Math 230.01, Fall 2012: HW 5 Solutions

Due Thursday, October 4th, 2012.

Problem 1 (p.158 #2)**.** Let X and Y be the numbers obtained in two draws at random from a box containing four tickets labeled $1, 2, 3, 4$. Display the joint distribution table for X and Y:

a) for sampling with replacement;

b) for sampling without replacement;

c) calculate $P(X \leq Y)$ in both cases.

SOLUTION. In sampling with replacement, every pair of numbers is equally likely, so we have

$$P(X = i, Y = j) = \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{16} \text{ for } i, j \in \{1, 2, 3, 4\}$$

And $P(X \leq Y) = \frac{1}{16}(1 + 2 + 3 + 4) = \frac{10}{16}$.

In sampling without replacement, we know X and Y can never be equal, and we have

$$P(X = i, Y = j) = \frac{1}{4} \cdot \frac{1}{3} = \frac{1}{12} \text{ for } i \neq j \text{ and } i, j \in \{1, 2, 3, 4\}$$
$$P(X = i, Y = i) = 0 \text{ for } i \in \{1, 2, 3, 4\}$$

In this case, $P(X \leq Y) = P(X < Y) = \frac{1}{2}$. This can be found either by computing, or by reasoning as follows: $P(X < Y) = P(X > Y)$ by symmetry, and since $P(X = Y) = 0$, $P(X < Y) + P(X > Y) = 1$ so $2P(X < Y) = 1$.

Problem 2 (p.159 #4)**.** Let X_1 and X_2 be the numbers obtained on two rolls of a fair die. Let $Y_1 = \max\{X_1, X_2\}$, $Y_2 = \min\{X_1, X_2\}$. Display joint distribution tables for

a) the pair (X_1, X_2);

b) the pair (Y_1, Y_2).

Since X_1 and X_2 are independent, we know that for any pair of integers (i, j) where $1 \leq i \leq 6$, $1 \leq j \leq 6$,

$$P(X_1 = i, X_2 = j) = P(X_1 = i) \cdot P(X_2 = j) = \frac{1}{36}$$

We also know that the minimum of (X_1, X_2) cannot be greater than the maximum of (X_1, X_2), so

$$P(Y_1 = i, Y_2 = j) = 0 \text{ for } j > i$$
$$P(Y_1 = i, Y_2 = i) = \frac{1}{36}$$
$$P(Y_1 = i, Y_2 = j) = \frac{2}{36} \text{ for } i > j$$

Problem 3 (p.159 #7)**.** Let A, B, and C be events that are independent, with probabilities a, b, and c, respectively. Let N be the random number of events that occur.

a) Express the event $(N = 2)$ in terms of the events A, B, and C.

b) Find $P(N = 2)$.

SOLUTION. The event $N = 2$ is the union of the following events

$$\{N = 2\} = [A \cap B \cap C^c] \cup [A \cap C \cap B^c] \cup [B \cap C \cap A^c]$$

and these are mutually exclusive.
Also, since A, B, C are mutually independent, we have

$$\begin{aligned} P(ABC^c) &= P(A)P(B)P(C^c) = ab(1-c) \\ P(AB^cC) &= P(A)P(B^c)P(C) = a(1-b)c \\ P(A^cBC) &= P(A^c)P(B)P(C) = (1-a)bc \end{aligned}$$

So we have $P(N = 2) = ab + ac + bc - 3abc$.

Problem 4 (p.182 #4)**.** Suppose all the numbers in a list of 100 numbers are nonnegative, and that the average of the numbers in the list is 2. Prove that at most 25 of the numbers in the list are greater than 8.

SOLUTION. We use Markov's inequality. Let X be a number drawn at random from the list. We know $E[X] = 2$. Hence

$$P(X > 8) \leq \frac{E[X]}{8} \leq \frac{1}{4}$$

so at most 1/4 of the 100 numbers, i.e. 25 of the numbers in the list, can be bigger than 8.

Problem 5 (p.182 #6)**.** Let X be the number of spades in 7 cards dealt from a well-shuffled deck of 52 cards containing 13 spades. Find $E[X]$.

