= \sqrt{2} \cdot \sqrt{\frac{1}{2}(X_1 - X_2)^2 + \frac{1}{2}(Y_1 - Y_2)^2} \\
&= \sqrt{2} \cdot \sqrt{X^2 + Y^2} \\
&= \sqrt{2} R
\end{aligned}
$$

where R has the Rayleigh distribution. As we computed in class, $ER = \frac{\sqrt{2\pi}}{2}$ so $EZ = \sqrt{\pi}$.

b) Find the variance of the distance between the points where the two shots strike.

SOLUTION: Starting with $Z = \sqrt{2} \cdot \sqrt{X^2 + Y^2}$, we have

$$
\begin{aligned}
&E(Z^2) = 2E(X^2) + 2E(Y^2) = 4 \\
&V(Z) = 4 - \pi.
\end{aligned}
$$

Problem 10. Let X be the number on a die roll, between 1 and 6. Let $Y \sim$Uniform(0,1), independent of X. Let $Z = 10X + 10Y$.

a) What is the distribution of Z? Explain.

SOLUTION: $Z \sim$ Uniform(10,70). To check, take $10 < x < y < 70$. Let k_x and k_y be such that $10k_x < x \leq 10k_x + 10$ and $10k_y < y \leq 10k_y + 10$. Then if $k_x < k_y$

$$
\begin{aligned}
P(x < Z < y) &= P(X = k_x, Y > \frac{x}{10} - k_x) + P(k_x + 1 \leq X \leq k_y - 1) + P(X = k_y, Y < \frac{y}{10} - k_y) \\
&= \frac{1}{6}(1 - \frac{x}{10} + k_x) + \frac{1}{6}(k_y - k_x - 1) + \frac{1}{6}(\frac{y}{10} - k_y) \\
&= \frac{y - x}{60}.
\end{aligned}
$$

A similar calculation gives the same result if $k_x = k_y$, which confirms that Z is uniform on $(10, 70)$.

b) Find $P(25 \leq Z \leq 48)$.

SOLUTION: $P(25 \leq Z \leq 48) = \frac{48-25}{60} = \frac{23}{60}$.

Problem 11.

a) Suppose X and Y are independent Exp(1) random variables. What is the distribution of $X+Y$?

SOLUTION: We know from class that the sum of exponentials with the same rate is a gamma, so $X+Y \sim$ Gamma(2,1). That is, the density is $f_{X+Y}(z) = ze^{-z}$.

b) Suppose U and V are independent Uniform(0,1) random variables. What is the density of $Z = UV$? [*Hint: Either read pages 382-383 regarding the density of $Z = Y/X$ and adapt the method to $Z = XY$, or try the following. Find the distribution (and density) of $W = -\log(Z)$ then make a change of variables noting that $Z = e^{-W}$.*]

SOLUTION: Let $W = -\log(Z) = -\log(U) - \log(V)$. Recall that $X = -\log(U) \sim \text{Exp}(1)$ and $Y = -\log(V) \sim \text{Exp}(1)$ since U and V are Uniform(0,1). Further, since U and V are independent, so are X and Y. Therefore, from part (a), $W = X+Y \sim$ Gamma(2,1). Now we can write $Z = e^{-W}$. Since $W > 0$ we know that $Z \in (0,1)$. So for $z \in (0,1)$

$$\begin{aligned}
F_Z(z) &= P(Z \leq z) \\
&= P(e^{-W} \leq z) \\
&= P(W \geq -\log z) \\
&= 1 - F_W(-\log z).
\end{aligned}$$

Differentiating gives

$$\begin{aligned}
f_Z(z) &= \frac{1}{z} f_W(-\log z) \\
&= \frac{1}{z}(-\log z)e^{\log z} \\
&= -\log z.
\end{aligned}$$

for $z \in (0,1)$ and $f_Z(z) = 0$ otherwise.

Problem 12 (p. 385 # 13)**.** Find the density of $Z = X - Y$ for independent Exp(λ) random variables X and Y.

SOLUTION: Using the independence of X and Y we can write the joint density as the product of the marginals. To compute the density of Z we use an analogue of the convolution for differences of random variables. Alternatively, we can take the convolution of the densities of X and $-Y$ by writing $Z = X + (-Y)$. The range of Z is all of $\mathbb{R}$. If $z \geq 0$.

$$\begin{aligned}
f_Z(z) &= \int_{-\infty}^{\infty} f_X(y+z) f_Y(y) dy \\
&= \int_0^{\infty} \lambda e^{-\lambda(y+z)} \lambda e^{-\lambda y} dy \\
&= \lambda^2 e^{-\lambda z} \int_0^{\infty} e^{-2\lambda y} dy \\
&= \frac{1}{2}\lambda e^{-\lambda z}.
\end{aligned}$$

If $z < 0$ then

$$\begin{aligned} f_Z(z) &= \int_{-\infty}^{\infty} f_X(y+z) f_Y(y) dy \\ &= \int_{-z}^{\infty} \lambda e^{-\lambda(y+z)} \lambda e^{-\lambda y} dy \\ &= \lambda^2 e^{-\lambda z} \int_{-z}^{\infty} e^{-2\lambda y} dy \\ &= \frac{1}{2} \lambda e^{\lambda z}. \end{aligned}$$

Therefore, $f_Z(z) = \frac{1}{2}\lambda e^{-\lambda |z|}$.