Problem 1. A woman has 2 children, one of whom is a boy born on a Tuesday. What is the probability that both children are boys? (You may assume that boys and girls are equally likely and independent, and that children are equally likely to be born on any given day of the week.)

SOLUTION. In this problem, one of the first questions that comes to mind is, “Why does Tuesday matter?” To determine whether this information matters, let us work from the definitions.

We are asked to find the probability that the woman has two boys, given that one of her two children is a boy born on a Tuesday. The problem does not tell us which of the two children is a boy. Let the days of the week be numbered 1, 2, 3, …, 7, where 1=Mon, 2=Tues, 7=Sunday. Since the woman has two children, the sample space Ω consists of ordered pairs (Bi, Bj), (Bi, Gj), (Gi, Bj), (Gi, Gj) where B or G represents the gender, and i and j each take on any integer value between 1 and 7 and represent the day of the week that the child is born. Furthermore, (Bi, Gj) represents the event that the firstborn child is a boy born on day i, and the secondborn is a girl born on day j, etc. Note that all of these outcomes are equally likely.

Since we know that one of the children was a boy born on a Tuesday, however, we can restrict ourselves to considering the following outcomes in our sample space: the 7 outcomes (B2, Gj), where j = 1, 2, …, 7; the 7 outcomes (Gj, B2) where j = 1, 2, …, 7, the 7 outcomes (B2, Bj), where j = 1, 2, …, 7, and the six outcomes (Bj, B2) where j = 1, 3, …, 7, because we already counted the pair (B2, B2). Hence the event C = {The woman has two children, one of whom is a boy born on a Tuesday} has a total of 27 outcomes.

The event D = {both children are boys} has 49 outcomes, but the event

\[ D \cap C = \{ \text{both children are boys, one of whom was born on a Tuesday} \} \]

has only 13 outcomes. Hence

\[ P(D|C) = \frac{P(D \cap C)}{P(C)} = \frac{13}{27} \]

which is slightly less than 1/2.

One way to think about this is that if you only know that the woman has two children, then the probability that both are boys is 1/4; if you know that one of them is a boy, the probability that both are boys rises to 1/3; and if you know that one of them is a boy born on a Tuesday, the probability that both are boys rises to nearly 1/2.

Problem 2 (p.90 #1). Answer the following:

a) How many sequences of zeros and ones of length 7 contain exactly 4 ones and 3 zeros?
SOLUTION. There are precisely \(^7C_4 = \binom{7}{4}\) ways. Note that it is not asking for the probability of this configuration of zeros and ones, just the number of ways.

b) If you roll 7 dice, what is the chance of getting exactly 4 sixes?

SOLUTION. The probability of success, i.e. getting a 6, one each trial is 1/6. Therefore,

\[
P[\text{4 sixes in 7 rolls}] = \binom{7}{4} \left(\frac{1}{6}\right)^4 \left(\frac{5}{6}\right)^3 \approx 0.0156.
\]

Problem 3 (p.91 #2). Suppose that in 4-child families, each child is equally likely to be a boy or girl, independently of the others. What is more common, 4-child families with 2 boys and 2 girls, or 4 child families with different numbers of boys and girls?

SOLUTION.

\[
P[\text{exactly two boys in a 4 child family}] = \frac{4!}{2!2!} \left(\frac{1}{2}\right)^4 = \frac{4 \times 3}{2} \left(\frac{1}{2}\right)^4 = \frac{3}{8}
\]

The probability of different numbers of boys and girls is therefore \(1 - \frac{6}{16} = \frac{10}{16}\), which is greater than the probability of exactly two boys and exactly two girls.

Problem 4 (p.91 #5). Given that there were 12 heads in 20 independent coin tosses, calculate

a) the chance that first toss landed heads;

SOLUTION. Let \(A = \{12 \text{ heads in 20 tosses}\}\). Then \(P(A)\) is the binomial probability:

\[
P(A) = \frac{20!}{12!8!} \left(\frac{1}{2}\right)^{20}
\]

Let \(B = \{\text{first toss landed heads}\}\). We want to find \(P(B|A)\). Note that

\[
B \cap A = \{\text{1st toss is H and 12 heads in 20 tosses}\}
= \{\text{1st toss is heads and 11 heads in remaining 19 tosses}\}
\]

