Shintani zeta-functions and Gross–Stark units for totally real fields

Samit Dasgupta
Department of Mathematics
Harvard University
1 Oxford St
Cambridge, MA 02138
dasgupta@post.harvard.edu

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Abstract

Let F be a totally real number field and let $\mathfrak{p}$ be a finite prime of F, such that $\mathfrak{p}$ splits completely in the finite abelian extension H of F. Stark has proposed a conjecture stating the existence of a $\mathfrak{p}$-unit in H with absolute values at the places above $\mathfrak{p}$ specified in terms of the values at zero of the partial zeta-functions associated to H/F. Gross proposed a refinement of Stark's conjecture which gives a conjectural formula for the image of Stark's unit in $F_{\mathfrak{p}}^{\times}/\widehat{E}$, where $F_{\mathfrak{p}}$ denotes the completion of F at p and $\widehat{E}$ denotes the topological closure of the group of totally positive units E of F. We propose a further refinement of Gross' conjecture by proposing a conjectural formula for the exact value of Stark's unit in $F_{\mathfrak{p}}^{\times}$.

Contents

1 Introduction

Let F be a number field and let H be a finite Galois extension of F. In the late 1970s, Stark stated a series of conjectures relating the leading terms at 0 of the partial zeta-functions of H/F to a certain regulator defined in terms of valuations of elements in $H^\times$ [21]. In the last two of these papers, Stark studied in greater detail the case where H/F is an abelian extension. Let v denote a place of F which splits completely in H. The "rank one abelian Stark conjecture" then purports the existence of an element $u \in H^\times$ whose valuations at the places above v are related via a precise formula to the derivatives at 0 of the partial zeta-functions of H/F. When v is a real place, Stark's formula is particularly striking because it provides an explicit formula for the image of u under the real embeddings of H. When u generates H, Stark's conjecture thus provides an "explicit class field theory" for H/F.

When v is complex, knowledge of the absolute value of u at the places above v does not provide an explicit analytic formula for u. Not only does this prevent one from giving an explicit class field theory using Stark's conjecture, but it creates computational difficulties as well (see [10]). Recently, some progress has been made on the problem of providing an explicit analytic formula for u, and not just its absolute value ([3] and [14]).

Stark's original papers focused on the case where v is an infinite prime, but the case where v is finite was incorporated into a uniform exposition by Tate [22]. The present article concerns the case when v is a finite prime. Suppose that v lies above the rational prime p. In this case, knowing the absolute values of u at the places above v also does not provide an analytic formula for u (even though Stark's precise conjecture, stated in Conjecture 2.5 below, determines u uniquely by specifying the valuations of u at all places and imposing an additional congruence). Let w be a place above v, which gives an embedding $H \subset H_w \cong F_v$. In [11] and [12], Gross stated a refinement of Stark's conjecture which can be used to provide a formula for $\mathrm{Norm}_{F_v/\mathbf{Q}_p} u \in \mathbf{Q}_p^\times$. This gives more p-adic analytic information about u, but does not provide an explicit formula for u itself. The goal of this article is to propose an exact v-adic analytic formula for $u \in F_v^\times$ and a description of the action of $\mathrm{Gal}(H/F)$ on u in analytic terms, as a form of "Shimura Reciprocity Law." Our conjectural formula may be viewed as a v-adic explicit class field theory for H/F.

Since Stark's unit is only non-trivial (in the finite v case) when F is totally real and H is totally complex containing a CM subfield, we will make this assumption for the remainder of the article. In fact in this case, the element u will lie in the maximal CM subfield of H. The case where F is real quadratic and H is a ring class field extension was addressed in [7], where a formula for Stark's unit was proposed using modular symbols in analogy with Darmon's definition of Stark–Heegner points [6]. The methodology in the present article is quite different than that of [7]. However it is proven in Section 8 that the formula for Stark's unit given in Conjecture 3.21 below agrees with that proposed in [7]. Thus the present article may be viewed as a generalization of [7] to arbitrary totally real fields (and arbitrary totally complex abelian extensions H).

We now briefly describe the methods and content of this article. In Section 2 we state the conjectures of Gross and Stark in our context. Gross stated his first p-adic conjecture in [11], and provided a simultaneous refinement of this conjecture and Stark's conjecture in [12]. In Section 3, we restate Gross's second conjecture in terms of v-adic measures, and use this formulation as motivation for our proposed formula for u. To express this more precisely, consider for a moment the case where H is the narrow Hilbert class field of F. Let $\mathcal{O}$ denote the ring of integers of F, and let $E \subset \mathcal{O}^\times$ denote the group of totally positive units of F. Let $\mathcal{O}_v$ denote the v-adic completion of $\mathcal{O}$, and let $\widehat{E}$ denote the topological closure of E in $\mathcal{O}_v^\times$. Gross's conjecture can be viewed as giving a formula for the image of u in $F_v^\times/\widehat{E}$, in terms of an integral on the compact group $\mathcal{O}_v^\times/\widehat{E}$ with respect to a certain measure μ (see Proposition 3.3). To define a formula for the image of u in $F_v^\times$, we lift the measure μ to a measure ν on $\mathcal{O}_v^\times$; this lifting depends on the choice of a fundamental domain $\mathcal{D}$ for the action of E on $F \otimes_{\mathbf{Q}} \mathbf{R}$. We follow Shintani by choosing $\mathcal{D}$ to be a union of simplicial cones (defined in Section 3.3). Shintani defined these fundamental domains in order to find explicit formulas for the values of the partial zeta-functions of H/F at nonpositive integers [19]. We use Shintani's domains and his formulas for the special values in order to prove that the lifted measures we define are in fact p-adically bounded. Our calculations are quite closely related to those of Cassou-Nogues [1], who used Shintani's methods to define the p-adic L-functions of H/F.

In Section 3 we present our conjectural formula for u, and in the following section we briefly describe a computation providing evidence for this conjecture in a case where F is a totally real cubic field. In Section 5 we discuss the dependence of our formula for u on the choice of Shintani domain $\mathcal{D}$; modulo some technicalities, we essentially show that our formula is independent of $\mathcal{D}$. We also prove (again modulo some technicalities) that Gross's Conjecture 2.6 is actually equivalent to our main Conjecture 3.21. In Section 6 we prove the p-adic boundedness of the lifted measure ν; in fact, we show that ν is $\mathbf{Z}$-valued. In Section 7 we remark on the consistency of our formula with the norm compatibility relations for Gross–Stark units. We conclude in Section 8 by studying the case where F is real quadratic, and comparing our formulas with those of [7].

We conclude this introduction by remarking that it remains to connect the constructions of the current article with work of others on the p-adic multiple Gamma-function. This function was studied using Shintani's methods and linked to Gross's conjectures by various authors, notably Cassou-Nogues [2] and Kashio–Yoshida [13].

It is a pleasure to thank the Centre de Recherches Mathématiques in Montreal, Quebec, where much of this research was conducted during a visit in the fall of 2005. In particular, I greatly benefited from discussions with Hugo Chapdelaine, Pierre Charollois, and Henri Darmon. I am also indebted to Pierre Colmez, whose suggestions were vital for the arguments of Section 5. Finally, I would like to acknowledge the extraordinary efforts of the anonymous referees, whose remarkably detailed comments and suggestions greatly improved the quality of the exposition.

2 The conjectures of Gross

Let F be a totally real field, and let $\mathfrak{p}$ be a prime of F lying above the rational prime p of $\mathbf{Q}$. Let H be a finite abelian extension of F such that $\mathfrak{p}$ splits completely in H. Let S be a finite set of primes of F containing the archimedean primes, the primes lying above p, and those ramifying in H. We assume throughout this article that $\#S \geq 3$, since the only cases that this excludes have $H = F = \mathbf{Q}$. Write $R = S - \{\mathfrak{p}\}$.

For $\sigma \in G = \mathrm{Gal}(H/F)$, define the partial zeta-function

$$\zeta_R(\sigma, s) = \sum_{\substack{(\mathfrak{a},R)=1 \\ \sigma_{\mathfrak{a}}=\sigma}} \mathrm{N}\mathfrak{a}^{-s}. \tag{1}$$

The sum is over all integral ideals $\mathfrak{a} \subset \mathcal{O}$ that are relatively prime to the elements of R and whose associated Frobenius element $\sigma_{\mathfrak{a}} \in G$ is equal to σ. The series (1) converges for $\mathrm{Re}(s) > 1$ and has a meromorphic continuation to $\mathbf{C}$, regular outside $s = 1$. The zeta-functions associated to the sets of primes S and R are related by the formula

$$\zeta_S(\sigma, s) = (1 - \mathrm{N}\mathfrak{p}^{-s})\zeta_R(\sigma, s).$$

Deligne and Ribet [9] and Cassou-Nogues [1] independently proved the existence of a $\mathbf{Q}_p$-valued function $\zeta_{S,p}(\sigma, s)$, meromorphic on $\mathbf{Z}_p$ and regular outside $s = 1$, such that

$$\zeta_{S,p}(\sigma, n) = \zeta_S(\sigma, n) \tag{2}$$

for nonpositive integers $n \equiv 0 \pmod{d}$, where $d = [F(\mu_{2p}) : F]$. In particular $\zeta_{S,p}(\sigma, 0) = 0$ for all $\sigma \in G$.

We remark that in (2) we have followed Gross's normalization of the p-adic zeta-function as in [11], rather than the more standard normalization of Serre [15]. To be precise, let χ be a complex character on G, and let ω denote the Teichmuller character. Serre's p-adic L-function attached to the character $\chi\omega$ satisfies the interpolation property

$$L_{S,p}(1 - n, \chi\omega) = L_S(1 - n, \chi\omega^{1-n})$$

for all integers $n \geq 1$. The function we have denoted $\zeta_{S,p}(\sigma, n)$ in (2) is then given by the formula

$$\zeta_{S,p}(\sigma, s) = \frac{1}{\#G} \sum_{\chi \in \hat{G}} \chi^{-1}(\sigma) L_{S,p}(s, \chi\omega),$$

where the sum is over all irreducible characters χ of G.

2.1 The first Gross conjecture

Define the group

$$U_{\mathfrak{p}} = \{u \in H^{\times} : |u|_{\mathfrak{P}} = 1 \text{ if } \mathfrak{P} \text{ does not divide } \mathfrak{p}\}. \tag{3}$$

Here $\mathfrak{P}$ ranges over all finite and archimedean places of H; in particular each complex conjugation in H acts as inversion on $U_{\mathfrak{p}}$. For each divisor $\mathfrak{P}$ of $\mathfrak{p}$ in H, extend the $\mathfrak{P}$-adic valuation $\mathrm{ord}_{\mathfrak{P}} : U_{\mathfrak{p}} \to \mathbf{Z}$ to the tensor product $\mathbf{Q} \otimes U_{\mathfrak{p}} \to \mathbf{Q}$. Gross proved:

Proposition 2.1 ([11], Proposition 3.8)**.** *For each divisor $\mathfrak{P}$ of $\mathfrak{p}$ in H, there is a unique element $u = u(\mathfrak{P}) \in \mathbf{Q} \otimes U_{\mathfrak{p}}$ such that*

$$\zeta_R(\sigma, 0) = \mathrm{ord}_{\mathfrak{P}}(u^{\sigma}) \textit{ for all } \sigma \in G.$$

Since $\mathfrak{p}$ splits completely in H, we have $H \subset H_{\mathfrak{P}} \cong F_{\mathfrak{p}}$. Let $\log_p : \mathbf{Q}_p^{\times} \to \mathbf{Z}_p$ be the branch of the Iwasawa p-adic logarithm for which $\log_p(p) = 0$. The map

$$u \mapsto \log_p \mathrm{Norm}_{F_{\mathfrak{p}}/\mathbf{Q}_p} u$$

from $U_{\mathfrak{p}}$ to $\mathbf{Z}_p$ may be extended to a map $\mathbf{Q} \otimes U_{\mathfrak{p}} \to \mathbf{Q}_p$ by tensoring with $\mathbf{Q}$. Gross stated:

Conjecture 2.2 ([11], Conjecture 2.12 and Proposition 3.8)**.** *Let $u = u(\mathfrak{P})$ be as above. Then*

$$\zeta'_{S,p}(\sigma, 0) = -\log_p \mathrm{Norm}_{F_{\mathfrak{p}}/\mathbf{Q}_p}(u^{\sigma}) \textit{ for all } \sigma \in G.$$

In order to state an integral version of Conjecture 2.2, we introduce an auxiliary finite set T of primes of F, disjoint from S. Define the partial zeta-function associated to the sets S and T by the group ring equation

$$\sum_{\sigma \in G} \zeta_{S,T}(\sigma, s)[\sigma] = \prod_{\eta \in T} (1 - [\sigma_{\eta}] \, \mathrm{N}\eta^{1-s}) \sum_{\sigma \in G} \zeta_S(\sigma, s)[\sigma]. \tag{4}$$

Suppose now that T satisfies the following assumption:

Assumption 2.3. *The set T contains at least two primes of different residue characteristic or at least one prime η with absolute ramification degree at most $\ell - 2$, where η lies above ℓ.*

When this assumption is satisfied, the values $\zeta_{S,T}(K/F,\sigma,0)$ are rational integers for any abelian extension K/F unramified outside S and any $\sigma\in\mathrm{Gal}(K/F)$ (see [12, Proposition 3.7] and the preceding discussion). In Section 3.3, we will make an even stronger assumption on T in order to ensure the integrality at 0 of a more general type of zeta-function. We now state an integral version of Conjecture 2.2. While this conjecture does not appear explicitly in the literature, we shall see shortly that it lies between Gross's first Conjecture 2.2 and his second Conjecture 2.6.

Conjecture 2.4. *There exists* $u_T=u_T(\mathfrak{P})\in U_{\mathfrak{p}}$ *such that* $\mathrm{ord}_{\mathfrak{P}}(u_T^\sigma)=\zeta_{R,T}(\sigma,0)$ *and*

$$\zeta'_{S,T,p}(\sigma,0)=-\log_p \mathrm{Norm}_{F_{\mathfrak{p}}/\mathbf{Q}_p}(u_T^\sigma)$$

for all $\sigma\in G$.

Here $\zeta_{S,T,p}(\sigma,s)$ is defined in terms of $\zeta_{S,p}(\sigma,s)$ as in (4), with $\mathrm{N}\eta$ replaced by $\langle \mathrm{N}\eta\rangle$; for $x\in\mathbf{Z}_p^\times$ we define $\langle x\rangle$ to be the unique element of $1+2p\mathbf{Z}_p$ whose ratio with x is a root of unity. The element

$$g_T:=\prod_{\eta\in T}(1-[\sigma_\eta]^{-1}\,\mathrm{N}\eta)$$

is invertible in the group ring $\mathbf{Q}[G]$, and letting $u=u_T^{g_T^{-1}}$ shows that Conjecture 2.4 implies Conjecture 2.2. Conversely, letting $u_T=u^{g_T}$ shows that Conjecture 2.2 implies the existence of an element $u_T\in U_{\mathfrak{p}}\otimes\mathbf{Q}$ (but not necessarily $U_{\mathfrak{p}}$) satisfying the conditions of Conjecture 2.4.

2.2 Stark's conjecture and the second Gross conjecture

We now remove the assumption on S that it contain all the primes of F lying above p. Thus S is required only to contain $\mathfrak{p}$, the archimedean primes, and those which ramify in H. We have the following conjecture of Stark, as formulated by Gross:

Conjecture 2.5 ([12], Conjecture 7.4)**.** *There exists an element* $u_T\in U_{\mathfrak{p}}$ *such that* $u_T\equiv 1 \pmod{T}$ *and for all* $\sigma\in G$ *we have*

$$\mathrm{ord}_{\mathfrak{P}}(u_T^\sigma)=\zeta_{R,T}(H/F,\sigma,0). \tag{5}$$

Assumption 2.3 implies that there are no non-trivial roots of unity in H which are congruent to 1 modulo T. Thus the $\mathfrak{p}$-unit u_T, if it exists, is unique. Note also that our u_T is actually the inverse of the u in [12, Conjecture 7.4]; we have made this choice to remain consistent with Proposition 2.1 and Gross's earlier paper [11].

In [12], Gross stated a conjecture which simultaneously strengthens Conjecture 2.4 and Conjecture 2.5. Let K be an auxiliary finite abelian extension of F containing H and unramified outside S. Let

$$\mathrm{rec}_{\mathfrak{p}}:F_{\mathfrak{p}}^\times\to\mathbf{A}_F^\times\to\mathrm{Gal}(K/F) \tag{6}$$

denote the reciprocity map of local class field theory. From $H\subset H_{\mathfrak{P}}\cong F_{\mathfrak{p}}$ we may evaluate $\mathrm{rec}_{\mathfrak{p}}$ on any element of $H^\times$; the image will be contained in $\mathrm{Gal}(K/H)$.

Conjecture 2.6 ([12], Conjecture 7.6)**.** *Assume Conjecture 2.5. For all $\sigma \in G$ we have*

$$\mathrm{rec}_{\mathfrak{p}}(u_T^\sigma) = \prod_{\substack{\tau \in \mathrm{Gal}(K/F) \\ \tau|_H = \sigma^{-1}}} \tau^{\zeta_{S,T}(K/F, \tau^{-1}, 0)}. \tag{7}$$

We remark that

$$\sum_{\substack{\tau \in \mathrm{Gal}(K/F) \\ \tau|_H = \sigma^{-1}}} \zeta_{S,T}(K/F, \tau, 0) = \zeta_{S,T}(H/F, \sigma^{-1}, 0) = 0,$$

so the right side of (7) lies in $\mathrm{Gal}(K/H)$. Also, the inverses in (7) appear because as noted above, our u_T is the inverse of Gross's u.

The conjectural element $u_T \in U_{\mathfrak{p}}$ satisfying Conjecture 2.6 is called the *Gross–Stark unit* for the data $(S, T, H, \mathfrak{P})$. Gross did not *prove* that Conjecture 2.6 was a strengthening of Conjecture 2.4, but he was certainly aware of this fact; for completeness, we include the proof in the next section.

3 Measures

The remainder of this paper is devoted to proposing a formula for the value in $F_{\mathfrak{p}}^\times$ of the element u_T of Conjecture 2.6. To set the stage for this formula, we will restate Conjecture 2.6 in terms of $\mathfrak{p}$-adic measures.

Definition 3.1. Let G be a compact open subset of a quotient of $\mathbf{A}_F^\times$, and let A be any abelian group. An *A-valued distribution μ on G* is an assignment $\mu(U) \in A$ to each compact open set $U \subset G$, such that $\mu(U \cup V) = \mu(U) + \mu(V)$ for disjoint compact opens U and V.

In this article, we will most often be concerned with distributions satisfying $\mu(G) = 0$.