SOLUTION. The probability distribution of X, $P(X = i)$, where i is an integer between 0 and 13, can be computed using the hypergeometric distribution:

$$\begin{aligned} P(X = i) &= \frac{\binom{13}{i}\binom{39}{13-i}}{\binom{52}{13}} \\ \Rightarrow E[X] &= \sum_{i=0}^{13} iP(X = i) \end{aligned}$$

Alternatively, one can solve this with the method of indicator functions: let I_j be the indicator function of the event $A_j = \{\text{jth card is a spade}\}$. Note that $P(A_j) = \frac{13}{52}$. If you are concerned about the fact that, say, the probability of the second card being a spade depends on the previous card, observe that you can condition:

$$\begin{aligned} P(A_2) &= P(A_2|A_1)P(A_1) + P(A_2|A_1^c)P(A_1^c) \\ &= \frac{12}{51}\frac{13}{52} + \frac{13}{51}\frac{39}{52} \\ &= \frac{13}{52} \end{aligned}$$

Now, if X is the total number of spades in the hand,

$$\begin{aligned} X &= \sum_{j=1}^{7} I_j \\ \Rightarrow E(X) &= E\left[\sum_{j=1}^{7} I_j\right] \\ &= \sum_{j=1}^{7} E[I_j] = \sum_{j=1}^{7} \frac{13}{52} = \frac{7}{4} \end{aligned}$$

Problem 6 (p.182 #7)**.** In a circuit containing n switches, the ith switch is closed with probability p_i, $i = 1, \cdots, n$. Let X be the total number of switches that are closed. Is it possible to calculate $E[X]$ without further assumptions? If so, what is $E[X]$?

SOLUTION. It is *not* possible to know the *probability distribution* of X without further assumptions. However, we can still compute the expected value of X by using the method of indicator functions. Let $A_i = \{\text{Switch } i \text{ is closed}\}$. Let I_{A_i} be the indicator function of A_i, so $I_{A_i} = 1$ if switch i is closed, and 0 if switch i is open. We note that

$$X = \sum_{i=1}^{n} I_{A_i}$$

because X is simply the total number of switches that are closed. Hence by the addition rule for expectation,

$$\begin{aligned} E[X] &= E\left[\sum_{i=1}^{n} I_{A_i}\right] \\ &= \sum_{i=1}^{n} E[I_{A_i}] \\ &= \sum_{i=1}^{n} [1P(A_i) + 0P(A_i^c)] \\ &= \sum_{i=1}^{n} P(A_i) \end{aligned}$$

Problem 7 (p.182#10)**.** Let A and B be independent events, with indicator random variables I_A and I_B.

a) Describe the distribution of the random variable $(I_A + I_B)^2$ in terms of $P(A)$ and $P(B)$.

b) What is $E\left[(I_A + I_B)^2\right]$?

SOLUTION. Observe that

$$\begin{aligned} I_A + I_B &= 0 \quad \text{if neither A nor B occurs, i.e. on } A^c \cap B^c; \\ I_A + I_B &= 1 \quad \text{if only one of either A or B occur; i.e. on } (B \cap A^c) \cup (A \cap B^c); \\ I_A + I_B &= 2 \quad \text{if both A and B occur, i.e., on } A \cap B \end{aligned}$$

Since A and B are independent, so are A and B^c and B and A^c (this was on Exam 1), and therefore the distribution of the function $(I_A + I_B)^2$ is

$$\begin{aligned}
P\left((I_A + I_B)^2 = 0\right) &= (1 - P(A))(1 - P(B)) \\
P\left((I_A + I_B)^2 = 1\right) &= P(B)(1 - P(A)) + P(A)(1 - P(B)) \\
P\left((I_A + I_B)^2 = 4\right) &= P(A)P(B)
\end{aligned}$$

One can compute $E\left[(I_A + I_B)^2\right]$ using the above distribution:

$$\begin{aligned}
E\left[(I_A + I_B)^2\right] &= 1 \cdot [P(B)(1 - P(A)) + P(A)(1 - P(B))] + 4 \cdot P(A)P(B) \\
&\quad + 0 \cdot (1 - P(A))(1 - P(B)) \\
&= [P(B)(1 - P(A)) + P(A)(1 - P(B))] + 4 \cdot P(A)P(B)
\end{aligned}$$

Problem 8 (p. 183 #14)**.** A building has ten floors above the basement. If 12 people get into an elevator at the basement, and each chooses a floor at random to get out, independently of the others, at how many floors do you expect the elevator to stop to let out one or more of these 12 people?

SOLUTION. Let X = the number of floors at which at least one person wants to get out. We want to find $E(X)$. However, computing $P(X = x)$ for $x = 1, 2, \cdots, 10$ is difficult. However, we don't need the distribution of X; instead, let

$$\begin{aligned}
&I_j = 1 \text{ if at least one person wants to get out at floor j, and 0 otherwise} \\
&\Rightarrow X = \sum_{j=1}^{10} I_j
\end{aligned}$$

The expected value of an indicator function is the probability of the set on which the indicator is 1.

$$\begin{aligned}
E(X) &= E\left[\sum_{j=1}^{10} I_j\right] \\
&= \sum_{j=1}^{10} E[I_j] \\
&= \sum_{j=1}^{10} \left[1 - \left(\frac{9}{10}\right)^{12}\right]
\end{aligned}$$

Problem 9 (p. 183, #18)**.** Suppose X is a random variable with just two possible values a and b. For $x = a$ and $x = b$, find a formula for $p(x) = P(X = x)$ in terms of a, b, and $\mu = E[X]$.