Let \(C\) denote the event \(C = \{11 \text{ heads in tosses 2 through 20}\}\). By the independence of the trials, the events \(B\) and \(C\) are independent. By the definition of \(C\),

\[
B \cap A = B \cap C.
\]

\[
P(B|A) = \frac{P(A \cap B)}{P(A)}
= \frac{P(B \cap C)}{P(A)}
\]

\[
P(B \cap C) = P(\text{1st toss is heads})P(11 \text{ heads in next 19 tosses})
= \left[\frac{1}{2}\right] \left[\frac{19!}{11!8!} \left(\frac{1}{2}\right)^{19}\right]
\]

\[
\Rightarrow P(B|A) = \frac{\left[\frac{1}{2}\right] \left[\frac{19!}{11!8!} \left(\frac{1}{2}\right)^{19}\right]}{\frac{20!}{12!8!} \left(\frac{1}{2}\right)^{20}}
\]
Alternatively, given that there are 12 heads in 20 tosses, and since all outcomes are equally likely, any ordering of 12 heads and 8 tails is equally likely. The number of orderings that have a head on the first toss is \( \binom{19}{11} \), since after fixing the first toss as a head, the other 11 heads must be assigned to the remaining 19 tosses. The total number of orderings is \( \binom{20}{12} \) giving

\[
P(B|A) = \frac{\binom{19}{11}}{\binom{20}{12}} = \frac{12}{20}.
\]

b) the chance that the first two tosses landed heads;

SOLUTION: Again, all orderings are equally likely, so after fixing the first two tosses as heads there are \( \binom{18}{10} \) ways to order the remaining 10 heads and 8 tails, so the probability that the first two tosses landed heads is

\[
\frac{\binom{18}{10}}{\binom{20}{12}} = \frac{12 \cdot 11}{20 \cdot 19}
\]

c) the chance that at least two of the first five tosses landed heads;

Let \( F = \{ \text{at least two of the first five tosses landed heads} \} \). We can write \( F \) as the union of the sets \( G_2, G_3, G_4, \) and \( G_5 \) where

\[
\begin{align*}
G_2 &= \{ \text{two of the first five tosses landed H} \} \\
G_3 &= \{ \text{three of the first five tosses landed H} \} \\
G_4 &= \{ \text{four of the first five tosses landed H} \} \\
G_5 &= \{ \text{five of the first five tosses landed H} \}
\end{align*}
\]

The sets \( G_i, i = 2, 3, 4, 5 \) are mutually exclusive. The event \( F \cap A \) is then

\[
\begin{align*}
F \cap A &= [\bigcup_{i=2}^5 G_i] \cap A \\
&= \bigcup_{i=2}^5 [G_i \cap A] \\
G_i \cap A &= \{ \text{i of the first five tosses land H and 12-i of the remaining 15 tosses land H} \}
\end{align*}
\]

Since the events \( G_i \) are mutually exclusive, we get

\[
P(F \cap A) = P \left( \bigcup_{i=2}^5 [G_i \cap A] \right)
= \sum_{i=2}^5 P(G_i \cap A)
\]

By independence of the trials, we have

\[
P(G_i \cap A) = P(\text{i of the first five tosses land H and 12-i of the remaining 15 tosses land H})
= P(G_i)P(12-i of 15 tosses lands H)
= \left[ \frac{5!}{i!(5-i)!} \left( \frac{1}{2} \right)^5 \right] \left[ \frac{15!}{(12-i)!(3+i)!} \left( \frac{1}{2} \right)^{15} \right]
\]

So the requisite probability is

\[
P(G|A) = \frac{P(G \cap A)}{P(A)}
= \sum_{i=2}^5 \left( \frac{5!}{i!(5-i)!} \left( \frac{1}{2} \right)^5 \frac{15!}{(12-i)!(3+i)!} \left( \frac{1}{2} \right)^{15} \right)
\]

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Problem 5 (p. 91 #8). For each positive integer, what is the largest value of $p$ such that zero is the most likely number of successes in $n$ independent trials with success probability $p$?

SOLUTION. Recall that the mode of the binomial $B(n, p)$ distribution is

$$\text{mode of } B(n, p) = \lfloor np + p \rfloor$$

where $\lfloor \rfloor$ denotes “greatest integer less than” $np + p$. In order for this to be zero, $np + p < 1$, i.e $p < \frac{1}{1+n}$.

(Note that $np + p$ can never equal 0 unless the probability of success is 0.)

If, however, $np + p = 1$, then there are two modes: 0 and 1, and both are equally likely. (Recall how we used $P(k)/P(k-1)$ to prove what the mode would be!) So we must have

$$p \leq \frac{1}{n + 1}$$

if zero is to be a mode, and hence $p = \frac{1}{n+1}$ is the largest $p$ can be.