Definition 3.2. A $\mathbf{Q}_p$-valued distribution on G is called a *p-adic measure* if it takes values in a p-adically bounded subgroup $A \subset \mathbf{Q}_p$.

We will consider two types of integrals on G.

The Additive Integral. Let $f : G \to \mathbf{Z}_p$ be a continuous map, and let μ be a $\mathbf{Z}_p$-valued measure on G. Define

$$\int_G f(x) d\mu(x) := \varprojlim \sum_{a \pmod{p^n}} a \cdot \mu(f^{-1}(a + p^n \mathbf{Z}_p)) \in \mathbf{Z}_p,$$

where the inverse limit is over all positive integers n.

The Multiplicative Integral. Let I be an abelian topological group which may be written as an inverse limit of discrete groups:

$$I = \varprojlim I_\alpha.$$

Denote the group operation on I multiplicatively. For each $i \in I_\alpha$, denote by U_i the open subset of I consisting of those elements which map to i in I_α. Let $f : G \to I$ be a continuous map and let μ be a $\mathbf{Z}$-valued measure. We define the multiplicative integral, written with a cross through the integration sign, by:

$$\times\!\!\!\!\!\!\int_G f(x)d\mu(x) = \varprojlim \prod_{i \in I_\alpha} i^{\mu(f^{-1}(U_i))} \in I. \tag{8}$$

Note that for each α, only finitely many of the sets $f^{-1}(U_i)$ are non-empty since G is compact and f is continuous; thus the product in (8) makes sense.

3.1 Restatement of Gross's conjecture

Let $\mathfrak{f}$ be an integral ideal of F relatively prime to $\mathfrak{p}$, and denote by $H_\mathfrak{f}$ the narrow ray class field of F of conductor $\mathfrak{f}$. The map which sends a fractional ideal $\mathfrak{b}$ relatively prime to $\mathfrak{f}$ to its Frobenius element $\sigma_\mathfrak{b}$ induces an isomorphism between the narrow ray class group of conductor $\mathfrak{f}$, denoted $G_\mathfrak{f}$, and the Galois group $\mathrm{Gal}(H_\mathfrak{f}/F)$.

As in Section 2, let H be a finite abelian extension of F unramified outside S, in which $\mathfrak{p}$ splits completely. Let H' be a subfield of H containing F. Suppose that $u_T \in U_\mathfrak{p}(H)$ satisfies Conjecture 2.6 for the data $(S, T, H, \mathfrak{P})$. Then it is easy to verify that $\mathrm{N}_{H/H'}(u_T)$ satisfies the conjecture for the data $(S, T, H', \mathfrak{P} \cap \mathcal{O}_{H'})$. This is the "norm compatibility relation" for Gross–Stark units.

In attempting to construct Gross–Stark units, it therefore suffices to consider, for every ideal $\mathfrak{f}$, the case where H is the maximal subextension of $H_\mathfrak{f}/F$ in which $\mathfrak{p}$ splits completely; we now fix this choice of H. The finite set S contains at least $\mathfrak{p}$, the primes dividing $\mathfrak{f}$, and the archimedean primes. Formula (7) of Conjecture 2.6 gives $\mathfrak{p}$-adic information about u_T when the extension K/F is ramified above $\mathfrak{p}$. Define $H_{\mathfrak{f}\mathfrak{p}^\infty}$ to be the union of the narrow ray class fields $H_{\mathfrak{f}\mathfrak{p}^m}$ for all positive integers m. We will now analyze what information about u_T can be gleaned from Conjecture 2.6 applied to all fields $K \subset H_{\mathfrak{f}\mathfrak{p}^\infty}$. At the end of this section, we will conduct a similar analysis for *all* possible fields K.

For a fractional ideal $\mathfrak{a}$ prime to S, denote by $\sigma_\mathfrak{a}$ the Frobenius automorphism attached to $\mathfrak{a}$ in $\mathrm{Gal}(H_{\mathfrak{f}\mathfrak{p}^\infty}/F)$. The map (6) induces an isomorphism

$$\mathrm{rec}_\mathfrak{p} : F_\mathfrak{p}^\times / \widehat{E_\mathfrak{p}(\mathfrak{f})} \cong \mathrm{Gal}(H_{\mathfrak{f}\mathfrak{p}^\infty}/H), \tag{9}$$

where $E_\mathfrak{p}(\mathfrak{f})$ denotes the group of totally positive $\mathfrak{p}$-units of F that are congruent to 1 modulo $\mathfrak{f}$, and $\widehat{E_\mathfrak{p}(\mathfrak{f})}$ denotes its closure in $F_\mathfrak{p}^\times$. (Note that if Leopoldt's conjecture holds, then $\widehat{E_\mathfrak{p}(\mathfrak{f})}$ is of finite index in $F_\mathfrak{p}^\times$ unless $\mathfrak{p}$ is the only prime above p; in this finite index case, the integral in Proposition 3.3 below is just a finite product and equation (13) is simply a restatement of (7) for the finite extension $K = H_{\mathfrak{f}\mathfrak{p}^\infty}$.)

Let e be the order of $\mathfrak{p}$ in $G_\mathfrak{f}$, and suppose that $\mathfrak{p}^e = (\pi)$ with $\pi \equiv 1 \pmod{\mathfrak{f}}$ and π totally positive. Then $E_\mathfrak{p}(\mathfrak{f}) \cong \langle \pi \rangle \times E(\mathfrak{f})$, where $E(\mathfrak{f})$ denotes the group of totally positive units of $\mathcal{O}$ that are congruent to 1 modulo $\mathfrak{f}$. Note that $\mathbf{O} := \mathcal{O}_\mathfrak{p} - \pi\mathcal{O}_\mathfrak{p} \subset F_\mathfrak{p}^\times$ is a fundamental domain for the action of $\langle \pi \rangle$ on $F_\mathfrak{p}^\times$.

Let $\mathfrak{b}$ be a fractional ideal of F relatively prime to S and T. Let z be an element of F with $z \in \mathfrak{b}^{-1}$ and $z \equiv 1 \pmod{\mathfrak{f}}$. (For a general element $z \in F^\times$, the congruence $z \equiv 1 \pmod{\mathfrak{f}}$ means that $z - 1 \in \mathfrak{f}\mathcal{O}_\mathfrak{f} \cap F$, where $\mathcal{O}_\mathfrak{f}$ is the $\mathfrak{f}$-adic completion of $\mathcal{O}$.) Normalize the $\mathfrak{p}$-adic norm by

$$|\alpha|_\mathfrak{p} = \mathrm{N}\mathfrak{p}^{-v_\mathfrak{p}(\alpha)}.$$

For each compact open set $U \subset \mathbf{O}/\widehat{E(\mathfrak{f})}$ and $\mathrm{Re}(s) > 1$ define

$$\zeta_S(\mathfrak{b}, U, s) = \sum_{\substack{\mathfrak{a} \subset \mathcal{O}, (\mathfrak{a}, S) = 1 \\ \sigma_\mathfrak{a} \in \sigma_\mathfrak{b} \cdot \mathrm{rec}_\mathfrak{p}(U)^{-1}}} \mathrm{N}\mathfrak{a}^{-s} = \mathrm{N}\mathfrak{b}^{-s} \sum_\alpha (\mathrm{N}\alpha |\alpha|_\mathfrak{p})^{-s}, \tag{10}$$

where the second sum ranges over distinct representatives mod $E(\mathfrak{f})$ of totally positive

$$\alpha \in (\mathfrak{b}^{-1}\mathfrak{f} + z) \cap U \text{ with } (\alpha, R) = 1.$$

The condition $\alpha \in \mathfrak{b}^{-1}\mathfrak{f} + z$ is equivalent to $\alpha \in \mathfrak{b}^{-1}, \ \alpha \equiv 1 \pmod{\mathfrak{f}}$, and it thus does not depend on the choice of z. The equality of the two sums in (10) follows from the change of variables $\mathfrak{a}\mathfrak{b}^{-1} = (\alpha)\mathfrak{p}^{-v_\mathfrak{p}(\alpha)}$ and the fact that $\mathrm{rec}_\mathfrak{p}(\alpha)^{-1} = \sigma_{(\alpha)\mathfrak{p}^{-v_\mathfrak{p}(\alpha)}}$. For a fixed compact open set U, the function $\zeta_S(\mathfrak{b}, U, s)$ is a finite sum of partial zeta-functions attached to some finite extension $H_{\mathfrak{f}\mathfrak{p}^m}/F$.

We define $\zeta_{S,T}(\mathfrak{b}, U, s)$ in analogy with (4); suppose that

$$\prod_{\eta \in T} (1 - [\eta]\, \mathrm{N}\eta^{1-s}) = \sum_\mathfrak{a} c_\mathfrak{a}(s)[\mathfrak{a}]$$

in the group ring of fractional ideals with coefficients in the ring of complex valued functions on $\mathbf{C}$, and define

$$\zeta_{S,T}(\mathfrak{b}, U, s) = \sum_\mathfrak{a} c_\mathfrak{a}(s) \zeta_S(\mathfrak{a}^{-1}\mathfrak{b}, U, s). \tag{11}$$

We may then extend $\zeta_{S,T}$ by analytic continuation and define a $\mathbf{Z}$-valued measure $\mu(\mathfrak{b})$ on $\mathbf{O}/\widehat{E(\mathfrak{f})}$ by:

$$\mu(\mathfrak{b}, U) := \zeta_{S,T}(\mathfrak{b}, U, 0). \tag{12}$$

Note that in particular

$$\mu(\mathfrak{b}, \mathbf{O}/\widehat{E(\mathfrak{f})}) = \zeta_{S,T}(H/F, \mathfrak{b}, 0) = 0.$$

Proposition 3.3. *If Conjecture 2.6 is true, then we have the formula*

$$u_T^{\sigma_\mathfrak{b}} = \pi^{\zeta_{R,T}(H_\mathfrak{f}/F, \mathfrak{b}, 0)} \times\!\!\!\!\!\!\int_{\mathbf{O}/\widehat{E(\mathfrak{f})}} x \, d\mu(\mathfrak{b}, x) \tag{13}$$

in $F_\mathfrak{p}^\times/\widehat{E(\mathfrak{f})}$.

The integrand x in (13) is the inclusion $\mathbf{O}/\widehat{E(\mathfrak{f})} \hookrightarrow F_\mathfrak{p}^\times/\widehat{E(\mathfrak{f})}$.

Proof. The contribution of the integral to the $\mathfrak{p}$-adic valuation of the right side of (13) is

$$\begin{aligned}\sum_{i=0}^{e-1} i\cdot\mu(\mathfrak{b},\mathfrak{p}^i\mathcal{O}_{\mathfrak{p}}^\times/\widehat{E(\mathfrak{f})}) &= \sum_{i=0}^{e-1} i\cdot\zeta_{S,T}(H_{\mathfrak{f}}/F,\mathfrak{b}\mathfrak{p}^{-i},0)\\ &= \sum_{i=0}^{e-1} i\cdot\big(\zeta_{R,T}(H_{\mathfrak{f}}/F,\mathfrak{b}\mathfrak{p}^{-i},0)-\zeta_{R,T}(H_{\mathfrak{f}}/F,\mathfrak{b}\mathfrak{p}^{-i-1},0)\big)\\ &= \left(\sum_{i=1}^{e}\zeta_{R,T}(H_{\mathfrak{f}}/F,\mathfrak{b}\mathfrak{p}^{-i},0)\right)-e\cdot\zeta_{R,T}(H_{\mathfrak{f}}/F,\mathfrak{b}\mathfrak{p}^{-e},0)\\ &= \zeta_{R,T}(H/F,\mathfrak{b},0)-e\cdot\zeta_{R,T}(H_{\mathfrak{f}}/F,\mathfrak{b},0). \qquad (14)\end{aligned}$$

Combining (14) with the power of π in the right side of (13), the entire expression has the correct $\mathfrak{p}$-adic valuation $\zeta_{R,T}(H/F,\mathfrak{b},0)$ as prescribed by (5).

It remains to check that the image in $F_{\mathfrak{p}}^\times/\widehat{E_{\mathfrak{p}}(\mathfrak{f})}$ of the integral in (13) equals the value proposed for $u_T^{\sigma_{\mathfrak{b}}}$ by Conjecture 2.6. Let $K=H_{\mathfrak{f}\mathfrak{p}^m}$, and let $U_{\mathfrak{p}^m}$ be the subgroup of $\mathcal{O}_{\mathfrak{p}}^\times$ consisting of those elements that are congruent to 1 (mod $\mathfrak{p}^m$). Then $\mathrm{rec}_{\mathfrak{p}}$ provides an isomorphism $\mathrm{rec}_{\mathfrak{p}}: F_{\mathfrak{p}}^\times/(U_{\mathfrak{p}^m}E_{\mathfrak{p}}(\mathfrak{f}))\to\mathrm{Gal}(K/H)$. Applying the inverse of $\mathrm{rec}_{\mathfrak{p}}$ to equation (7) and using the change of variables $\alpha=\mathrm{rec}_{\mathfrak{p}}^{-1}(\tau\sigma_{\mathfrak{b}})$ yields

$$u_T^{\sigma_{\mathfrak{b}}}\equiv\prod_{\alpha\in F_{\mathfrak{p}}^\times/(U_{\mathfrak{p}^m}E_{\mathfrak{p}}(\mathfrak{f}))}\alpha^{\zeta_{S,T}(K/F,\sigma_{\mathfrak{b}}\cdot\mathrm{rec}_{\mathfrak{p}}\alpha^{-1},0)}\pmod{U_{\mathfrak{p}^m}E_{\mathfrak{p}}(\mathfrak{f})}.$$

To compare with the right side of (13), note that each element $\alpha\in F_{\mathfrak{p}}^\times/(U_{\mathfrak{p}^m}E_{\mathfrak{p}}(\mathfrak{f}))$ has a unique representative $\tilde{\alpha}\in\mathbf{O}/(U_{\mathfrak{p}^m}E(\mathfrak{f}))$. It remains to prove that if $U_{\tilde{\alpha}}$ denotes the inverse image of $\tilde{\alpha}$ in $\mathbf{O}/\widehat{E(\mathfrak{f})}$, then

$$\mu(\mathfrak{b},U_{\tilde{\alpha}})=\zeta_{S,T}(K/F,\sigma_{\mathfrak{b}}\cdot\mathrm{rec}_{\mathfrak{p}}\alpha^{-1},0).$$

But this is exactly how the measure $\mu(\mathfrak{b})$ has been defined. □

We now present a version of equation (13) which is *equivalent* to Conjecture 2.6. Equation (13) packages together equation (7) for all extensions $K\subset H_{\mathfrak{f}\mathfrak{p}^\infty}$; to obtain an equality which encapsulates equation (7) for all possible extensions K, we must allow ramification at all primes in S. Let $\mathfrak{g}$ denote the product of the finite primes in S which do not divide $\mathfrak{f}\mathfrak{p}$. Then the compositum of fields K for which Conjecture 2.6 may be applied is $H_S:=H_{(\mathfrak{f}\mathfrak{p}\mathfrak{g})^\infty}$. For $v|\mathfrak{f}\mathfrak{g}$, let $U_{v,\mathfrak{f}}$ denote the group of elements of $\mathcal{O}_v^\times$ that are congruent to 1 modulo $\mathfrak{f}\mathcal{O}_v$; in particular $U_{v,\mathfrak{f}}=\mathcal{O}_v^\times$ for $v|\mathfrak{g}$. Let $\mathcal{U}=\prod_{v|\mathfrak{f}\mathfrak{g}}U_{v,\mathfrak{f}}$. The reciprocity map induces an isomorphism

$$\mathrm{rec}_S:(F_{\mathfrak{p}}^\times\times\mathcal{U})/\overline{E_{\mathfrak{p}}(\mathfrak{f})}\cong\mathrm{Gal}(H_S/H). \qquad (15)$$

Here we have denoted by $\overline{E_{\mathfrak{p}}(\mathfrak{f})}$ the closure of $E_{\mathfrak{p}}(\mathfrak{f})$ diagonally embedded in $F_{\mathfrak{p}}^\times\times\mathcal{U}$, to distinguish it from $\widehat{E_{\mathfrak{p}}(\mathfrak{f})}\subset F_{\mathfrak{p}}^\times$. We may now carry over our previous methods with $\mathbf{O}/\widehat{E(\mathfrak{f})}$ replaced by $(\mathbf{O}\times\mathcal{U})/\overline{E(\mathfrak{f})}$. More precisely, define a measure $\mu(\mathfrak{b})$ on this latter space via equations (10), (11), and (12) with the notational change that in equation (10), U is a compact open subset of $(\mathbf{O} \times \mathcal{U})/\overline{E(\mathfrak{f})}$, the map $\mathrm{rec}_{\mathfrak{p}}$ is replaced by rec_S, and $\sigma_{\mathfrak{a}}, \sigma_{\mathfrak{b}}$ are elements of $\mathrm{Gal}(H_S/F)$. Then we have:

Proposition 3.4. *Conjecture 2.6 is equivalent to the existence of an element $u_T \in U_{\mathfrak{p}}$ with $u_T \equiv 1 \pmod{T}$ and*

$$(u_T^{\sigma_{\mathfrak{b}}}, 1) = \pi^{\zeta_{R,T}(H_{\mathfrak{f}}/F,\mathfrak{b},0)} \times\!\!\!\!\!\!\int_{\mathbf{O}\times\mathcal{U}/\overline{E(\mathfrak{f})}} x \, d\mu(\mathfrak{b}, x) \qquad (16)$$

in $(F_{\mathfrak{p}}^{\times} \times \mathcal{U})/\overline{E(\mathfrak{f})}$, for all fractional ideals $\mathfrak{b}$ relatively prime to S.

Define

$$D(\mathfrak{f}, \mathfrak{g}) = \{x \in F_{\mathfrak{p}}^{\times} : (x, 1) \in \overline{E(\mathfrak{f})} \subset F_{\mathfrak{p}}^{\times} \times \mathcal{U}\}.$$

Proposition 3.4 may be interpreted as stating that Gross's conjecture is equivalent to a formula for the image of u_T in $F_{\mathfrak{p}}^{\times}/D(\mathfrak{f}, \mathfrak{g})$. If S does not contain any primes lying above p other than $\mathfrak{p}$, then $D(\mathfrak{f}, \mathfrak{g})$ has finite index in $\widehat{E(\mathfrak{f})}$. Even without this assumption, Leopoldt's conjecture implies that $D(\mathfrak{f}, \mathfrak{g})$ will have positive dimension as a p-adic Lie group (i.e. $D(\mathfrak{f}, \mathfrak{g}) \otimes \mathbf{Q}_p$ will be nonzero) unless S contains all of the primes above p and $F_{\mathfrak{p}} = \mathbf{Q}_p$. Note that this is precisely the case when Gross's *first* Conjecture 2.4 already determines the image of u_T in $F_{\mathfrak{p}}^{\times}$ up to a root of unity.