SOLUTION. Since X has two possible values, $X = a$ or $X = b$, we know

$$P(X = a) + P(X = b) = 1.$$

Therefore

$$\begin{aligned}
E(X) &= aP(X = a) + bP(X = b) \\
\Rightarrow \mu &= aP(X = a) + b(1 - P(X = a)) \\
\Rightarrow P(X = a) &= \frac{\mu - b}{a - b}
\end{aligned}$$

and from this $P(X = b)$ is immediate.

Problem 10 (p.184 #21)**.** Let I_A be the indicator of an event A. Show that

a) $I_{A^c} = 1 - I_A$;

SOLUTION. The function $1 - I_A = 0$ if A occurs and $1 - I_A$=1 if A does not occur; this is the exactly the indicator function of A^c.

b) $I_{AB} = I_A I_B$;

SOLUTION. Observe that $I_{AB} = 1$ if and only if both A and B occur, and 0 otherwise. Now I_A is 1 if only if A occurs and 0 otherwise, and I_B is 1 if and only if B occurs, and 0 otherwise. Therefore the product $I_A I_B$ is 1 if and only if both A and B occur, and 0 otherwise; that is, $I_{AB} = I_A I_B$.

c) For any collection of events $A_1, \cdots A_n$, the indicator of their union is

$$I_{\cup A_i} = 1 - (1 - I_{A_1})(1 - I_{A_2}) \cdots (1 - I_{A_n}).$$

SOLUTION. Note that

$$[\cup_{i=1}^n A_i]^c = \cap_{i=1}^n A_i^c$$

From the previous parts, we know that $I_{A_i^c} = 1 - I_{A_i}$, and

$$I_{[\cap_{i=1}^n A_i^c]} = I_{A_1^c} I_{A_2^c} \cdots I_{A_n^c}.$$

So

$$\begin{aligned} I_{[\cup_{i=1}^n A_i]^c} &= I_{[\cap_{i=1}^n A_i^c]} \\ &= I_{A_1^c} I_{A_2^c} \cdots I_{A_n^c} \\ \Rightarrow I_{\cup A_i} &= 1 - I_{[\cup_{i=1}^n A_i]^c} \\ &= 1 - \left[I_{A_1^c} I_{A_2^c} \cdots I_{A_n^c} \right] \\ &= 1 - (1 - I_{A_1})(1 - I_{A_2}) \cdots (1 - I_{A_n}) \end{aligned}$$

d) Expand the product in the last formula and use the rules of expectation to derive the inclusion exclusion formula.

SOLUTION. This is simple algebra and an application of the fact that the expected value of I_A is $P(A)$.

Problem 11 (St Petersburg Paradox)**.** Suppose you have the opportunity to play the following game. You flip a coin, and if it comes up heads on the first flip, then you win \$1. If not, then you flip again. If it comes up heads on the second flip, then you win \$2, and if not you flip again. On the third flip, a heads pays \$4, on the fourth \$8, and so on. That is, each time you get tails, you flip again and your prize doubles, and you get paid the first time you flip heads.

a) How much should you be willing to pay to play this amazing game? In other words, compute the expected payout from playing this game.

SOLUTION: Let X be the payout from this game.

$$\begin{aligned}
E(X) &= \sum_{k=1}^{\infty} 2^{k-1} P(\text{first head is on } k^{\text{th}} \text{ flip}) \\
&= \sum_{k=1}^{\infty} 2^{k-1} \left(\frac{1}{2}\right)^k \\
&= \sum_{k=1}^{\infty} \frac{1}{2} = \infty
\end{aligned}$$

According to this, you should be willing to pay any amount to play this game!

b) Now suppose the casino (or wherever you're playing this game) has a limited bankroll of \$$2^n$. So, if you get tails n times in a row, then the game is over automatically and you are paid \$$2^n$. Now what is the expected payout? How much should you pay to play the game if $n = 10$?

SOLUTION:

$$\begin{aligned}
E(X) &= \sum_{k=1}^{n} 2^{k-1} P(\text{first head is on } k^{\text{th}} \text{ flip}) + 2^n P(n \text{ tails in a row}) \\
&= \sum_{k=1}^{n} \frac{1}{2} + 1 \\
&= \frac{n+2}{2}
\end{aligned}$$

The expected payout is \$6.