Problem 6 (p.91 #10). Suppose a fair coin is tossed $n$ times. Find simple formulae in terms of $n$ and $k$ for

a) $P(k-1 \text{ heads} | k-1 \text{ or } k \text{ heads})$

SOLUTION. Let $A_k = \{k \text{ heads}\}, A_{k-1} = \{k-1 \text{ heads}\}$. Observe that $A_k$ and $A_{k-1}$ are mutually exclusive because there cannot simultaneously be $k$ and $k-1$ heads. Also recall

$$P(A_k) = \frac{n!}{k!(n-k)!} \left[ \frac{1}{2} \right]^n$$
$$P(A_{k-1}) = \frac{n!}{(k-1)!(n-k+1)!} \left[ \frac{1}{2} \right]^n$$

We want to find $P(A_{k-1} | A_k \cup A_{k-1})$. We have

$$P(A_{k-1} | A_k \cup A_{k-1}) = P(A_{k-1} \cap [A_k \cup A_{k-1}]) / P(A_k \cup A_{k-1})$$

$$= P(A_{k-1}) / (P(A_{k-1}) + P(A_k))$$

$$= \frac{n!}{(k-1)!(n-k+1)!} \left[ \frac{1}{2} \right]^n / \left[ \frac{n!}{k!(n-k)!} + \frac{n!}{(k-1)!(n-k+1)!} \right] \left[ \frac{1}{2} \right]^n$$

$$= \frac{k}{n+1}$$

b) $P(k \text{ heads} | k-1 \text{ or } k \text{ heads})$

SOLUTION. Since $P(A_k \cup A_{k-1} | A_k \cup A_{k-1}) = 1$ and $A_k \cap A_{k-1} = \emptyset$, we have:

$$P(A_k | A_k \cup A_{k-1}) = 1 - P(A_{k-1} | A_k \cup A_{k-1})$$

$$= \frac{n-k+1}{n+1}$$

Problem 7 (p.108 #3). A fair coin is tossed repeatedly. Consider the following two possible outcomes:

a) 55 or more heads in the first 100 tosses;

b) 220 or more heads in the first 400 tosses.
Without calculating, which probability is more likely? Why?
Confirm this by using the normal approximation to the binomial distribution to estimate the probabilities of both events.

SOLUTION. The first probability is more likely—to see why, without calculating, note that when we multiply the number of tosses by 4, the mean increases by a factor of 4 ($\mu = n/2 \rightarrow 4n/2 = 4\mu$), but the standard deviation only increases by a factor of 2 ($\sigma = \sqrt{n/2} \rightarrow \sqrt{4n/2} = \sqrt{n} = 2\sigma$). So, while the relative fraction of heads above the mean remains constant (everything is multiplied by 4: $5/100 = 20/400$), the number of standard deviations above the mean has increased, since the standard deviation did not increase by the same factor of 4.

Next, we use the normal approximation to the binomial to compare the two probabilities. Let $k$ = number of heads in $n$ tosses, each with success probability $p$ and failure probability $q = 1 - p$. First, since the number of successes can only take on integer values between 0 and $n$, for any integer $a$ between 0 and $n$,

$$P(k \geq a) = P(k \geq a - 0.5)$$

Therefore, by the normal approximation to the binomial, we have

$$P(k \geq a) = P\left(k - np \geq \frac{a - 0.5 - np}{\sqrt{npq}}\right) \approx 1 - \Phi\left(\frac{a - 0.5 - np}{\sqrt{npq}}\right)$$

where $\Phi$ is the standard normal cumulative distribution function. Using the table in Appendix 5 of Pitman with the values $a = 55, n = 100, p = 0.5$ gives 0.1841 for the approximate probability of 55 or more heads in 100 tosses; plugging in the values $a = 220, n = 400, p = 0.5$ gives 0.0256 for the probability of 220 or more heads in 400 tosses.

**Problem 8** (p.109 #9). An airline knows that over the long run, 90% of passengers who reserve seats will show up for their flights. On a particular flight with 300 seats, the airline accepts 324 reservations. Use the normal approximation to the binomial distribution to answer the following.

a) Assuming that passengers show up independently of one another, what is the chance that the flight will be overbooked?

SOLUTION. The airline will be overbooked if more than 301 or more passengers show up. Since passengers show up independently of one another and each passenger has probability 0.9 of showing up, the probability of more than 301 (inclusive) passengers showing up is a binomial probability. Let $S$ = the number of passengers who show up. Since $n = 324$, the number of trials is large, and we can use the normal approximation. As in the previous problem, we have

$$P(S \geq 301) = P(S \geq 301 - 0.5)$$

$$P(S \geq 301) = P\left(\frac{S - (324)(0.9)}{\sqrt{324(0.9)(0.1)}} \geq \frac{301 - 0.5 - (324)(0.9)}{\sqrt{324(0.9)(0.1)}}\right)$$

$$\approx 1 - \Phi\left(\frac{301 - 0.5 - (324)(0.9)}{\sqrt{324(0.9)(0.1)}}\right) = 0.0495$$

by evaluating $\Phi$ from the table in Appendix 5.