By expanding the set S in an appropriate way, one can shrink the subgroup $D(\mathfrak{f}, \mathfrak{g})$ to gain more $\mathfrak{p}$-adic information about u_T. Repeating this process indefinitely one can specify u_T in $F_{\mathfrak{p}}^{\times}$ to any specified degree of $\mathfrak{p}$-adic accuracy. However, there is a certain lack of explicitness involved in this process. To specify u_T modulo $\mathfrak{p}^m$, one must first adjoin enough primes to S such that $D(\mathfrak{f}, \mathfrak{g}) \subset 1 + \mathfrak{p}^m \mathcal{O}_{\mathfrak{p}}$. Then one must calculate the integral in the right-side of (16), and find a representative mod $\overline{E(\mathfrak{f})}$ of the form $(x, 1)$. Then x is uniquely determined mod $\mathfrak{p}^m$, and if the primes adjoined to S were chosen appropriately one can determine u_T mod $\mathfrak{p}^m$ from x. This is discussed in greater detail in Theorem 5.18.

The goal of this article is to provide one concise formula for u_T in all cases, avoiding the process of artificially enlarging S and the technicalities introduced therein. We conclude this section by proving:

Proposition 3.5. *Conjecture 2.6 implies Conjecture 2.2.*

Proof. We will prove Conjecture 2.2 via Conjecture 2.4. To place ourselves in the setting where these two conjectures apply, assume that S contains all the primes dividing p. Let $\mathrm{N}\colon F_{\mathfrak{p}}^{\times} \times \mathcal{U} \to \mathbf{Z}_p^{\times}$ denote the norm map, which sends an element $(x_v)_{v\in S}$ to the p-adic unit part of $\prod_{v|p} \mathrm{Norm}_{F_v/\mathbf{Q}_p} x_v$. (By p-adic unit part we mean that the appropriate power of p is divided out to obtain an element of $\mathbf{Z}_p^{\times}$.) For a totally positive element $x \in \mathcal{O}$ whose diagonally embedded image in $\prod_{v\in S} F_v^{\times}$ lies in $F_{\mathfrak{p}}^{\times} \times \mathcal{U}$, the value of $\mathrm{N}x$ is the p-adic unit part of the image of $\mathrm{Norm}_{F/\mathbf{Q}}\, x$ in $\mathbf{Z}_p$. In particular, $\overline{E_{\mathfrak{p}}(\mathfrak{f})}$ lies in the kernel of N.

Deligne and Ribet [9] and Cassou-Nogues [1] proved that the measures μ on $\mathbf{O} \times \mathcal{U}/\overline{E(\mathfrak{f})}$ have the following interpolation property:

$$\zeta_{S,T}(\sigma_{\mathfrak{b}}, s) = \int_{\mathbf{O}\times\mathcal{U}/\overline{E(\mathfrak{f})}} (\mathrm{N}x)^{-s} d\mu(\mathfrak{b}, x) \qquad (17)$$

for nonpositive integers s. For a precise reference for (17) in the case where T is the set containing one ideal c, see the last two subsections of the introduction in [9]. The ideal $\mathfrak{f}$ of [9] is played by our $\mathfrak{f}\mathfrak{g}\mathfrak{p}$, and the function ϵ on the narrow ray class group of conductor $\mathfrak{f}\mathfrak{g}\mathfrak{p}$ is the characteristic function of the set of ideals $\mathfrak{a}$ such that $\sigma_{\mathfrak{a}}|_H = \sigma_{\mathfrak{b}}|_H$. In [9], $\mu_{c,1}$ is a certain measure on $\mathrm{Gal}(H_S/F)$, and via

$$(\mathbf{O} \times \mathcal{U})/\overline{E(\mathfrak{f})} \hookrightarrow (F_{\mathfrak{p}}^\times \times \mathcal{U})/\overline{E_{\mathfrak{p}}(\mathfrak{f})} \cong \mathrm{Gal}(H_S/H) \hookrightarrow \mathrm{Gal}(H_S/F)$$

one checks that we have an equality of measures $\mu(\mathfrak{b}) = \epsilon \cdot \mu_{c,1}$. Equation (17) above is then a restatement of the equation at the top of page 232 in [9].

When $s \equiv 0 \pmod{2(p-1)}$, the left side of (17) equals the p-adic zeta-function $\zeta_{S,T,p}(\sigma_{\mathfrak{b}}, s)$, by definition. Furthermore, for such s, the function $(\mathrm{N}x)^{-s}$ may be replaced by $\langle \mathrm{N}x\rangle^{-s}$. Thus we have

$$\zeta_{S,T,p}(\sigma_{\mathfrak{b}}, s) = \int_{\mathbf{O}\times\mathcal{U}/\overline{E(\mathfrak{f})}} \langle \mathrm{N}x\rangle^{-s} d\mu(\mathfrak{b}, x)$$

for all $s \in 2\mathbf{Z}_p$. Differentiating, we obtain

$$\zeta'_{S,T,p}(\sigma_{\mathfrak{b}}, 0) = -\int_{\mathbf{O}\times\mathcal{U}/\overline{E(\mathfrak{f})}} \log_p \mathrm{N}x \; d\mu(\mathfrak{b}, x).$$

Proposition 3.4 then shows that Conjecture 2.6 implies Conjecture 2.4 and hence Conjecture 2.2. □

On the Galois side, the map N in the proof of Proposition 3.5 is simply the restriction from $\mathrm{Gal}(H_S/H)$ to $\mathrm{Gal}(\mathbf{Q}(\mu_{p^\infty})/\mathbf{Q}) \cong \mathbf{Z}_p^\times$. Thus Conjecture 2.4 is equivalent to Conjecture 2.6 for K restricted to lie in the cyclotomic $\mathbf{Z}_p$-extension of H.

3.2 A lifted measure

We now introduce our method to find an exact formula for u_T in $F_{\mathfrak{p}}^\times$, rather than just modulo $\widehat{E(\mathfrak{f})}$ or $D(\mathfrak{f}, \mathfrak{g})$. As above, write $\mathfrak{p}^e = (\pi)$ with π totally positive and $\pi \equiv 1 \pmod{\mathfrak{f}}$. Let $\mathfrak{b}$ be a fractional ideal of F relatively prime to S and T. Motivated by [7], we will attempt to define a measure $\nu(\mathfrak{b})$ on $\mathbf{O}$ such that its push forward to $\mathbf{O}/\widehat{E(\mathfrak{f})}$ under the natural projection is equal to $\mu(\mathfrak{b})$; in other words, for a compact open $U \subset \mathbf{O}/\widehat{E(\mathfrak{f})}$ with inverse image $\widetilde{U}$ in $\mathbf{O}$, we will require $\nu(\mathfrak{b}, \widetilde{U}) = \mu(\mathfrak{b}, U)$. We will then propose a formula of the form

$$u_T^{\sigma_{\mathfrak{b}}} = \epsilon \cdot \pi^{\zeta_{R,T}(H_{\mathfrak{f}}/F,\mathfrak{b},0)} \times\!\!\!\!\!\!\int_{\mathbf{O}} x \; d\nu(\mathfrak{b}, x) \tag{18}$$

in $F_{\mathfrak{p}}^\times$, where $\epsilon \in E(\mathfrak{f})$ is an "error term" defined in such a way that the right hand side of (18) depends only on $\sigma_{\mathfrak{b}} \in \mathrm{Gal}(H/F)$; in particular our formula will be independent of the choice of π generating $\mathfrak{p}^e$. The fact that $\nu(\mathfrak{b})$ pushes forward to $\mu(\mathfrak{b})$ implies that (18) is compatible with (13).

Ideally, we would define the measure ν by reconsidering, for $U \subset \mathbf{O}$, equation (10). However, the condition on α in the sum of (10) is only well-defined modulo $E(\mathfrak{f})$. Our solution to this dilemma is to restrict the sum to a particular fundamental domain for the action of $E(\mathfrak{f})$ on $\mathfrak{b}^{-1}\mathfrak{f} + z$; the condition $\alpha \in U$ will then be well-defined for α in the fundamental domain and $U \subset \mathbf{O}$. The fundamental domain we will take will be the intersection of $\mathfrak{b}^{-1}\mathfrak{f} + z$ with a union of simplicial cones, as introduced by Shintani [19] and described below in Section 3.3. Shintani's motivation was to prove the rationality of the partial zeta-functions of F at nonpositive integers (which had been proven earlier by Siegel [16], [17], [18]), and to provide explicit formulas for these values.

The idea of using Shintani's method to construct the p-adic zeta-functions of totally real fields goes back to Cassou-Nogues [1].

3.3 Shintani's method

Suppose that $[F : \mathbf{Q}] = n$, and let I denote the set of real embeddings of F. The field F may be embedded in $\mathbf{R}^I$ by $x \mapsto (x^\iota)_{\iota \in I}$. Under this embedding, any fractional ideal of F is a lattice in $\mathbf{R}^I$. The group $F^\times$ acts on $\mathbf{R}^I$ with $x \in F$ acting by multiplication by x^ι on the ι-component of any vector in $\mathbf{R}^I$. Denote by Q the positive "quadrant" $(\mathbf{R}^+)^I$. For linearly independent $v_1, \ldots, v_r \in Q$, define the *simplicial cone*

$$C(v_1, v_2, \ldots, v_r) = \left\{ \sum_{i=1}^{r} c_i v_i \in Q : c_i > 0 \right\}.$$

Definition 3.6. A *Shintani cone* is a simplicial cone $C(v_1, v_2, \ldots, v_r)$ generated by elements $v_i \in F \cap Q$. A *Shintani set* is a subset of Q which can be written as a finite disjoint union of Shintani cones. For a cone $C(v_1, \ldots, v_r)$ with $v_i \in F$, the v_i are specified uniquely if we require that they lie in $\mathcal{O}$ and are not divisible in $\mathcal{O}$ by any rational integer; these v_i are called the *generators* of the cone C.

Proposition 3.7 (Shintani [19], Proposition 4)**.** *There exists a Shintani set $\mathcal{D}$ which is a fundamental domain for the action of $E(\mathfrak{f})$ on Q, i.e. such that*

$$Q = \bigcup_{\epsilon \in E(\mathfrak{f})} \epsilon \mathcal{D} \qquad (\textit{disjoint union}).$$

We call a set $\mathcal{D}$ satisfying Proposition 3.7 a *Shintani domain*. For example, when F is real quadratic, we may take $\mathcal{D} = C(1, \epsilon) \cup C(1)$ for a generator ϵ of $E(\mathfrak{f})$.

Definition 3.8. A prime ideal η of F is called *good* for a Shintani cone C if:

- $\mathrm{N}\eta$ is a rational prime ℓ;
- the cone C may be written $C = C(v_1, \ldots, v_r)$ with $v_i \in \mathcal{O}$ and $v_i \notin \eta$.

Definition 3.9. A prime η is called *good* for a Shintani set $\mathcal{D}$ if $\mathcal{D}$ may be written as a finite disjoint union of Shintani cones for which η is good.

In order to demonstrate the consistency of these two definitions, we prove:

Proposition 3.10. *Let C be a Shintani cone, and let $C = \bigsqcup C_\alpha$ be a decomposition of C as a finite disjoint union of other Shintani cones. Then each generator of C appears as a generator of one of the C_α.*

Proof. Let the generators of C be $v_1, \ldots, v_r$. The topological closure of C in $\mathbf{R}^I$ is

$$\overline{C} = \left\{ \sum_{i=0}^{r} c_i v_i : c_i \geq 0 \right\}.$$

The point v_1 lies in $\overline{C}$ and hence lies in some $\overline{C}_\alpha$. If $C_\alpha = C(w_1, \ldots, w_s)$, we therefore have

$$v_1 = \sum_{i=1}^{s} c_i w_i \quad \text{with } c_i \geq 0.$$

Let i be an index of the sum above such that $c_i > 0$. Then $v = v_1 - \frac{c_i}{2} w_i$ lies in $\overline{C}_\alpha$, and hence lies in $\overline{C}$. But w_i lies in $\overline{C}_\alpha$, hence also $\overline{C}$, and can therefore be written

$$w_i = \sum_{k=1}^{r} a_k v_k \quad \text{with } a_k \geq 0.$$

Plugging this expression into the definition of v, we see that the expression of v as a linear combination of the v_k has a negative coefficient of v_k for all $k \neq 1$ with $a_k > 0$. As this would contradict $v \in \overline{C}$, we must have $a_k = 0$ for $k > 1$. Thus $w_i = a_1 v_1$ as desired. □

We now impose the following important assumption on the sets S and T:

Assumption 3.11. *Assume that no prime of S has the same residue characteristic as any prime in T, and that no two primes in T have the same residue characteristic.*

Let $\mathfrak{b}$ be a fractional ideal of F relatively prime to S and to the residue characteristic of every prime in T (we will write this as "$\mathfrak{b}$ is prime to $\operatorname{char} T$" from now on). Let $z \in \mathfrak{b}^{-1}$ be such that $z \equiv 1 \pmod{\mathfrak{f}}$, and let $\mathcal{D}$ be a Shintani set. For each compact open $U \subset \mathcal{O}_\mathfrak{p}$, define for $\operatorname{Re}(s) > 1$:

$$\zeta_R(\mathfrak{b}, \mathcal{D}, U, s) = \mathrm{N}\mathfrak{b}^{-s} \sum_{\substack{\alpha \in (\mathfrak{b}^{-1}\mathfrak{f} + z) \cap \mathcal{D} \\ \alpha \in U, (\alpha, R) = 1}} \mathrm{N}\alpha^{-s}. \tag{19}$$

Define $\zeta_{R,T}(\mathfrak{b}, \mathcal{D}, U, s)$ as in (11); in particular, if $\mathrm{N}\eta = \ell$ and $T = \{\eta\}$, we have

$$\zeta_{R,T}(\mathfrak{b}, \mathcal{D}, U, s) := \zeta_R(\mathfrak{b}, \mathcal{D}, U, s) - \ell^{1-s} \zeta_R(\mathfrak{b}\eta^{-1}, \mathcal{D}, U, s).$$

It follows from Shintani's work in [19] that the function $\zeta_{R,T}(\mathfrak{b}, \mathcal{D}, U, s)$ has a meromorphic continuation to $\mathbf{C}$. Indeed, in Section 6, we will deduce this fact from [19, Proposition 1 of §1.1], and show:

Proposition 3.12. *If T contains a prime η that is good for a Shintani cone C and $\mathrm{N}\eta = \ell$, then we have*

$$\zeta_{R,T}(\mathfrak{b}, C, U, 0) \in \mathbf{Z}[1/\ell].$$

Furthermore, the denominator of $\zeta_{R,T}(\mathfrak{b}, C, U, 0)$ is at most $\ell^{n/(\ell-1)}$.

To ensure integrality of our zeta-functions at 0, we propose:

Definition 3.13. The set T is *good* for a Shintani set $\mathcal{D}$ if $\mathcal{D}$ can be written as a finite disjoint union of Shintani cones $\mathcal{D} = \bigcup C_i$ such that for each cone C_i there are at least two primes in T that are good for C_i (necessarily of different residue characteristic by Assumption 3.11), or one prime $\eta \in T$ which is good for C_i such that $\mathrm{N}\eta \geq n + 2$.

Let us assume that T is good for $\mathcal{D}$. Then Proposition 3.12 implies that

$$\zeta_{R,T}(\mathfrak{b}, \mathcal{D}, U, 0) \in \mathbf{Z}.$$

Define a $\mathbf{Z}$-valued measure $\nu(\mathfrak{b}, \mathcal{D})$ on $\mathcal{O}_{\mathfrak{p}}$ by

$$\nu(\mathfrak{b}, \mathcal{D}, U) := \zeta_{R,T}(\mathfrak{b}, \mathcal{D}, U, 0). \tag{20}$$

For any Shintani domain $\mathcal{D}$, it is clear that the push forward to $\mathbf{O}/\widehat{E(\mathfrak{f})}$ of the restriction of $\nu(\mathfrak{b}, \mathcal{D})$ to $\mathbf{O}$ is equal to $\mu(\mathfrak{b})$. In particular we have

$$\nu(\mathfrak{b}, \mathcal{D}, \mathbf{O}) = \zeta_{S,T}(H/F, \mathfrak{b}, 0) = 0. \tag{21}$$

Note also

$$\nu(\mathfrak{b}, \mathcal{D}, \mathcal{O}_{\mathfrak{p}}) = \zeta_{R,T}(H_{\mathfrak{f}}/F, \mathfrak{b}, 0) \quad \text{and} \quad \nu(\mathfrak{b}, \mathcal{D}, \mathcal{O}_{\mathfrak{p}}^{\times}) = \zeta_{S,T}(H_{\mathfrak{f}}/F, \mathfrak{b}, 0). \tag{22}$$

In order to define the "error term" ϵ appearing in (18), and also later to consider the dependence of our constructions on choice of Shintani domain $\mathcal{D}$, it will be necessary to consider the intersections between distinct Shintani domains.

Lemma 3.14. *The intersection of two Shintani sets is a Shintani set. For two Shintani sets $\mathcal{D}$ and $\mathcal{D}'$ there exists a finite number of $\epsilon \in E(\mathfrak{f})$ such that $\epsilon\mathcal{D} \cap \mathcal{D}'$ is nonempty.*

Proof. The first claim of the lemma was proven by Shintani [19, Corollary to Lemma 2]. For the second claim, define a topological isomorphism $\log : Q \to \mathbf{R}^I$ by $\log((x_\iota)_{\iota \in I}) = (\log x_\iota)_{\iota \in I}$. The log map restricts to give a topological isomorphism between

$$Z_0 = \left\{ (x_\iota)_{\iota \in I} \in Q : \prod_{\iota \in I} x_\iota = 1 \right\}.$$

and the hyperplane

$$Z = \left\{ (x_\iota)_{\iota \in I} : \sum_{\iota \in I} x_\iota = 0 \right\} \subset \mathbf{R}^I.$$

Define a map $\lambda : Q \to Z$ by the formula

$$\lambda : x = (x_\iota)_{\iota \in I} \mapsto \log(x) - \left(\frac{1}{n} \sum_{\iota \in I} \log x_\iota \right) (1, 1, \ldots, 1).$$

In other words, λ is the composition of the natural retraction from Q to Z_0 (which sends a vector in Q to the intersection of the real line it generates with Z_0) with $\log : Z_0 \to Z$.

Define Y and Y' to be the topological closures of $\lambda(\mathcal{D})$ and $\lambda(\mathcal{D}')$, respectively. The subgroup $\log E(\mathfrak{f})$ is a discrete lattice in Z, so there are at most finitely many $\epsilon \in E(\mathfrak{f})$ such that the compact sets $Y + \log \epsilon$ and Y' intersect. This gives the desired result. □

The following proposition appears already as Lemma 2 in [23], but we include the proof for completeness.

Proposition 3.15. *Let $\mathcal{D}$ and $\mathcal{D}'$ be Shintani domains. We may write $\mathcal{D}$ and $\mathcal{D}'$ as finite disjoint unions of the same number of simplicial cones*

$$\mathcal{D} = \bigcup_{i=1}^{d} C_i, \quad \mathcal{D}' = \bigcup_{i=1}^{d} C_i', \tag{23}$$

with $C_i' = \epsilon_i C_i$ for some $\epsilon_i \in E(\mathfrak{f})$, $i = 1, \ldots, d$.

Proof. Write $\mathcal{D}$ and $\mathcal{D}'$ as finite disjoint unions of simplicial cones $\mathcal{D} = \bigcup B$, $\mathcal{D}' = \bigcup B'$. By Lemma 3.14, for each B and B' there exists a finite number of $\epsilon \in E(\mathfrak{f})$ such that $B \cap \epsilon B'$ is nonempty; for each such ϵ we can decompose $B \cap \epsilon B'$ as a finite disjoint union of simplicial cones C. Letting $C' = \epsilon^{-1} C$, we have decompositions $\mathcal{D} = \bigcup C$ and $\mathcal{D}' = \bigcup C'$ as desired. □

Definition 3.16. A decomposition as in (23) is called a *simultaneous decomposition* of the Shintani domains $(\mathcal{D}, \mathcal{D}')$. A set T is *good* for the pair $(\mathcal{D}, \mathcal{D}')$ if there is a simultaneous decomposition as in (23) such that such that for each cone C_i there are at least two primes in T that are good for C_i, or one prime $\eta \in T$ which is good for C_i such that $\mathrm{N}\eta \geq n + 2$. If $\beta \in F^\times$ is totally positive, then T is *β-good* for $\mathcal{D}$ if it is good for the pair $(\mathcal{D}, \beta^{-1}\mathcal{D})$.

Assume now that T is π-good for the Shintani domain $\mathcal{D}$. This property is independent of choice of π generating $\mathfrak{p}^e$. Furthermore, all but finitely many primes η with $\mathrm{N}\eta$ prime are π-good for $\mathcal{D}$. In particular, the set of such primes has Dirichlet density 1. In the special case $\mathfrak{p} = (p)$, the condition that η is π-good for $\mathcal{D}$ reduces to the condition that η is good for $\mathcal{D}$.

Definition 3.17. Define the "error term"

$$\epsilon(\mathfrak{b}, \mathcal{D}, \pi) := \prod_{\epsilon \in E(\mathfrak{f})} \epsilon^{\nu(\mathfrak{b}, \epsilon\mathcal{D} \cap \pi^{-1}\mathcal{D}, \mathcal{O}_\mathfrak{p})}. \tag{24}$$

Lemma 3.14 implies that only finitely many of the exponents in (24) are nonzero. Proposition 3.12 and the assumption that T is π-good for $\mathcal{D}$ imply that the exponents are integers. We may now state our putative formula for the Gross–Stark unit u_T.

Definition 3.18. Let $\mathcal{D}$ be a Shintani domain and assume that T is π-good for $\mathcal{D}$. Define

$$u_T(\mathfrak{b}, \mathcal{D}) := \epsilon(\mathfrak{b}, \mathcal{D}, \pi) \cdot \pi^{\zeta_{R,T}(H_\mathfrak{f}/F, \mathfrak{b}, 0)} \times\!\!\!\!\!\!\int_{\mathbf{O}} x \, d\nu(\mathfrak{b}, \mathcal{D}, x) \in F_\mathfrak{p}^\times. \tag{25}$$

As our notation suggests, we have:

Proposition 3.19. *The element $u_T(\mathfrak{b}, \mathcal{D})$ does not depend on the choice of generator π of $\mathfrak{p}^e$.*

Before proving the proposition we prove the following general lemma, which will be extremely useful in future change of variable computations.

Lemma 3.20. *Let $\mathcal{D}$ be a Shintani set such that T is good for $\mathcal{D}$, and let U be a compact open subset of $\mathcal{O}_\mathfrak{p}$. Let $\mathfrak{b}$ be a fractional ideal of F, and let $\beta \in F^\times$ be totally positive such that $\beta \equiv 1 \pmod{\mathfrak{f}}$ and $\mathrm{ord}_\mathfrak{p}(\beta) \geq 0$. Suppose that $\mathfrak{b}$ and β are relatively prime to R and $\mathrm{char}\, T$. Let $\mathfrak{q} = (\beta)\mathfrak{p}^{-\mathrm{ord}_\mathfrak{p}(\beta)}$. Then*

$$\nu(\mathfrak{bq}, \mathcal{D}, U) = \nu(\mathfrak{b}, \beta\mathcal{D}, \beta U)$$

and hence if $U \subset \mathbf{O}$ we have

$$\times\!\!\!\!\!\!\int_U x \; d\nu(\mathfrak{bq}, \mathcal{D}, x) = \beta^{-\nu(\mathfrak{b},\beta\mathcal{D},\beta U)} \times\!\!\!\!\!\!\int_{\beta U} x \; d\nu(\mathfrak{b}, \beta\mathcal{D}, x).$$

Proof. Recall the definition

$$\zeta_R(\mathfrak{bq}, \mathcal{D}, U, s) = \mathrm{N}(\mathfrak{bq})^{-s} \sum_{\substack{\alpha \in (\mathfrak{b}^{-1}\beta^{-1}\mathfrak{f}+z)\cap\mathcal{D} \\ \alpha \in U, (\alpha, R)=1}} \mathrm{N}\alpha^{-s},$$

where $z \in \mathfrak{b}^{-1}\mathfrak{q}^{-1}$ and $z \equiv 1 \pmod{\mathfrak{f}}$. Note that we have replaced the condition $\alpha \in \mathfrak{b}^{-1}\mathfrak{q}^{-1}$ by $\alpha \in \mathfrak{b}^{-1}\beta^{-1}$, since the condition $\mathrm{ord}_\mathfrak{p}(\alpha) \geq 0$ is already ensured by $\alpha \in U$. Letting $\alpha' = \alpha\beta$ and noting that $\beta \equiv 1 \pmod{\mathfrak{f}}$, the condition on α can be written: $\alpha' \in (\mathfrak{b}^{-1}\mathfrak{f}+z')\cap\beta\mathcal{D}\cap\beta U$, where $z' \in \mathfrak{b}^{-1}$ and $z' \equiv 1 \pmod{\mathfrak{f}}$. Thus we have

$$\zeta_R(\mathfrak{bq}, \mathcal{D}, U, s) = \zeta_R(\mathfrak{b}, \beta\mathcal{D}, \beta U, s)$$

and the first claim of the lemma follows. For the second, we calculate

$$\begin{aligned}
\times\!\!\!\!\!\!\int_U x \; d\nu(\mathfrak{bq}, \mathcal{D}, x) &= \times\!\!\!\!\!\!\int_U x \; d\nu(\mathfrak{b}, \beta\mathcal{D}, \beta x) \\
&= \times\!\!\!\!\!\!\int_{\beta U} (\beta^{-1}y) \; d\nu(\mathfrak{b}, \beta\mathcal{D}, y) \\
&= \beta^{-\nu(\mathfrak{b},\beta\mathcal{D},\beta U)} \times\!\!\!\!\!\!\int_{\beta U} y \; d\nu(\mathfrak{b}, \beta\mathcal{D}, y).
\end{aligned}$$

□

We may now demonstrate the independence of the definition of $u_T(\mathfrak{b}, \mathcal{D})$ on the choice of π.

Proof of Proposition 3.19. Consider the effect on the formula for u_T when π is replaced by $\pi\gamma$ for $\gamma \in E(\mathfrak{f})$. Lemma 3.20 implies that that for any Shintani set $\mathcal{D}'$ for which T is good, we have

$$\nu(\mathfrak{b}, \gamma^{-1}\mathcal{D}', \mathcal{O}_\mathfrak{p}) = \nu(\mathfrak{b}, \mathcal{D}', \gamma\mathcal{O}_\mathfrak{p}) = \nu(\mathfrak{b}, \mathcal{D}', \mathcal{O}_\mathfrak{p}).$$

Thus

$$\begin{aligned}
\epsilon(\mathfrak{b},\mathcal{D},\pi\gamma) &= \prod_{\epsilon\in E(\mathfrak{f})} \epsilon^{\nu(\mathfrak{b},\epsilon\mathcal{D}\cap\pi^{-1}\gamma^{-1}\mathcal{D},\mathcal{O}_\mathfrak{p})} \\
&= \prod_{\epsilon\in E(\mathfrak{f})} \epsilon^{\nu(\mathfrak{b},\epsilon\gamma\mathcal{D}\cap\pi^{-1}\mathcal{D},\mathcal{O}_\mathfrak{p})} \\
&= \epsilon(\mathfrak{b},\mathcal{D},\pi)\cdot\prod_{\epsilon\in E(\mathfrak{f})} \gamma^{-\nu(\mathfrak{b},\epsilon\gamma\mathcal{D}\cap\pi^{-1}\mathcal{D},\mathcal{O}_\mathfrak{p})} \\
&= \epsilon(\mathfrak{b},\mathcal{D},\pi)\cdot\gamma^{-\nu(\mathfrak{b},\pi^{-1}\mathcal{D},\mathcal{O}_\mathfrak{p})} \\
&= \epsilon(\mathfrak{b},\mathcal{D},\pi)\cdot\gamma^{-\zeta_{R,T}(H_\mathfrak{f}/F,\mathfrak{b},0)},
\end{aligned}$$

by (22). This demonstrates the independence of u_T on the choice of π. □

3.4 The conjectural formula for the Gross–Stark unit

Since $\mathfrak{p}$ splits completely in H, we have an embedding $H\subset F_\mathfrak{p}$. We now propose:

Conjecture 3.21. *Let e be the order of $\mathfrak{p}$ in $G_\mathfrak{f}$, and suppose that $\mathfrak{p}^e=(\pi)$ with π totally positive and $\pi\equiv 1 \pmod{\mathfrak{f}}$. Let $\mathcal{D}$ be a Shintani domain and let T be π-good for $\mathcal{D}$. Let $\mathfrak{b}$ be a fractional ideal of F relatively prime to S and $\operatorname{char} T$. We have:*

1. *The element $u_T(\mathfrak{b},\mathcal{D})\in F_\mathfrak{p}^\times$ depends only on the class of $\mathfrak{b}\in G_\mathfrak{f}/\langle\mathfrak{p}\rangle$ and no other choices, including the choice of $\mathcal{D}$, and hence may be denoted $u_T(\sigma_\mathfrak{b})$ where $\sigma_\mathfrak{b}\in \operatorname{Gal}(H/F)$.*
2. *$u_T(\sigma_\mathfrak{b})\in U_\mathfrak{p}$ and $u_T(\sigma_\mathfrak{b})\equiv 1 \pmod{T}$.*
3. *(Shimura Reciprocity Law) For any fractional ideal $\mathfrak{a}$ of F prime to S and $\operatorname{char} T$,*

$$u_T(\sigma_{\mathfrak{a}\mathfrak{b}})=u_T(\sigma_\mathfrak{b})^{\sigma_\mathfrak{a}}. \tag{26}$$

In Section 5 we will analyze the dependence of $u_T(\mathfrak{b},\mathcal{D})$ on the choices of $\mathfrak{b}$ and $\mathcal{D}$. We prove that when $\mathcal{D}$ is restricted to a certain subset of all possible Shintani domains for which T is π-good, then $u_T(\mathfrak{b},\mathcal{D})$ indeed depends only on $\sigma_\mathfrak{b}\in\operatorname{Gal}(H/F)$, up to multiplication by a root of unity. When $n=2$, the restriction on the domain $\mathcal{D}$ and the root of unity ambiguity do not occur, i.e., we prove part (1) of Conjecture 3.21 in this case.

Theorem 3.22. *Conjecture 3.21 implies Conjecture 2.6.*

Proof. The embedding $H\subset F_\mathfrak{p}$ corresponds to a choice of a prime $\mathfrak{P}$ of H above $\mathfrak{p}$. The fact that $\nu(\mathfrak{b},\mathcal{D})|_\mathbf{O}$ pushes forward to $\mu(\mathfrak{b})$ implies that the element $u_T((1),\mathcal{D})$ satisfies (13) if Conjecture 3.21 is true. We must show, however, that the stronger equality (16) holds.

To this end, extend the definition of ν to a measure on $\mathcal{O}_\mathfrak{p}\times\mathcal{U}$ by defining $\nu(\mathfrak{b},\mathcal{D},U)\in\mathbf{Z}$ via equations (19) and (20), for compact open $U\subset\mathcal{O}_\mathfrak{p}\times\mathcal{U}$. The push forward of $\nu(\mathfrak{b},\mathcal{D})|_{\mathbf{O}\times\mathcal{U}}$ to $\mathbf{O}\times\mathcal{U}/\overline{E(\mathfrak{f})}$ is $\mu(\mathfrak{b})$. The projection of

$$\epsilon(\mathfrak{b},\mathcal{D},\pi)\cdot\pi^{\zeta_{R,T}(H_\mathfrak{f}/F,\mathfrak{b},0)}\times\!\!\!\!\!\!\int_{\mathbf{O}\times\mathcal{U}} x\ d\nu(\mathfrak{b},\mathcal{D},x)\in F_\mathfrak{p}^\times\times\mathcal{U} \tag{27}$$

onto $F_{\mathfrak{p}}^{\times}$ is $u_T(\mathfrak{b},\mathcal{D})$. To show that (16) holds for $u_T(\mathfrak{b},\mathcal{D})$, we must prove that the projection of (27) onto $\mathcal{U}$ is $(1,\dots,1)$. Let v be a finite prime in R. Define two measures $\nu_0(\mathfrak{b},\mathcal{D})$ and $\nu_1(\mathfrak{b},\mathcal{D})$ on $U_{\mathfrak{f},v}$ by the rules:

$$\nu_0(\mathfrak{b},\mathcal{D},U) = \nu(\mathfrak{b},\mathcal{D},\mathcal{O}_{\mathfrak{p}} \times U \times \prod_{w \in R-\{v\}} U_{\mathfrak{f},w})$$

and

$$\nu_1(\mathfrak{b},\mathcal{D},U) = \nu(\mathfrak{b},\mathcal{D},\pi\mathcal{O}_{\mathfrak{p}} \times U \times \prod_{w \in R-\{v\}} U_{\mathfrak{f},w}).$$

We must show that

$$\times\!\!\!\!\!\!\int_{U_{\mathfrak{f},v}} x \, d\nu_1(\mathfrak{b},\mathcal{D},x) \div \times\!\!\!\!\!\!\int_{U_{\mathfrak{f},v}} x \, d\nu_0(\mathfrak{b},\mathcal{D},x) = \epsilon(\mathfrak{b},\mathcal{D},\pi)\cdot \pi^{\zeta_{R,T}(H_{\mathfrak{f}}/F,\mathfrak{b},0)}$$

in $U_{\mathfrak{f},v}$. It follows from an argument nearly identical to the proof of Lemma 3.20 that

$$\nu_1(\mathfrak{b},\mathcal{D},U) = \nu_0(\mathfrak{b},\pi^{-1}\mathcal{D},\pi^{-1}U).$$

Thus

$$\begin{aligned} \times\!\!\!\!\!\!\int_{U_{\mathfrak{f},v}} x \, d\nu_1(\mathfrak{b},\mathcal{D},x) &= \times\!\!\!\!\!\!\int_{U_{\mathfrak{f},v}} x \, d\nu_0(\mathfrak{b},\pi^{-1}\mathcal{D},\pi^{-1}x) \\ &= \pi^{\nu_0(\mathfrak{b},\pi^{-1}\mathcal{D},U_{\mathfrak{f},v})} \times\!\!\!\!\!\!\int_{U_{\mathfrak{f},v}} x \, d\nu_0(\mathfrak{b},\pi^{-1}\mathcal{D},x) \\ &= \pi^{\zeta_{R,T}(H_{\mathfrak{f}}/F,\mathfrak{b},0)} \times\!\!\!\!\!\!\int_{U_{\mathfrak{f},v}} x \, d\nu_0(\mathfrak{b},\pi^{-1}\mathcal{D},x), \qquad (28) \end{aligned}$$

where (28) follows from (22). We are therefore reduced to proving

$$\times\!\!\!\!\!\!\int_{U_{\mathfrak{f},v}} x \, d\nu_0(\mathfrak{b},\pi^{-1}\mathcal{D},x) \div \times\!\!\!\!\!\!\int_{U_{\mathfrak{f},v}} x \, d\nu_0(\mathfrak{b},\mathcal{D},x) = \epsilon(\mathfrak{b},\mathcal{D},\pi). \qquad (29)$$

Since $\mathcal{D}$ is a fundamental domain for the action of $E(\mathfrak{f})$ on Q, we have

$$\times\!\!\!\!\!\!\int_{U_{\mathfrak{f},v}} x \, d\nu_0(\mathfrak{b},\pi^{-1}\mathcal{D},x) = \prod_{\epsilon \in E(\mathfrak{f})} \times\!\!\!\!\!\!\int_{U_{\mathfrak{f},v}} x \, d\nu_0(\mathfrak{b},\epsilon\mathcal{D}\cap\pi^{-1}\mathcal{D},x) \qquad (30)$$

and similarly

$$\begin{aligned} \times\!\!\!\!\!\!\int_{U_{\mathfrak{f},v}} x \, d\nu_0(\mathfrak{b},\mathcal{D},x) &= \prod_{\epsilon \in E(\mathfrak{f})} \times\!\!\!\!\!\!\int_{U_{\mathfrak{f},v}} x \, d\nu_0(\mathfrak{b},\mathcal{D}\cap\epsilon^{-1}\pi^{-1}\mathcal{D},x) \\ &= \prod_{\epsilon \in E(\mathfrak{f})} \times\!\!\!\!\!\!\int_{U_{\mathfrak{f},v}} x \, d\nu_0(\mathfrak{b},\epsilon\mathcal{D}\cap\pi^{-1}\mathcal{D},\epsilon x) \\ &= \prod_{\epsilon \in E(\mathfrak{f})} \epsilon^{-\nu_0(\mathfrak{b},\epsilon\mathcal{D}\cap\pi^{-1}\mathcal{D},U_{\mathfrak{f},v})} \times\!\!\!\!\!\!\int_{U_{\mathfrak{f},v}} x \, d\nu_0(\mathfrak{b},\epsilon\mathcal{D}\cap\pi^{-1}\mathcal{D},x). \qquad (31) \end{aligned}$$

Combining (30) and (31), we obtain (29) as desired. □

4 A computation

We are indebted to Kaloyan Slavov for producing the following computational evidence for Conjecture 3.21. For the details, including the algorithm and code used to practically implement the theoretical constructions of this article, we refer to [20].