b) Suppose that people tend to travel in groups, and a group travels only if everyone in the group shows up. Would that increase or decrease the probability of overbooking? Explain your answer.
SOLUTION. Suppose people travel in groups of size $k$. Then each of the $324/k$ groups (assuming $k$ divides 324) will travel independently with probability $(.9)^k$ (the probability that all $k$ members show up). So, we will have $\mu = (324/k)(.9)^k$ as the mean number of groups that will travel. This means the mean number of people that will travel is $k\mu = 324(.9)^k$ since each group has $k$ people. By similar reasoning $\sigma = \sqrt{(324/k)(.9)^k(1 - (.9)^k)}$, and the standard deviation of the number of people is again $\sqrt{324(.9)^k(1 - (.9)^k)}$. At first glance, the answer here is “it depends on $k$” since the mean has decreased (by taking powers of .9) and the standard deviation may have increased for small $k$ or decreased for larger $k$. Heuristically, though, the slight increase in the standard deviation will not compensate for the larger decrease in the mean as long as $n$ is large, as it is here. (Since the increase in $\sigma$ happens in square-root land, while the decrease in $\mu$ does not.) Therefore, we expect the probability of overbooking to decrease.

c) Redo in the calculation in part (a), now assuming that passengers always travel in pairs (and travel only if both people in the pair show up). Check that your answers to the three parts are consistent.

SOLUTION. Suppose passengers travel in pairs and that the probability that the pair travels is $(.9)^2 = 0.81$, and the probability that pair does not travel is $0.19$. We have 162 pairs, each with probability 0.81 of showing up, so $\mu = 131.22$ and $\sigma = \sqrt{162(0.81)(0.19)} = 4.993$. If the airline is to avoid being overbooked, 150 pairs or fewer must show up. We get

$$P(S \geq 151) = P(S \geq 151 - 0.5)$$

$$P(S \geq 151) = P \left( \frac{S - \mu}{\sigma} \geq \frac{151 - 0.5 - \mu}{\sigma} \right)$$

$$\approx 1 - \Phi \left( \frac{151 - 0.5 - 131.22}{4.993} \right) = 1 - \Phi(3.86) \approx 0.0001$$

since 3.86 is off the chart, I estimated the last value.

Problem 9 (p. 109 #13). A pollster wishes to know the percentage $p$ of people in a population who intend to vote for a particular candidate. How large must a random sample with replacement be in order to be at least 95% sure that the sample percentage $\hat{p}$ is within one percentage point of $p$?

SOLUTION. Let $n$ be the sample size. Note that $\hat{p}$ is a random number and the true fraction $p$, of people who want to vote for a particular candidate, is unknown. The (random) number of people in the sample who intend to vote for a particular candidate is $np$, and has a binomial distribution with parameters $n$ and $p$. Therefore the mean is $\mu = np$ and the standard deviation is $\sigma = \sqrt{np(1-p)}$, and the distribution is well-approximated by a normal random variable with the same mean and standard deviation.

We need to find $z$ such that

$$\Phi(z) - \Phi(-z) = 0.95$$

And by symmetry, this gives

$$2\Phi(z) - 1 = 0.95$$

From the tables, we get $z = 1.96$.

We don’t know $p$, of course. But observe that we have

$$P \left( -z \leq \frac{\hat{p} - np}{\sqrt{np(1-p)}} \leq z \right) \geq 0.95$$

$$\Rightarrow P \left( -z \left[ \frac{\sqrt{p(1-p)}}{\sqrt{n}} \right] \leq \hat{p} - p \leq z \left[ \frac{\sqrt{p(1-p)}}{\sqrt{n}} \right] \right)$$

and although we don’t know $p$, it is a number between 0 and 1. Simple calculus shows that the maximum of $p(1-p)$ for $p \in [0,1]$ is achieved at $p = 1/2$, so $\sqrt{p(1-p)} \leq 0.5$. Therefore, with probability at least 0.95,
the difference between $p$ and $\hat{p}$ will be less than

$$1.96 \times \frac{0.5}{\sqrt{n}}$$

Since we want this difference to be less than 0.01, we simply set

$$1.96 \times \frac{0.5}{\sqrt{n}} = 0.01$$

and solve for $n$. (You may have noticed that I dropped the $-1/2$ and $+1/2$ needed to make the Normal approximation to the Binomial accurate. In this problem, dropping these terms is justifiable since I want .95 to be a lower bound on the probability, and leaving off these corrections gives a lower bound. Further, since $n$ will be relatively large, the error made from the optimal solution will be relatively small.)