Let $F = \mathbf{Q}(w)$ be the totally real cubic field defined by the equation $w^3+2w-6w-1=0$. Let $\mathfrak{f} = \mathfrak{q}^2$, where $\mathfrak{q}$ is the unique prime of F with $\mathrm{N}\mathfrak{q} = 2$. The narrow ray class field $H_\mathfrak{f}$ has degree 4 over F and Galois group

$$G_\mathfrak{f} = \langle (3), \mathfrak{q}' \rangle \cong \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}.$$

Let $\mathfrak{p} = (5)$ and let η be the unique prime of F with $\mathrm{N}\eta = 11$. We choose the minimal $S = \{\infty_1, \infty_2, \infty_3, \mathfrak{q}, \mathfrak{p}\}$ and let $T = \{\eta\}$. Since $p \equiv 1 \pmod{\mathfrak{f}}$, we have $H = H_\mathfrak{f}$. Slavov computed:

$$\zeta_{R,T}(H/F, \mathfrak{b}, 0) = \begin{cases} -10, & \text{if} \quad \mathfrak{b} = 1, \\ 10, & \text{if} \quad \mathfrak{b} = (3), \\ -10, & \text{if} \quad \mathfrak{b} = \mathfrak{q}', \\ 10, & \text{if} \quad \mathfrak{b} = (3)\mathfrak{q}'. \end{cases} \tag{32}$$

The fact that $\zeta_{R,T}(H/F, \mathfrak{b}, 0) = \zeta_{R,T}(H/F, \mathfrak{b}\mathfrak{q}', 0)$ for all $\mathfrak{b}$ implies that the Gross–Stark unit should satisfy $u_T = u_T^{\sigma_{\mathfrak{q}'}}$, i.e. u_T should be defined over a quadratic extension of F in H.

Choosing a Shintani domain $\mathcal{D}$ of the form described by Colmez in [4], Slavov computed:

$$\begin{aligned} A &= \times\!\!\!\!\!\!\int_{\mathcal{O}_\mathfrak{p}^\times} x \, d\nu(\mathfrak{b}, \mathcal{D}, x) \\ &\equiv 14138w^2 + 10366w + 10366 \pmod{5^6} \end{aligned}$$

in $\mathcal{O}_\mathfrak{p} = \mathbf{Z}_5[w]$ for $\mathfrak{b} = (\mathfrak{q}')^2$, a representative of the trivial class. Interestingly, this calculation was faster than that for $\mathfrak{b} = (1)$. Conjecture 3.21 would imply that the minimal polynomial of $u_T(\mathfrak{b}, \mathcal{D})$ over F is

$$x^2 - \left(\frac{A}{5^{10}} + \frac{5^{10}}{A} \right) x + 1. \tag{33}$$

The fact that the values in (32) are multiples of 10 led to the hope that $u_T(\mathfrak{b}, \mathcal{D})$ is actually a 10th power in H. By taking a 10th root, we could reduce the size of the coefficients involved in its minimal polynomial. The value A has two 10th roots in $F_\mathfrak{p}^\times$, denoted $\pm A^{1/10}$. A 10th root $u_T(\mathfrak{b}, \mathcal{D})^{1/10}$ should have minimal polynomial

$$x^2 - \left(\frac{A^{1/10}}{5} + \frac{5}{A^{1/10}} \right) x + 1 \equiv x^2 - \frac{1}{5}(w^2 - w - 10)x + 1 \pmod{5^5}.$$

Note that the p-adic accuracy has decreased upon taking the 10th root. One may now check that the polynomial $x^2 - \frac{1}{5}(w^2 - w - 10)x + 1$ indeed defines a quadratic extension of F contained in $H_\mathfrak{f}$, and that the 10th power of a root satisfies the conditions of Stark's Conjecture 2.5. The computations here imply that it satisfies Gross's Conjecture 2.6 up to an accuracy of 5^6.

We conclude by noting that we were helped in this computation by the fact that u_T turned out to be a 10th power. The Gross–Stark unit u_T satisfies the polynomial

$$x^2 + \frac{1}{5^{10}}(-1154763w^2 - 6369741w + 5739634)x + 1, \tag{34}$$

and since these coefficients are larger than $5^6 = 15625$, they would have been impossible to recognize from our low 5-adic precision estimate. However we would still be able to see that the middle coefficient of (33) was congruent to that of (34) modulo 5^6. Since $\mathcal{O}/5^6$ has 5^{18} elements, this would still be significant evidence for the conjecture in this case.

5 Dependence of u_T on choices

We now analyze how u_T depends on the choices of $\mathfrak{b}$ and $\mathcal{D}$; we first consider how u_T changes as the fractional ideal $\mathfrak{b}$ varies within its class in $G_{\mathfrak{f}}/\langle\mathfrak{p}\rangle$.

5.1 Dependence on $\mathfrak{b}$

Proposition 5.1. *Let $\beta \in F^\times$ be totally positive with $\beta \equiv 1 \pmod{\mathfrak{f}}$ and β relatively prime to S and $\operatorname{char} T$. We have*

$$u_T(\mathfrak{b}(\beta), \mathcal{D}) = u_T(\mathfrak{b}, \beta\mathcal{D}).$$

Proof. From Lemma 3.20 and equation (21) we have

$$\begin{aligned} \times\!\!\!\!\!\!\int_{\mathbf{O}} x \, d\nu(\mathfrak{b}(\beta), \mathcal{D}, x) &= \beta^{-\nu(\mathfrak{b},\beta\mathcal{D},\mathbf{O})} \times\!\!\!\!\!\!\int_{\mathbf{O}} y \, d\nu(\mathfrak{b}, \beta\mathcal{D}, y) \\ &= \times\!\!\!\!\!\!\int_{\mathbf{O}} y \, d\nu(\mathfrak{b}, \beta\mathcal{D}, y). \end{aligned} \tag{35}$$

Similarly from Lemma 3.20 one checks that

$$\epsilon(\mathfrak{b}(\beta), \mathcal{D}, \pi) = \epsilon(\mathfrak{b}, \beta\mathcal{D}, \pi),$$

which proves the desired result. □

Proposition 5.1 deals with changing $\mathfrak{b}$ within its class in $G_{\mathfrak{f}}$. We now study what happens when $\mathfrak{b}$ is multiplied by an element equivalent to $\mathfrak{p}$ in $G_{\mathfrak{f}}$.

Proposition 5.2. *Let $\mathfrak{q}$ be relatively prime to S and $\operatorname{char} T$, such that $\mathfrak{q} = \mathfrak{p}(\rho)$ where $\rho \in F^\times$ is totally positive and $\rho \equiv 1 \pmod{\mathfrak{f}}$. We have*

$$u_T(\mathfrak{b}\mathfrak{q}, \mathcal{D}) = u_T(\mathfrak{b}, \rho\mathcal{D}).$$

Proof. By definition, we have

$$u_T(\mathfrak{b}, \rho\mathcal{D}) = \epsilon(\mathfrak{b}, \rho\mathcal{D}, \pi) \cdot \pi^{\zeta_{R,T}(H_{\mathfrak{f}}/F, \mathfrak{b}, 0)} \times\!\!\!\!\!\!\int_{\mathbf{O}} x \, d\nu(\mathfrak{b}, \rho\mathcal{D}, x) \tag{36}$$

and

$$u_T(\mathfrak{bq}, \mathcal{D}) = \epsilon(\mathfrak{bq}, \mathcal{D}, \pi) \cdot \pi^{\zeta_{R,T}(H_\mathfrak{f}/F, \mathfrak{bq}, 0)} \times\!\!\!\!\!\!\int_{\mathbf{O}} x \; d\nu(\mathfrak{bq}, \mathcal{D}, x). \tag{37}$$

The ratio of the π-power terms in (36) and (37) is

$$\pi^{\zeta_{R,T}(H_\mathfrak{f}/F, \mathfrak{b}, 0) - \zeta_{R,T}(H_\mathfrak{f}/F, \mathfrak{bq}, 0)} = \pi^{-\zeta_{S,T}(H_\mathfrak{f}/F, \mathfrak{bq}, 0)}. \tag{38}$$

Lemma 3.20 (with the role of $(\mathfrak{b}, \mathcal{D}, \beta, \mathfrak{q})$ in the lemma being played by $(\mathfrak{bq}, \rho\mathcal{D}, \rho^{-1}, \mathfrak{q}^{-1})$) implies that

$$\begin{aligned} \times\!\!\!\!\!\!\int_{\mathbf{O}} x \; d\nu(\mathfrak{b}, \rho\mathcal{D}, x) &= \rho^{\nu(\mathfrak{b}, \rho\mathcal{D}, \mathbf{O})} \times\!\!\!\!\!\!\int_{\rho^{-1}\mathbf{O}} x \; d\nu(\mathfrak{bq}, \mathcal{D}, x) \\ &= \times\!\!\!\!\!\!\int_{\rho^{-1}\mathbf{O}} x \; d\nu(\mathfrak{bq}, \mathcal{D}, x), \end{aligned}$$

by (21). The ratio of the integral terms in (36) and (37) is therefore equal to

$$\frac{\times\!\!\!\!\!\!\int_{\rho^{-1}\mathbf{O}} x \; d\nu(\mathfrak{bq}, \mathcal{D}, x)}{\times\!\!\!\!\!\!\int_{\mathbf{O}} x \; d\nu(\mathfrak{bq}, \mathcal{D}, x)} = \frac{\times\!\!\!\!\!\!\int_{\pi\mathcal{O}_\mathfrak{p}^\times} x \; d\nu(\mathfrak{bq}, \mathcal{D}, x)}{\times\!\!\!\!\!\!\int_{\mathcal{O}_\mathfrak{p}^\times} x \; d\nu(\mathfrak{bq}, \mathcal{D}, x)}, \tag{39}$$

by canceling the intersection of the domains of integration. Another application of Lemma 3.20 allows us to rewrite the numerator of the right side of (39) as:

$$\times\!\!\!\!\!\!\int_{\pi\mathcal{O}_\mathfrak{p}^\times} x \; d\nu(\mathfrak{bq}, \mathcal{D}, x) = \pi^{\nu(\mathfrak{bq}, \pi^{-1}\mathcal{D}, \mathcal{O}_\mathfrak{p}^\times)} \times\!\!\!\!\!\!\int_{\mathcal{O}_\mathfrak{p}^\times} x \; d\nu(\mathfrak{bq}, \pi^{-1}\mathcal{D}, x). \tag{40}$$

Now the π-power term in (40) exactly cancels that of (38), by (22). It therefore remains to prove:

$$\frac{\times\!\!\!\!\!\!\int_{\mathcal{O}_\mathfrak{p}^\times} x \; d\nu(\mathfrak{bq}, \pi^{-1}\mathcal{D}, x)}{\times\!\!\!\!\!\!\int_{\mathcal{O}_\mathfrak{p}^\times} x \; d\nu(\mathfrak{bq}, \mathcal{D}, x)} = \frac{\epsilon(\mathfrak{bq}, \mathcal{D}, \pi)}{\epsilon(\mathfrak{b}, \rho\mathcal{D}, \pi)}. \tag{41}$$

Since $\mathcal{D}$ is π-good, we have a simultaneous decomposition

$$\pi^{-1}\mathcal{D} = \bigcup C_i \quad \text{and} \quad \mathcal{D} = \bigcup \gamma_i C_i,$$

with $\gamma_i \in E(\mathfrak{f})$, such that T is good for each C_i. We have

$$\begin{aligned} \times\!\!\!\!\!\!\int_{\mathcal{O}_\mathfrak{p}^\times} x \; d\nu(\mathfrak{bq}, \mathcal{D}, x) &= \prod_i \times\!\!\!\!\!\!\int_{\mathcal{O}_\mathfrak{p}^\times} x \; d\nu(\mathfrak{bq}, \gamma_i C_i, x) \\ &= \prod_i \left(\gamma_i^{\nu(\mathfrak{bq}, C_i, \mathcal{O}_\mathfrak{p}^\times)} \times\!\!\!\!\!\!\int_{\mathcal{O}_\mathfrak{p}^\times} x \; d\nu(\mathfrak{bq}, C_i, x) \right) \qquad (42) \\ &= \left(\prod_i \gamma_i^{\nu(\mathfrak{bq}, C_i, \mathcal{O}_\mathfrak{p}^\times)} \right) \times\!\!\!\!\!\!\int_{\mathcal{O}_\mathfrak{p}^\times} x \; d\nu(\mathfrak{bq}, \pi^{-1}\mathcal{D}, x), \qquad (43) \end{aligned}$$

where (42) follows from Lemma 3.20. We now analyze the right side of (41).

$$\frac{\epsilon(\mathfrak{b}\mathfrak{q},\mathcal{D},\pi)}{\epsilon(\mathfrak{b},\rho\mathcal{D},\pi)} = \frac{\prod \epsilon^{\nu(\mathfrak{b}\mathfrak{q},\epsilon\mathcal{D}\cap\pi^{-1}\mathcal{D},\mathcal{O}_\mathfrak{p})}}{\prod \epsilon^{\nu(\mathfrak{b},\epsilon\rho\mathcal{D}\cap\pi^{-1}\rho\mathcal{D},\mathcal{O}_\mathfrak{p})}}, \tag{44}$$

where both products run over all $\epsilon \in E(\mathfrak{f})$. By Lemma 3.20, the exponents in the denominator of (44) satisfy

$$\nu(\mathfrak{b},\epsilon\rho\mathcal{D}\cap\pi^{-1}\rho\mathcal{D},\mathcal{O}_\mathfrak{p}) = \nu(\mathfrak{b}\mathfrak{q},\epsilon\mathcal{D}\cap\pi^{-1}\mathcal{D},\mathfrak{p}\mathcal{O}_\mathfrak{p}),$$

as $\mathrm{ord}_\mathfrak{p}(\rho^{-1}) = 1$. The fraction in (44) therefore simplifies to

$$\prod \epsilon^{\nu(\mathfrak{b}\mathfrak{q},\epsilon\mathcal{D}\cap\pi^{-1}\mathcal{D},\mathcal{O}_\mathfrak{p}^\times)} \tag{45}$$

It is clear from the fact that the C_i are disjoint and inequivalent mod $E(\mathfrak{f})$ that for each $\epsilon \in E(\mathfrak{f})$, the intersection $\epsilon\mathcal{D}\cap\pi^{-1}\mathcal{D}$ is equal to the union of the C_i such that $\gamma_i = \epsilon^{-1}$. Therefore (45) may be written

$$\prod_i \gamma_i^{-\nu(\mathfrak{b}\mathfrak{q},C_i,\mathcal{O}_\mathfrak{p}^\times)}.$$

Combining with (43), we obtain (41); this concludes the proof. □

5.2 Dependence on $\mathcal{D}$

We now study the dependence of $u_T(\mathfrak{b},\mathcal{D})$ on the choice of Shintani domain $\mathcal{D}$.

Theorem 5.3. *Let $\mathcal{D}$ and $\mathcal{D}'$ be two Shintani domains such that T is π-good for $\mathcal{D}$ and for $\mathcal{D}'$. If T is good for $(\mathcal{D},\mathcal{D}')$, then $u_T(\mathfrak{b},\mathcal{D}) = u_T(\mathfrak{b},\mathcal{D}')$.*

Proof. Since T is good for $(\mathcal{D},\mathcal{D}')$, we have a simultaneous decomposition

$$\mathcal{D} = \bigcup C_i \quad \text{and} \quad \mathcal{D}' = \bigcup \gamma_i C_i, \tag{46}$$

with $\gamma_i \in E(\mathfrak{f})$, such that T is good for each C_i. It suffices to demonstrate that $u_T(\mathfrak{b},\mathcal{D})$ is unchanged when in the decomposition $\mathcal{D} = \bigcup C_j$, one cone $C = C_i$ is replaced by γC for some $\gamma \in E(\mathfrak{f})$. In other words, if we write $\mathcal{G} = \bigcup_{j\neq i} C_i$, so that $\mathcal{D}$ is the disjoint union of $\mathcal{G}$ and C, it suffices to prove $u_T(\mathfrak{b},\mathcal{D}) = u_T(\mathfrak{b},\mathcal{D}')$ in the case where $\mathcal{D}' = \mathcal{G}\cup\gamma C$. This implies the general case (46), because we can "move" from $\mathcal{D}$ to $\mathcal{D}'$ with a finite series of such operations, replacing the cones C_i by $\gamma_i C_i$ one at a time.

Therefore, suppose that $\mathcal{D} = \mathcal{G}\cup C$ and $\mathcal{D}' = \mathcal{G}\cup\gamma C$. Consider first the integral in the defining equation (25) of u_T. By Lemma 3.20 we have

$$⨍_{\mathbf{O}} x \, d\nu(\mathfrak{b},\gamma C,x) = \gamma^{\nu(\mathfrak{b},C,\mathbf{O})} \cdot ⨍_{\mathbf{O}} x \, d\nu(\mathfrak{b},C,x),$$

and thus

$$⨍_{\mathbf{O}} x \, d\nu(\mathfrak{b},\mathcal{D}',x) = \gamma^{\nu(\mathfrak{b},C,\mathbf{O})} \cdot ⨍_{\mathbf{O}} x \, d\nu(\mathfrak{b},\mathcal{D},x). \tag{47}$$

Let us for the moment assume that T is good for each intersection $\epsilon C_i \cap \pi^{-1} C_j$, for all $\epsilon \in E(\mathfrak{f})$. Then $\epsilon(\mathfrak{b}, \mathcal{D}, \pi)$ may be decomposed into four components:

$$\begin{aligned}
\epsilon(\mathfrak{b}, \mathcal{D}, \pi) &= \prod_{\epsilon} \epsilon^{\nu(\mathfrak{b}, \epsilon\mathcal{G}\cap\pi^{-1}\mathcal{G}, \mathcal{O}_{\mathfrak{p}})} && (48)\\
&\quad \times \prod_{\epsilon} \epsilon^{\nu(\mathfrak{b}, \epsilon\mathcal{G}\cap\pi^{-1}C, \mathcal{O}_{\mathfrak{p}})} && (49)\\
&\quad \times \prod_{\epsilon} \epsilon^{\nu(\mathfrak{b}, \epsilon C\cap\pi^{-1}\mathcal{G}, \mathcal{O}_{\mathfrak{p}})} && (50)\\
&\quad \times \prod_{\epsilon} \epsilon^{\nu(\mathfrak{b}, \epsilon C\cap\pi^{-1}C, \mathcal{O}_{\mathfrak{p}})}. && (51)
\end{aligned}$$

In the corresponding decomposition of $\epsilon(\mathfrak{b}, \mathcal{D}', \pi)$ for $\mathcal{D}' = \mathcal{G} \cup \gamma C$, the terms from (48) and (51) are unchanged (to see this for (51), one uses Lemma 3.20). For the term corresponding to (49), we find

$$\begin{aligned}
\prod_{\epsilon} \epsilon^{\nu(\mathfrak{b}, \epsilon\mathcal{G}\cap\pi^{-1}\gamma C, \mathcal{O}_{\mathfrak{p}})} &= \prod_{\epsilon} \epsilon^{\nu(\mathfrak{b}, \epsilon\gamma^{-1}\mathcal{G}\cap\pi^{-1}C, \mathcal{O}_{\mathfrak{p}})} \\
&= \gamma^{\sum_{\epsilon} \nu(\mathfrak{b}, \epsilon\mathcal{G}\cap\pi^{-1}C, \mathcal{O}_{\mathfrak{p}})} \cdot \prod_{\epsilon} \epsilon^{\nu(\mathfrak{b}, \epsilon\mathcal{G}\cap\pi^{-1}C, \mathcal{O}_{\mathfrak{p}})}. \qquad (52)
\end{aligned}$$

Similarly, for the term corresponding to (50) we have

$$\prod_{\epsilon} \epsilon^{\nu(\mathfrak{b}, \epsilon\gamma C\cap\pi^{-1}\mathcal{G}, \mathcal{O}_{\mathfrak{p}})} = \gamma^{-\sum_{\epsilon} \nu(\mathfrak{b}, C\cap\epsilon^{-1}\pi^{-1}\mathcal{G}, \mathcal{O}_{\mathfrak{p}})} \cdot \prod_{\epsilon} \epsilon^{\nu(\mathfrak{b}, \epsilon C\cap\pi^{-1}\mathcal{G}, \mathcal{O}_{\mathfrak{p}})}. \qquad (53)$$

Combining (52) and (53) we obtain

$$\epsilon(\mathfrak{b}, \mathcal{D}', \pi) = \epsilon(\mathfrak{b}, \mathcal{D}, \pi) \cdot \gamma^{\sum_{\epsilon} \nu(\mathfrak{b}, \epsilon\mathcal{G}\cap\pi^{-1}C, \mathcal{O}_{\mathfrak{p}}) - \nu(\mathfrak{b}, C\cap\epsilon^{-1}\pi^{-1}\mathcal{G}, \mathcal{O}_{\mathfrak{p}})}. \qquad (54)$$

Adding and subtracting

$$\nu(\mathfrak{b}, \epsilon C \cap \pi^{-1} C, \mathcal{O}_{\mathfrak{p}}) = \nu(\mathfrak{b}, C \cap \epsilon^{-1}\pi^{-1} C, \mathcal{O}_{\mathfrak{p}})$$

from each term in the exponent of (54), we find

$$\begin{aligned}
\text{exponent in (54)} &= \sum_{\epsilon} \nu(\mathfrak{b}, \epsilon\mathcal{D} \cap \pi^{-1} C, \mathcal{O}_{\mathfrak{p}}) - \nu(\mathfrak{b}, C \cap \epsilon^{-1}\pi^{-1}\mathcal{D}, \mathcal{O}_{\mathfrak{p}}) \\
&= \nu(\mathfrak{b}, \pi^{-1} C, \mathcal{O}_{\mathfrak{p}}) - \nu(\mathfrak{b}, C, \mathcal{O}_{\mathfrak{p}}) && (55)\\
&= \nu(\mathfrak{b}, C, \pi\mathcal{O}_{\mathfrak{p}}) - \nu(\mathfrak{b}, C, \mathcal{O}_{\mathfrak{p}}) && (56)\\
&= -\nu(\mathfrak{b}, C, \mathbf{O}). && (57)
\end{aligned}$$

Equation (55) results from the fact that $\mathcal{D}$ and $\pi^{-1}\mathcal{D}$ are fundamental domains for the action of $E(\mathfrak{f})$ on Q. Equation (56) follows from Lemma 3.20. Combining (54), and (57), we obtain

$$\epsilon(\mathfrak{b}, \mathcal{D}', \pi) = \epsilon(\mathfrak{b}, \mathcal{D}, \pi) \cdot \gamma^{\nu(\mathfrak{b}, C, \mathbf{O})}. \qquad (58)$$

This along with (47) gives the desired result $u_T(\mathfrak{b}, \mathcal{D}) = u_T(\mathfrak{b}, \mathcal{D}')$.

If our assumption that T is good for each intersection $\epsilon C_i \cap \pi^{-1}C_j$, does not hold, then the decomposition of $\epsilon(\mathfrak{b}, \mathcal{D}, \pi)$ in (48)–(51) ceases to make sense because the exponents may not be integers. However, these exponents are still rational numbers (see section 6) and only finitely many are non-zero, so they have a common denominator $M \in \mathbf{Z}$. Thus we can make sense of (48)–(51) if we raise both sides to the power M; the rest of the argument then shows that (58) holds if we raise both sides to the power M. However, both sides of (58) are elements of the torsion free group $E(\mathfrak{f})$, so we obtain (58) in this case as well; this completes the proof. □

If $n = 2$, and a set T is good for two simplicial cones, then it is good for their intersection. Thus if T is good for a Shintani domain $\mathcal{D}$, it is automatically π-good for $\mathcal{D}$. Furthermore, if T is good for $\mathcal{D}$ and $\mathcal{D}'$, it is good for the pair $(\mathcal{D}, \mathcal{D}')$. Thus Theorem 5.3 applies whenever T is good for $\mathcal{D}$ and $\mathcal{D}'$ individually. Combined with Propositions 5.1 and 5.2, this implies that part (1) of Conjecture 3.21 holds when $n = 2$. Unfortunately, if $n > 2$, then T may not be good for the pair $(\mathcal{D}, \mathcal{D}')$ even if T is π-good for $\mathcal{D}$ and $\mathcal{D}'$ individually. This is because the process of intersecting two simplicial cones of dimension greater than 2 may introduce new generators which lie in primes in T. Therefore, we cannot yet conclude that $u_T(\mathfrak{b}, \mathcal{D})$ is independent of $\mathcal{D}$ or choice of $\mathfrak{b}$ within its class in $G_{\mathfrak{f}}/\langle\mathfrak{p}\rangle$. However, note that for a given pair $(\mathcal{D}, \mathcal{D}')$, all but finitely many η with $\mathrm{N}\eta$ prime are good for $(\mathcal{D}, \mathcal{D}')$. The next section shows that if we restrict the possible choices for $\mathcal{D}$, then up to multiplication by a root of unity, the element $u_T(\mathfrak{b}, \mathcal{D})$ is independent of $\mathcal{D}$. Part of our argument will involve adjoining primes to the set T, so we will need the following lemma.

Lemma 5.4. *Suppose T is π-good for $\mathcal{D}$, and let η be a prime of F that is relatively prime to S and* $\operatorname{char} T$. *Then*

$$u_{T \cup \{\eta\}}(\mathfrak{b}, \mathcal{D}) = \frac{u_T(\mathfrak{b}, \mathcal{D})}{u_T(\mathfrak{b}\eta^{-1}, \mathcal{D})^{\mathrm{N}\eta}}$$

for all $\mathfrak{b}$ relatively prime to S, $\operatorname{char} T$, *and* $\operatorname{char} \eta$.

Proof. This follows directly from the formula

$$\zeta_{R, T \cup \{\eta\}}(\mathfrak{b}, \mathcal{D}, U, s) = \zeta_{R,T}(\mathfrak{b}, \mathcal{D}, U, s) - \mathrm{N}\eta^{1-s} \zeta_{R,T}(\mathfrak{b}\eta^{-1}, \mathcal{D}, U, s) \tag{59}$$

and the definition of u_T. □

5.3 Special domains

Before we delve into the details of this section, we provide some motivation for the definitions to follow. We would like to show that if $\mathcal{D}$ and $\mathcal{D}'$ are Shintani domains for which T is π-good, then $u_T(\mathfrak{b}, \mathcal{D}) = u_T(\mathfrak{b}, \mathcal{D}')$. Unfortunately, Theorem 5.3 requires that T is good for the pair $(\mathcal{D}, \mathcal{D}')$ and uses this fact in a crucial way. Suppose, however, that we can show that there are infinitely many primes (β) with $\mathrm{N}\beta$ prime, β totally positive, and $\beta \equiv 1 \pmod{\mathfrak{f}}$, such that T is good for the pair $(\mathcal{D}, \beta\mathcal{D})$ and also for the pair $(\mathcal{D}', \beta\mathcal{D}')$. Then by Proposition 5.1 and Theorem 5.3 we will have

$$u_T(\mathfrak{b}(\beta), \mathcal{D}) = u_T(\mathfrak{b}, \beta\mathcal{D}) = u_T(\mathfrak{b}, \mathcal{D}) \tag{60}$$

and similarly

$$u_T(\mathfrak{b}(\beta), \mathcal{D}') = u_T(\mathfrak{b}, \mathcal{D}'). \tag{61}$$

Now all but finitely many of the (β) will be good for the pair $(\mathcal{D}, \mathcal{D}')$, so we will have

$$u_{T\cup\{(\beta)\}}(\mathfrak{b}, \mathcal{D}) = u_{T\cup\{(\beta)\}}(\mathfrak{b}, \mathcal{D}').$$

Using Lemma 5.4, we can write this equation as

$$\frac{u_T(\mathfrak{b}, \mathcal{D})}{u_T(\mathfrak{b}(\beta), \mathcal{D})^{\mathrm{N}\beta}} = \frac{u_T(\mathfrak{b}, \mathcal{D})}{u_T(\mathfrak{b}(\beta), \mathcal{D})^{\mathrm{N}\beta}}.$$

Combining this equation with (60) and (61) yields

$$u_T(\mathfrak{b}, \mathcal{D})^{1-\mathrm{N}\beta} = u_T(\mathfrak{b}, \mathcal{D}')^{1-\mathrm{N}\beta}.$$

Thus we would have proven that $u_T(\mathfrak{b}, \mathcal{D})$ is independent of $\mathcal{D}$ up to a root of unity in $F_{\mathfrak{p}}^{\times}$.

Unfortunately, we cannot show the existence of infinitely many such β in general. Instead, suppose that $\mathcal{D}''$ is such that T is good for the pair $(\mathcal{D}, \mathcal{D}'')$. We will show that if $\mathcal{D}$ and $\mathcal{D}''$ intersect nicely (see Definition 5.7 below), β is sufficiently congruent to 1 modulo powers of the primes in $\operatorname{char} T$, and $\lambda(\beta)$ is sufficiently close to 0, then T is good for the pair $(\beta\mathcal{D}, \mathcal{D}'')$. The motivation for this is that the conditions on the intersection of $\mathcal{D}$ and $\mathcal{D}''$ and on $\lambda(\beta)$ imply that the intersections of the cones in $\beta\mathcal{D}$ and $\mathcal{D}''$ are close to those of $\mathcal{D}$ and $\mathcal{D}''$. The condition on β being ℓ-adically close to 1 for $\ell \in \operatorname{char} T$ implies that the generators of the cones comprising the intersection of $\beta\mathcal{D}$ and $\mathcal{D}''$ will not lie in $\eta \in T$ if the generators of the cones comprising the intersection of $\mathcal{D}$ and $\mathcal{D}''$ do not lie in η.

The condition that T is good for $(\mathcal{D}, \mathcal{D}'')$ and $(\beta\mathcal{D}, \mathcal{D}'')$ implies that

$$u_T(\mathfrak{b}, \mathcal{D}) = u_T(\mathfrak{b}, \mathcal{D}'') = u_T(\mathfrak{b}, \beta\mathcal{D}),$$

by Theorem 5.3. Now we can ignore $\mathcal{D}''$ and note the equality of the outer terms in this equation; choosing a common β for $\mathcal{D}$ and $\mathcal{D}'$ we can argue as above to conclude that $u_T(\mathfrak{b}, \mathcal{D})$ equals $u_T(\mathfrak{b}, \mathcal{D}')$ up to a root of unity. Thus we will have proven our independence result as long as we restrict to the set of $\mathcal{D}$ such that there exists an auxiliary $\mathcal{D}''$ with the properties above; for such $\mathcal{D}$ we will say that T is *special* (see Definition 5.10). Finally, in practice we will show the existence of infinitely many β as above, but β will be the ratio of two primes rather than a prime itself; a slight modification of the argument will ensue. We now proceed with the formal definitions.

Definition 5.5. For a Shintani set $\mathcal{D}$, let $\operatorname{Sp}(\mathcal{D})$ be the smallest $\mathbf{Q}$-vector subspace of F containing $F \cap \mathcal{D}$.

For a simplicial cone $C = C(v_1, \ldots, v_r)$ with $v_i \in F$, $\operatorname{Sp}(C)$ is the $\mathbf{Q}$-span of the v_i.

Definition 5.6. For a simplicial cone $C = C(v_1, \ldots, v_r)$, a *face* of C is the cone generated by any subset of the v_i.

If B is a face of C, we write $B \prec C$. Note that $B \not\subset C$ unless $B = C$.

Definition 5.7. Two Shintani cones C and C' are said to *intersect transversely* if for each pair of faces $B = C(v_1, \ldots, v_r) \prec C$ and $B' = C(w_1, \ldots, w_s) \prec C'$ such that B and B' intersect, the $\mathbf{Q}$-vector space dimension of $\mathrm{Sp}(B) \cap \mathrm{Sp}(B')$ is $\max(r + s - n, 0)$, i.e., the minimal possible dimension of intersection of subspaces of dimensions r and s.

Let $\mathcal{D}$ and $\mathcal{D}'$ be two Shintani domains with decompositions

$$\mathcal{D} = \bigcup C_i, \quad \mathcal{D}' = \bigcup C'_j$$

into disjoint unions of simplicial cones. The decompositions are said to *intersect transversely* if for every C_i, C_j, and $\epsilon \in E(\mathfrak{f})$, the cones ϵC_i and C'_j intersect transversely.

Note that if B and B' intersect transversely and nontrivially, then

$$\mathrm{Sp}(B \cap B') = \mathrm{Sp}(B) \cap \mathrm{Sp}(B').$$

We will require the use of the following lemma from linear algebra, whose proof was communicated to us by Gil Alon and Hugo Chapdelaine.

Lemma 5.8. *Let V and W be $\mathbf{Q}$-vector subspaces of F, of dimensions r and s, respectively. There exists a $\beta \in F^\times$ such that βV and W intersect transversely, i.e. in dimension* $\max(r+s-n, 0)$. *Furthermore, if we view $F^\times/\mathbf{Q}^\times$ as a projective space over $\mathbf{Q}$ of dimension $n-1$, the set of such β is a Zariski-open subset.*

Proof. One immediately reduces to the case $r + s = n$. Then βV and W having trivial intersection is equivalent to the non-vanishing of a certain determinant, which one easily checks is a homogeneous polynomial in the coordinates of β. This implies the second sentence of the lemma. To show that such a β exists, consider a surjective $\mathbf{Q}$-linear map $\varphi : F \to W$ with kernel V. For each $\beta \in F$, define $\varphi_\beta \in \mathrm{End}_{\mathbf{Q}}(W)$ by the rule $\varphi_\beta(w) = \varphi(\beta w)$. We must show that there exists $\beta \in F^\times$ such that φ_β is an isomorphism. The set $\{\varphi_\beta\}$ forms a $\mathbf{Q}$-vector subspace $B \subset \mathrm{End}_{\mathbf{Q}}(W)$ with the property that:

$$\text{for any nonzero } v, w \in W, \text{ there exists } b \in B \text{ such that } b(v) = w. \tag{62}$$

This is clear from the surjectivity of φ and the fact that F is a field. It is a general fact that for any finite dimensional vector space W over an infinite field, and any space B of endomorphisms of W satisfying (62), that the space B contains an isomorphism.

We prove this by induction on the dimension of W, the case $\dim W = 1$ being trivial. In the general case, choose a projection $\gamma : W \to W'$ onto a codimension 1 subspace of W. By the induction hypothesis, the set $\{(\gamma \circ b)|_{W'} : b \in B\}$ contains an isomorphism of W', say $(\gamma \circ \delta)|_{W'}$. Let g generate the kernel of γ and let $d \neq 0$ lie in the kernel of δ. (If δ has no kernel, we are done.) By (62), there exists $b \in B$ such that $b(d) = g$. Consider now elements of the form $b + t \cdot \delta \in B$, for scalars t. The determinant of the endomorphism $\gamma \circ (b + t \cdot \delta)$ of W' is a polynomial in t with leading coefficient $\det(\gamma \circ \delta) \neq 0$. Thus for all but finitely many t, the map γ induces a surjection from the image of $b + t \cdot \delta$ to W'. By construction, d lies in the kernel of $\gamma \circ (b + t \cdot \delta)$ but not of $b + t \cdot \delta$. Thus the image of $b + t \cdot \delta$ has dimension strictly larger than that of W', and hence must equal W. $\square$

Lemma 5.9. *Let $\mathcal{D}$ be a Shintani domain with decomposition $\mathcal{D} = \bigcup C$. The set of totally positive $\beta \in F$ such that $\beta \equiv 1 \pmod{\mathfrak{f}}$ and that the decompositions*

$$\mathcal{D} = \bigcup C \text{ and } \beta\mathcal{D} = \bigcup \beta C$$

intersect transversely is dense in Q.

Proof. Let $X \subset Q$ be a closed ball. It suffices to prove that the set of $\beta \in F \cap X$ satisfying the lemma is dense in X. By compactness, there are only finitely many ϵ such that $\epsilon\mathcal{D}$ and $\beta\mathcal{D}$ can intersect for any $\beta \in X$. By Lemma 5.8, all $\beta \in F \cap Q$ lying outside a proper Zariski closed subset satisfy the property that $\mathrm{Sp}(\epsilon B)$ and $\mathrm{Sp}(\beta B')$ intersect transversely for each triple (B, B', ϵ); for each such β it follows that $\mathcal{D}$ and $\beta\mathcal{D}$ intersect transversely. Thus the set of such β is dense in Q. Finally, given any such β, the elements $\beta + (1-\beta)/(1+f^n)$ approach β and are congruent to $1 \pmod{\mathfrak{f}}$ for n large enough, where $f\mathbf{Z} = \mathfrak{f} \cap \mathbf{Z}$ (and for n large enough, they will satisfy the property of the lemma since β does). □

Definition 5.10. Let $\mathcal{D}$ and $\mathcal{D}'$ be Shintani domains. The set T is *special* for the pair $(\mathcal{D}, \mathcal{D}')$ if there are decompositions

$$\mathcal{D} = \bigcup C_i, \quad \mathcal{D}' = \bigcup C'_j \tag{63}$$

that intersect transversely such that for each intersection $\mathcal{D}_{i,j,\epsilon} = \epsilon C_i \cap C'_j$, the set T contains two primes that are good for $\mathcal{D}_{i,j,\epsilon}$ or one prime η that is good for $\mathcal{D}_{i,j,\epsilon}$ such that $\mathrm{N}\eta \geq n+2$. The set T is *special* for $\mathcal{D}$ if there exists a Shintani domain $\mathcal{D}'$ such that T is special for the pair $(\mathcal{D}, \mathcal{D}')$.

Note that it is clear from the definitions that if T is special for the pair $(\mathcal{D}, \mathcal{D}')$, then it is good for the pair $(\mathcal{D}, \mathcal{D}')$, and in particular it is good for $\mathcal{D}$ and $\mathcal{D}'$ individually.

Lemma 5.9 gives a plentiful supply of β such that $\mathcal{D} = \bigcup C$ and $\beta\mathcal{D} = \bigcup \beta C$ intersect transversely. For any such β, all but finitely many primes η with $\mathrm{N}\eta$ prime will be special for $\mathcal{D}$ (i.e. any set T containing η will be special for $\mathcal{D}$) using the choice $\mathcal{D}' = \beta\mathcal{D}$. However, it is not clear whether η being good for $\mathcal{D}$ implies that it is special for $\mathcal{D}$; it seems plausible that this is the case if $\mathrm{N}\eta$ is large enough.

Lemma 5.11. *If two Shintani cones C and C' intersect transversely, then the intersection $C \cap C'$ can be written as a finite disjoint union of cones*

$$C \cap C' = \bigsqcup C_\alpha$$

such that the generators of the C_α are precisely the nonempty intersections of r-dimensional faces $B \prec C$ with s-dimensional faces $B' \prec C'$, with $r + s = n + 1$ (or more precisely, the unique elements of $\mathcal{O}$ not divisible by any integer in those intersections). Furthermore, any expression of $C \cap C'$ as a disjoint union of cones C_α contains this set of vectors among the generators of the C_α.

Proof. Let us intersect C and C' with the plane $Y = \{x \in \mathbf{R}^I : \sum x_i = 1\}$. Note that the $n-1$ dimensional space Y intersects every Shintani cone transversely. The sets $C_Y = C \cap Y$ and $C'_Y = C' \cap Y$ are interiors of simplices in Y. The faces of these simplices are exactly the intersections $B_Y = B \cap Y$ and $B'_Y = B' \cap Y$ of the faces of C and C', respectively, with the resulting dimensions reduced by 1. The intersection $C_Y \cap C'_Y$ is the interior of a convex polyhedron G_Y in Y. We must show that the vertices of G_Y are precisely the nonempty intersections of r-dimensional faces B_Y of C_Y with s-dimensional faces B'_Y of C'_Y, with $r + s = n - 1$. Then the result about the existence of the partition $C \cap C' = \bigsqcup C_\alpha$ follows from the standard procedure of triangulation of polyhedra.

The vertices of G_Y lie in the closure of C_Y and of C'_Y, and are specified by the property that they are not contained in the interior of any line segment in both closures. Thus each vertex v lies in some face B_Y of C_Y and some face B'_Y of C'_Y. If the dimensions of these faces are r and s, respectively, then by transversality we must have $r + s \geq n - 1$. Furthermore, if $r + s > n - 1$ then $\mathrm{Sp}(B_Y) \cap \mathrm{Sp}(B'_Y)$ would have dimension ≥ 1, yielding a line segment in $\overline{C}_Y \cap \overline{C}'_Y$ containing v. Thus $r + s = n - 1$ as desired. This proves that every vertex of G_Y is of the desired form.

Conversely, if faces B_Y of dimension r and B'_Y of dimension s intersect with $r+s = n-1$, their intersection point v cannot lie in the interior of any line segment in $\overline{C}_Y \cap \overline{C}'_Y$. Indeed, by convexity any such line segment would necessarily be contained in a face B''_Y of C_Y and a face B'''_Y of C_Y; but if v is on this line segment we must have $B_Y = B''_Y$ and $B'_Y = B'''_Y$ because the faces are disjoint. By transversality B_Y and B'_Y intersect only in v, and hence cannot contain a line segment. Thus every such v is indeed a vertex of G_Y.

Finally, if we had an expression of G_Y as a union of simplices which did not include v as a vertex, then v would have to be contained in the interior of one these simplices and we would again reach the same contradiction. This proves the final statement of the lemma. □

Proposition 5.12. *Let $\mathcal{D}$ and $\mathcal{D}'$ be Shintani domains with T special for $(\mathcal{D}, \mathcal{D}')$. There exists a positive integer m such that if $\beta \in F \cap Q$ is such that $\beta \equiv 1 \pmod{m}$ and $\lambda(\beta) \in Z$ is sufficiently close to 0, then T is good for $(\beta\mathcal{D}, \mathcal{D}')$.*

Proof. Suppose that $\mathcal{D} = \bigcup C_i$ and $\mathcal{D}' = \bigcup C'_j$ are decompositions as in (63). Let $B \prec C_i$ and $B' \prec C'_j$ be any two faces. By taking $\lambda(\beta)$ sufficiently close to 0, we can ensure that $\beta B \cap \epsilon B'$ is nonempty only if $B \cap \epsilon B'$ is nonempty. Furthermore, by taking $\lambda(\beta)$ sufficiently close to 0, we can ensure that in this case $\mathrm{Sp}(\beta B)$ and $\mathrm{Sp}(\epsilon B')$ intersect transversely, since this is the case for $\lambda(\beta) = 0$ by assumption. By Lemma 5.11 and the proof of Proposition 3.15, a generator in a simultaneous decomposition of the pair $(\beta\mathcal{D}, \mathcal{D}')$ arises when an r-dimensional face βB of a βC_i intersects an s-dimensional face $\epsilon B'$ of an $\epsilon C'_j$, with $r + s = n + 1$; the ray of intersection (or more precisely the unique element of $\mathcal{O}$ on that ray indivisible by an integer) between these faces will be the generator. By our choice of β, the faces B and $\epsilon B'$ intersect in a line with a unique totally positive generator $g \in \mathcal{O}$ not divisible by an integer. We claim that it suffices to prove that m can be chosen such that for every $\eta \in T$ such that $g \notin \eta$, we have $v \notin \eta$. Indeed, we know that T contains two primes that are good for $C_i \cap \epsilon C'_j$ or one prime η that is good with $\mathrm{N}\eta \geq n + 2$. By Lemma 5.11 we know that any expression of this intersection as a finite disjoint union of cones will contain g as a generator. Thus T

contains two primes η or one prime η with $\mathrm{N}\eta \geq n+2$ such that $g \notin \eta$ for all g (with i, j, ϵ fixed). Thus if we know that $v \notin \eta$ as well, it will follow that T is good for $\beta C_i \cap \epsilon C'_j$.

Let us now suppose that $g \notin \eta$ for $\eta \in T$ with $\mathrm{N}\eta = \ell$. Define V_ℓ and W_ℓ to be the $\mathbf{Z}_\ell$-submodules of $\mathcal{O}_\ell$ generated by $V \cap \mathcal{O}$ and $W \cap \mathcal{O}$, respectively, where $\mathcal{O}_\ell$ is the ℓ-adic completion of $\mathcal{O}$. The ring $\mathcal{O}_\ell$ is a free $\mathbf{Z}_\ell$-module of rank n, and V_ℓ and W_ℓ are free submodules of ranks r and s. Now $V_\ell \cap W_\ell = \mathbf{Z}_\ell \cdot g$. Let t be large enough such that the $\mathbf{Z}_\ell$-module generated by V_ℓ and W_ℓ contains $\ell^{t-1}\mathcal{O}_\ell$. Denote by $\tilde{V}_\ell$ and $\tilde{W}_\ell$ the reductions of V_ℓ and W_ℓ mod ℓ^t. Then the mod ℓ reduction of $\tilde{V}_\ell \cap \tilde{W}_\ell$ equals $(\mathbf{Z}/\ell) \cdot g$.

Now $\beta V_\ell \cap W_\ell = \mathbf{Z}_\ell \cdot v$. By reducing mod ℓ^t, the condition $\beta \equiv 1 \pmod{\ell^t}$ implies that

$$\beta\tilde{V}_\ell \cap \tilde{W}_\ell = \tilde{V}_\ell \cap \tilde{W}_\ell$$

in $\mathcal{O}_\ell/\ell^t$. Since v does not reduce to 0 mod ℓ, we have that v reduces mod ℓ to $c_\beta \cdot g$ for some $c_\beta \in (\mathbf{Z}/\ell)^\times$. Now the reduction mod η map is a functional $\mathcal{O}/\ell\mathcal{O} \to \mathbf{Z}/\ell\mathbf{Z}$, so it vanishes on v if and only if it vanishes on g; thus $g \notin \eta$ implies $v \notin \eta$. Letting m be the product of ℓ^t over all η and g completes the proof. □

Corollary 5.13. *Suppose that T is special for $\mathcal{D}$. There exists a positive integer m and a $\delta > 0$ such that if $\beta \in F \cap Q$ is such that $\beta \equiv 1 \pmod{\mathfrak{f}m}$ and $\lambda(\beta) < \delta$, then*

$$u_T(\mathfrak{b}, \mathcal{D}) = u_T(\mathfrak{b}(\beta), \mathcal{D}).$$

Proof. Suppose that T is special for the pair $(\mathcal{D}, \mathcal{D}')$. If β is as in Proposition 5.12, then T is good for $(\beta\mathcal{D}, \mathcal{D}')$. Theorem 5.3 implies

$$u_T(\mathfrak{b}, \mathcal{D}) = u_T(\mathfrak{b}, \mathcal{D}') = u_T(\mathfrak{b}, \beta\mathcal{D}).$$

Proposition 5.1 allows us to rewrite this equation as

$$u_T(\mathfrak{b}, \mathcal{D}) = u_T(\mathfrak{b}(\beta), \mathcal{D})$$

as desired. □

Lemma 5.14. *Given an integer m and a $\delta > 0$, there exists a pair (β_1, β_2) of totally positive elements of $\mathcal{O}$ with:*

- $\mathrm{N}\beta_i$ *is prime and* $\beta_i \equiv 1 \pmod{\mathfrak{f}m}$ *for* $i = 1, 2$*;*
- $\mathrm{N}\beta_1 \neq \mathrm{N}\beta_2$*;*
- $|\lambda(\beta_1) - \lambda(\beta_2)| < \delta$.

Proof. Let $H_{\mathfrak{f}m}$ be the narrow ray class field of F of conductor $\mathfrak{f}m$. There exist infinitely many primes q of $\mathbf{Q}$ which split completely in $H_{\mathfrak{f}m}$. Any prime of F lying above such a prime q can be written (β), where β is totally positive, $\beta \equiv 1 \pmod{\mathfrak{f}m}$, and $\mathrm{N}\beta = q$ is prime. Now β can be multiplied by any element of $E(\mathfrak{f}m)$, and these properties will still be satisfied. Thus we can choose β to lie in a Shintani domain $\mathcal{D}_{\mathfrak{f}m}$ for the action of $E(\mathfrak{f}m)$ on the totally positive quadrant Q. Since $\lambda(\mathcal{D}_{\mathfrak{f}m})$ is a bounded subset of Z, there exists a sequence of distinct such β such that $\lambda(\beta)$ converges to some element of Z, by compactness. Taking β_1 and β_2 close enough in this sequence gives the desired result. □

Let W_0 denote the finite group of roots of unity in $F_{\mathfrak{p}}^\times$, and define

$$W := \{w \in W_0 : (w, 1) \in \overline{E(\mathfrak{f})}\},$$

where $\overline{E(\mathfrak{f})}$ is the closure of $E(\mathfrak{f})$ in $\mathcal{O}_{\mathfrak{p}}^\times \times \mathcal{U}$ as in Section 3.1.

Theorem 5.15. *Let T be special and π-good for a Shintani domain $\mathcal{D}$. Then the image of $u_T(\mathfrak{b}, \mathcal{D})$ in $F_{\mathfrak{p}}^\times/W$ does not depend on $\mathcal{D}$, and depends only on the class of $\mathfrak{b}$ in $G_{\mathfrak{f}}/\langle \mathfrak{p} \rangle$.*

Proof. Let $\mathcal{D}$ and $\mathcal{D}'$ be distinct Shintani domains, such that T is special for both domains. Let β_1 and β_2 be as in Lemma 5.14, with m and δ chosen to satisfy Corollary 5.13 for both $\mathcal{D}$ and $\mathcal{D}'$. Since $\beta = \beta_1/\beta_2$ satisfies the conditions of Corollary 5.13, the corollary for $\mathfrak{b}(\beta_1)^{-1}$ yields

$$u_T(\mathfrak{b}(\beta_1)^{-1}, \mathcal{D}) = u_T(\mathfrak{b}(\beta_2)^{-1}, \mathcal{D}) \quad \text{and} \quad u_T(\mathfrak{b}(\beta_1)^{-1}, \mathcal{D}') = u_T(\mathfrak{b}(\beta_2)^{-1}, \mathcal{D}'). \tag{64}$$

Let us also suppose that (β_1) and (β_2) are both good for $(\mathcal{D}, \mathcal{D}')$; all but finitely many primes with prime norm are good for this pair, so Lemma 5.14 provides the existence of such β_i. Then Theorem 5.3 implies that

$$u_{T \cup \{\beta_i\}}(\mathfrak{b}, \mathcal{D}) = u_{T \cup \{\beta_i\}}(\mathfrak{b}, \mathcal{D}')$$

for $i = 1, 2$. From Lemma 5.4 we obtain

$$\frac{u_T(\mathfrak{b}, \mathcal{D})}{u_T(\mathfrak{b}(\beta_i)^{-1}, \mathcal{D})^{\mathrm{N}\beta_i}} = \frac{u_T(\mathfrak{b}, \mathcal{D}')}{u_T(\mathfrak{b}(\beta_i)^{-1}, \mathcal{D}')^{\mathrm{N}\beta_i}}. \tag{65}$$

Raise equation (65) for β_1 to the $\mathrm{N}\beta_2$ power, and for β_2 to the $\mathrm{N}\beta_1$ power. Dividing the resulting equations and combining with (64) yields

$$u_T(\mathfrak{b}, \mathcal{D})^{\mathrm{N}\beta_2 - \mathrm{N}\beta_1} = u_T(\mathfrak{b}, \mathcal{D}')^{\mathrm{N}\beta_2 - \mathrm{N}\beta_1}.$$

Thus $u_T(\mathfrak{b}, \mathcal{D})$ and $u_T(\mathfrak{b}, \mathcal{D}')$ differ (multiplicatively) by an element of W_0. In Theorem 3.22 we showed that $(u_T(\mathfrak{b}, \mathcal{D}), 1) \in F_{\mathfrak{p}}^\times \times \mathcal{U}$ always reduces mod $\overline{E(\mathfrak{f})}$ to the same element, namely, the one predicted by Gross's conjecture. Thus the ratio of $u_T(\mathfrak{b}, \mathcal{D})$ and $u_T(\mathfrak{b}, \mathcal{D}')$ lies in the subgroup $W \subset W_0$.

Now that we have demonstrated the independence of domain (mod W), the independence of choice of ideal $\mathfrak{b}$ within its class in $G_{\mathfrak{f}}/\langle \mathfrak{p} \rangle$ follows from Propositions 5.1 and 5.2. □

Remark 5.16. The statement of parts (2) and (3) of Conjecture 3.21 rely on part (1). However, these more essential claims may be salvaged if part (1) is false by replacing part (2) with:

> *(2') Let T be π-good and special for $\mathcal{D}$. The class of $u_T(\mathfrak{b}, \mathcal{D})$ in $F_{\mathfrak{p}}^\times/W$ contains a representative $u_T(\sigma_{\mathfrak{b}}) \in F_{\mathfrak{p}}^\times$ satisfying (2).*

Such a representative $u_T(\sigma_{\mathfrak{b}})$ is necessarily unique, so part (3) may then be left unchanged.

5.4 Relationship with Gross's conjecture

In Section 3.4, we proved that our main Conjecture 3.21 implies Gross's conjecture 2.6. In this section, we prove a partial converse to this result. For the remainder of this section, we assume that the image of $u_T(\mathfrak{b}, \mathcal{D})$ in $F_\mathfrak{p}^\times/W$ depends only on the class of $\mathfrak{b}$ in $G_\mathfrak{f}/\langle\mathfrak{p}\rangle$. For example, this is the case when T is special for $\mathcal{D}$. Assume that $p \neq 2$. Also assume that H is linearly disjoint from $F(\zeta_{p^m})$ over F, for all m; if $\mathfrak{p}$ is unramified over $\mathbf{Q}$, then this is automatic since $\mathfrak{p}$ splits completely in H. Finally, we impose the condition that S contains a finite prime $\mathfrak{q}$ which is unramified in H and whose associated Frobenius $\sigma_\mathfrak{q}$ is a complex conjugation in H.

Lemma 5.17. *Let $m \geq 0$. There exists a finite set of prime ideals $\{\mathfrak{r}_1, \ldots, \mathfrak{r}_s\}$ in the narrow ray class of $\mathfrak{q}$ modulo $\mathfrak{f}$ such that the reduction of $D(\mathfrak{f}, \mathfrak{r}_1\mathfrak{r}_2\cdots\mathfrak{r}_s)$ modulo $\mathfrak{p}^m$ is contained in the image of the group W_0 of roots of unity in $F_\mathfrak{p}^\times$.*

Proof. To ensure that $D(\mathfrak{f}, \mathfrak{r}_1\mathfrak{r}_2\cdots\mathfrak{r}_s) \subset W_0 \pmod{\mathfrak{p}^m}$, it suffices to choose the $\mathfrak{r}_i$ such that if $\epsilon \in E = E(1)$ with $\epsilon \equiv 1 \pmod{\mathfrak{r}_i}$ for all i, then $\epsilon^{\mathrm{N}\mathfrak{p}-1} \equiv 1 \pmod{\mathfrak{p}^m}$. Let $\epsilon_1, \ldots, \epsilon_{n-1}$ be a basis for E. Let $a = (a_1, \ldots, a_{n-1})$ be a tuple of integers. We will choose the $\mathfrak{r}_i$ such that if

$$\epsilon^a := \prod_{j=1}^{n-1} \epsilon_j^{a_j}$$

is congruent to 1 modulo $\mathfrak{r}_i$ for all i, then $\mathrm{N}\mathfrak{p}^m$ divides a_j for all j. This will give the result.

Choose a representative $a \in \mathbf{Z}^{n-1}$ for each non-zero class in $(\mathbf{Z}/\,\mathrm{N}\mathfrak{p}^m\mathbf{Z})^{n-1}$. Let p^t be the highest power of p dividing all of the a_i, so $p^t < \mathrm{N}\mathfrak{p}^m$. Let

$$L = F(\zeta_p, (\epsilon^a)^{1/p^{t+1}}).$$

Let $\mathfrak{r}$ be an ideal of F such that:

- $\mathrm{Frob}(H/F, \mathfrak{r}) = \mathrm{Frob}(H/F, \mathfrak{q})$;
- $\mathrm{Frob}(F(\zeta_{\mathrm{N}\mathfrak{p}^m})/F, \mathfrak{r}) = 1$;
- $\mathrm{Frob}(L/F, \mathfrak{r})$ acts nontrivially on $(\epsilon^a)^{1/p^{t+1}}$.

These conditions are not mutually exclusive, since we have assumed that $H \cap F(\zeta_{\mathrm{N}\mathfrak{p}^m}) = F$, and since $(\epsilon^a)^{1/p^{t+1}}$ is not contained in any abelian extension of F. Thus such an $\mathfrak{r}$ exists by Cebotarev. The second condition on $\mathfrak{r}$ implies that $\mathrm{N}\mathfrak{r} \equiv 1 \pmod{\mathrm{N}\mathfrak{p}^m}$. Choosing a generator for the cyclic group $(\mathcal{O}/\mathfrak{r})^\times$ gives a projection

$$\mathrm{pr}_\mathfrak{r} : (\mathcal{O}/\mathfrak{r})^\times \cong \mathbf{Z}/(\mathrm{N}\mathfrak{r} - 1)\mathbf{Z} \longrightarrow \mathbf{Z}/\,\mathrm{N}\mathfrak{p}^m\mathbf{Z}.$$

The last condition on $\mathfrak{r}$ implies that ϵ^{a/p^t} is not a p-th power in $(\mathcal{O}/\mathfrak{r})^\times$, and hence that $\mathrm{pr}_\mathfrak{r}(\epsilon^a) \not\equiv 0 \pmod{\mathrm{N}\mathfrak{p}^m}$. For any other $a' \in \mathbf{Z}^{n-1}$ equivalent to a in $(\mathbf{Z}/\,\mathrm{N}\mathfrak{p}^m\mathbf{Z})^{n-1}$ the same will be true, and hence $\epsilon^{a'} \not\equiv 1 \pmod{\mathfrak{r}}$. Thus letting the $\mathfrak{r}_i$ consist of such an ideal $\mathfrak{r}$ for each representative a, we will have the desired result. □

Theorem 5.18. *Given the assumptions stated at the start of section 5.4, we have that Conjecture 2.6 implies Conjecture 3.21 up to a root of unity. More precisely, if Conjecture 2.6 is true then $u_{S,T}(\mathfrak{b},\mathcal{D})$ equals the Gross–Stark unit $u_{S,T}^{\sigma_{\mathfrak{b}}}$ in $F_{\mathfrak{p}}^{\times}/W$.*

Proof. Fix a positive integer m. Let $\{\mathfrak{r}_1,\dots,\mathfrak{r}_s\}$ be a finite set of prime ideals as in Lemma 5.17, and let $\mathfrak{r}$ be one of the $\mathfrak{r}_i$. It follows from

$$\zeta_{R\cup\{\mathfrak{r}\}}(H/F,\sigma,s)=\zeta_R(H/F,\sigma,s)-\mathrm{N}\mathfrak{r}^{-s}\zeta_R(H/F,\sigma\sigma_{\mathfrak{r}}^{-1},s)$$

that the Gross–Stark units attached to S and $S\cup\{\mathfrak{r}\}$ are related by

$$u_{S\cup\{\mathfrak{r}\},T}=\frac{u_{S,T}}{u_{S,T}^{\sigma_{\mathfrak{r}}^{-1}}}=u_{S,T}^2,$$

where this last equation follows from the fact that any complex conjugation acts as inversion on $U_{\mathfrak{p}}$. Thus if we let $S':=S\cup\{\mathfrak{r}_1,\cdots,\mathfrak{r}_s\}$, then we inductively obtain:

$$u_{S',T}=u_{S,T}^{2^s}. \tag{66}$$

Similarly, if we write $S''=S-\{\mathfrak{q}\}$, then from the equation

$$\zeta_{R\cup\{\mathfrak{r}\}}(\mathfrak{b},\mathcal{D},U,s)=\zeta_R(\mathfrak{b},\mathcal{D},U,s)-\mathrm{N}\mathfrak{r}^{-s}\zeta_R(\mathfrak{b}\mathfrak{r}^{-1},\mathcal{D},U,s)$$

one calculates that

$$\begin{aligned}u_{S\cup\{\mathfrak{r}\},T}(\mathfrak{b},\mathcal{D}) &= \frac{u_{S,T}(\mathfrak{b},\mathcal{D})}{u_{S,T}(\mathfrak{b}\mathfrak{r}^{-1},\mathcal{D})}\\ &= \frac{u_{S'',T}(\mathfrak{b},\mathcal{D})}{u_{S'',T}(\mathfrak{b}\mathfrak{q}^{-1},\mathcal{D})}\Big/\frac{u_{S'',T}(\mathfrak{b}\mathfrak{r}^{-1},\mathcal{D})}{u_{S'',T}(\mathfrak{b}\mathfrak{r}^{-1}\mathfrak{q}^{-1},\mathcal{D})}\\ &\equiv u_{S,T}(\mathfrak{b},\mathcal{D})^2 \pmod{W_0},\end{aligned}$$

where the last equation follows from the assumption that $u_{S'',T}(\mathfrak{b},\mathcal{D})$ mod W_0 depends only on the class of $\mathfrak{b}$ in $G_{\mathfrak{f}}/\langle\mathfrak{p}\rangle$. We thus inductively obtain

$$u_{S',T}(\mathfrak{b},\mathcal{D})\equiv u_{S,T}(\mathfrak{b},\mathcal{D})^{2^s} \pmod{W_0}. \tag{67}$$

We showed in Theorem 3.22 that $u_{S',T}(\mathfrak{b},\mathcal{D})$ satisfies Gross's formula, and in Proposition 3.4 that Gross's formula specifies $u_{S',T}^{\sigma_{\mathfrak{b}}}$ uniquely modulo $D(\mathfrak{f},\mathfrak{r}_1\cdots\mathfrak{r}_s)$. By the choice of the $\mathfrak{r}_i$, it follows that $u_{S',T}(\mathfrak{b},\mathcal{D})\equiv w\cdot u_{S',T}^{\sigma_{\mathfrak{b}}} \pmod{\mathfrak{p}^m}$ for some $w\in W_0$. Using equations (66) and (67), and noting that m was arbitrary, we obtain $u_{S,T}(\mathfrak{b},\mathcal{D})=w\cdot u_{S,T}^{\sigma_{\mathfrak{b}}}$ for some possibly different $w\in W_0$. Since $u_{S,T}(\mathfrak{b},\mathcal{D})$ always agrees with Gross's formula modulo $\overline{E(\mathfrak{f})}$, we must have $w\in W$. This proves the result. □

6 Integrality of the measure

The goal of this section is to prove Proposition 3.12, which we restate below.

Proposition 6.1. *Let $\eta\in T$ be good for a simplicial cone C of dimension r. Let $\mathfrak{b}$ be a fractional ideal of F relatively prime to S and* $\mathrm{char}\,T$. *For any compact open set $U\subset\mathcal{O}_{\mathfrak{p}}$, the function $\zeta_{R,T}(\mathfrak{b},C,U,s)$ extends to a meromorphic function on* $\mathbf{C}$. *Furthermore, $\zeta_{R,T}(\mathfrak{b},C,U,0)\in\mathbf{Z}[1/\ell]$, and the denominator of $\zeta_{R,T}(\mathfrak{b},C,U,0)$ is at most $\ell^{r/(\ell-1)}$.*

6.1 Proof of integrality

We will compute $\zeta_{R,T}(\mathfrak{b}, C, U, s)$ following the method of Shintani [19, §1.4]. Let A be an $r \times n$ matrix of positive real numbers. Denote by L_j, for $j = 1, \ldots, r$, the linear form in n variables given by

$$L_j(t_1, \ldots, t_n) = \sum_{i=1}^{n} a_{ji} t_i,$$

and by L_j^*, for $j = 1, \ldots, n$, the linear form in r variables given by

$$L_j^*(t_1, \ldots, t_r) = \sum_{i=1}^{r} a_{ij} t_i.$$

Let $x = (x_1, \ldots, x_r)$ be an r-tuple of positive real numbers, and let $\chi = (\chi_1, \ldots, \chi_r)$ be an r-tuple of complex numbers of absolute value at most 1. Define the function

$$\zeta(A, x, \chi, s) := \sum_{z_1, \ldots, z_r = 0}^{\infty} \prod_{i=1}^{r} \chi_i^{z_i} \prod_{j=1}^{n} L_j^*(z + x)^{-s}. \tag{68}$$

Proposition 6.2. *[19, Proposition 1, §1.1] The Dirichlet series $\zeta(A, x, \chi, s)$ is absolutely convergent for* $\mathrm{Re}(s) > r/n$ *and has an analytic continuation to a meromorphic function in the whole complex plane. Furthermore, if $\chi_i \neq 1$ for $i = 1, \ldots, r$, we have*

$$\zeta(A, x, \chi, 0) = \prod_{i=1}^{r} \frac{1}{1 - \chi_i}.$$

Proof of Proposition 6.1. From equation (59), we can use induction on the size of T to reduce to the case $T = \{\eta\}$. Note that since we always assume that T contains no two primes of the same residue characteristic, the property that $\mathfrak{b}$ is relatively prime to $\operatorname{char} T$ is maintained during the induction.

Recall the definition

$$\zeta_R(\mathfrak{b}, C, U, s) = \mathrm{N}\mathfrak{b}^{-s} \sum_{\alpha \in V \cap C} \mathrm{N}\alpha^{-s},$$

where

$$V := \{\alpha \in \mathfrak{b}^{-1} \cap U : \alpha \equiv 1 \pmod{\mathfrak{f}}, (\alpha, R) = 1\}.$$

The set V can be written as a finite disjoint union

$$V = \bigcup_{i=1}^{d} (\mathfrak{a} + y_i),$$

with $\mathfrak{a}$ a fractional ideal supported only at the primes of S and those supporting $\mathfrak{b}\mathfrak{f}$, and some $y_i \in F$. More precisely, we can take

$$\mathfrak{a} = \mathfrak{b}^{-1}\mathfrak{f}\mathfrak{p}^e \prod_{\mathfrak{m} \in R} \mathfrak{m},$$

where e is large enough so that U can be written as a finite disjoint union of translates of $\mathfrak{p}^e\mathcal{O}_{\mathfrak{p}}$. In particular, the assumptions that S and $\mathfrak{b}$ are prime to $\ell = \mathrm{N}\eta$ imply that $\mathfrak{a}$ is relatively prime to all primes dividing ℓ. Without loss of generality, we may also choose $y_i \in \eta$. Then to prove the proposition, it suffices to show that if

$$Z(\mathfrak{a}, y, C, s) := \sum_{\alpha \in (\mathfrak{a}+y)\cap C} \mathrm{N}\alpha^{-s}, \tag{69}$$

then $Z(\mathfrak{a}, y, C, s)$ extends to a meromorphic function on $\mathbf{C}$, and that

$$Z(\mathfrak{a}, y, C, 0) - \ell \cdot Z(\mathfrak{a}\eta, y, C, 0) \in \mathbf{Z}[1/\ell]$$

with denominator at most $\ell^{r/(\ell-1)}$.

Write $C = C(v_1, \ldots, v_r)$, with $v_i \in \mathcal{O}$, $v_i \notin \eta$. By multiplying by an appropriate integer, we can assume that $v_i \in \mathfrak{a}$, for $i = 1, \ldots, r$. Using the crucial fact that $\mathfrak{a}$ is prime to ℓ, the property $v_i \notin \mathfrak{a}\eta$ may be maintained after this integer multiplication. Any element $\alpha \in C$ may be written uniquely as

$$\alpha = \sum_{i=1}^{r} (x_i + z_i) v_i$$

for positive real numbers $x_i \leq 1$ and non-negative integers z_i. Since $v_i \in \mathfrak{a}$, the element α will lie in $\mathfrak{a} + y$ if and only if $\sum x_i v_i$ does. Thus if we let

$$\Omega(\mathfrak{a}, y, v) = \left\{ x \in \mathfrak{a} + y : x = \sum x_i v_i \text{ with } 0 < x_i \leq 1 \right\},$$

then

$$\begin{aligned} Z(\mathfrak{a}, y, C, s) &= \sum_{x \in \Omega(\mathfrak{a},y,v)} \sum_{z_1,\ldots,z_r=0}^{\infty} \mathrm{N}\Big(\sum (x_i + z_i) v_i\Big)^{-s} \\ &= \sum_{x \in \Omega(\mathfrak{a},y,v)} \zeta(A_v, x, \{1\}_{i=1}^{r}, s), \end{aligned} \tag{70}$$

where A_v is the $r \times n$ matrix whose jth row contains $\iota(v_j)$ for $\iota \in I$. Note that $\Omega(\mathfrak{a}, y, v)$ is finite, since the image of $\mathfrak{a}+y$ is discrete in $\mathbf{R}^I$, and $\{\sum x_i v_i : 0 \leq x_i \leq 1\}$ is compact. Thus Proposition 6.2 implies that $Z(\mathfrak{a}, y, C, s)$ has a meromorphic continuation to $\mathbf{C}$.

Now let $\chi : \mathfrak{a}/\mathfrak{a}\eta \cong \mathbf{Z}/\ell\mathbf{Z} \to \mathbf{C}^\times$ be a non-trivial character. For $x \in \mathfrak{a}$ we have the standard orthogonality relation

$$\sum_{t=0}^{\ell-1} \chi(x)^t = \begin{cases} \ell & \text{if } x \in \mathfrak{a}\eta \\ 0 & \text{otherwise.} \end{cases}$$

Thus

$$\begin{aligned} \ell \cdot Z(\mathfrak{a}\eta, y, C, s) &= \sum_{x \in \Omega(\mathfrak{a},y,v)} \sum_{t=0}^{\ell-1} \sum_{z_1,\ldots,z_r=0}^{\infty} \mathrm{N}\Big(\sum (x_i + z_i) v_i\Big)^{-s} \chi\left(y - \sum (x_i + z_i) v_i\right)^t \\ &= \sum_{x \in \Omega(\mathfrak{a},y,v)} \sum_{t=0}^{\ell-1} \chi(y - x)^t \zeta(A_v, x, \{\chi(v_i)^t\}, s). \end{aligned} \tag{71}$$

Noting that (70) equals the term of (71) with $t = 0$, we find

$$Z(\mathfrak{a}, y, C, s) - \ell \cdot Z(\mathfrak{a}\eta, y, C, s) = -\sum_{x \in \Omega(\mathfrak{a}, y, v)} \sum_{t=1}^{\ell-1} \chi(y - x)^t \zeta(A_v, x, \{\chi(v_i)^t\}, s). \tag{72}$$

Since $v_i \notin \mathfrak{a}\eta$ and t ranges from 1 to $\ell - 1$, the values $\chi(v_i)^t$ are non-trivial ℓth roots of unity. Thus Proposition 6.2 gives the value of (72) at $s = 0$:

$$Z(\mathfrak{a}, y, C, 0) - \ell \cdot Z(\mathfrak{a}\eta, y, C, 0) = -\sum_{x \in \Omega(\mathfrak{a}, y, v)} \operatorname{Tr}_{\mathbf{Q}(\mu_\ell)/\mathbf{Q}} \left(\frac{\chi(y - x)}{\prod_{i=1}^r (1 - \chi(v_i))} \right).$$

The algebraic integer $1 - \chi(v_i)$ lies above ℓ and has valuation $\ell^{1/(\ell-1)}$. This gives the desired result. □

6.2 An alternate formula

We now provide an alternate formula for $Z(\mathfrak{a}, y, C, s)$ which will be useful in Section 8 for relating the element u_T to the constructions of [7]. We begin with the following calculation of Shintani.

Proposition 6.3. *[19, Proposition 1 and Corollary, §1.1] With the notation as in (68), we have*

$$(-1)^r \cdot \zeta(A, x, \{1\}, 0) = \prod_{i=1}^r B_1(x_i) + \frac{1}{n} \sum_q \left(\prod_{q_j > 0} \frac{B_{q_j}(x_j)}{q_j!} \right) \sum_k c_k(q, A). \tag{73}$$

Here $B_i(x)$ is the standard Bernoulli polynomial. The sum in (73) is taken over all tuples $(q_1, \ldots, q_r)$ of non-negative integers with $\sum q_i = r$ and at least one q_i equal to 0. The value $c_k(q, A)$ is the constant coefficient in the Taylor expansion of

$$\prod_{j=1}^r L_j(t_1, t_2, \ldots, t_{k-1}, 1, t_{k+1}, \ldots, t_n)$$

at the origin.

The precise form of the second summand in (73) will not be relevant for us. We will only use the fact that

$$\zeta(A, x, \{1\}, 0) = \delta(x) + \sum_q d(q, x, A) \tag{74}$$

where

$$\delta(x) = (-1)^r \cdot \prod_{i=1}^r B_1(x_i) = (-1)^r \cdot \prod_{i=1}^r \left(x_i - \frac{1}{2} \right)$$

and $d(q, x, A)$ does not depend on the value of the x_i with $q_i = 0$.

As in the proof of Proposition 6.2, write $C = C(v_1, \ldots, v_r)$ with $v_i \in \mathfrak{a}$, $v_i \notin \mathfrak{a}\eta$. Define $w_i = \ell \cdot v_i$, so that $w_i \in \mathfrak{a}\eta$. Then equation (70) with $\{v_i\}$ replaced by $\{w_i\}$ may be used to calculate both $Z(\mathfrak{a}, y, C, s)$ and $Z(\mathfrak{a}\eta, y, C, s)$; combining the resulting equations with (74) yields

$$Z(\mathfrak{a}, y, C, 0) - \ell Z(\mathfrak{a}\eta, y, C, 0) = \left\{ \sum_{x \in \Omega(\mathfrak{a}, y, w)} \delta(x) - \sum_{x' \in \Omega(\mathfrak{a}\eta, y, w)} \ell \cdot \delta(x') \right\} \quad (75)$$

$$+ \sum_q \left\{ \sum_{x \in \Omega(\mathfrak{a}, y, w)} d(q, x, A_w) - \sum_{x' \in \Omega(\mathfrak{a}\eta, y, w)} \ell \cdot d(q, x', A_w) \right\}. \quad (76)$$

We claim that each term in braces in (76) vanishes. Indeed, for each fixed q, choose any index j with $q_j = 0$. For every $x \in \Omega(\mathfrak{a}, y, w)$, there is a unique $x' \in \Omega(\mathfrak{a}\eta, y, w)$ such that $x'_i = x_i$ for $i \neq j$, and $x'_j \equiv x_j \pmod{\frac{1}{\ell}\mathbf{Z}}$. This follows from the fact that $v_j = w_j/\ell \in \mathfrak{a}$, $v_j \notin \mathfrak{a}\eta$, and $\mathfrak{a}/\mathfrak{a}\eta \cong \mathbf{Z}/\ell\mathbf{Z}$. The map $x \mapsto x'$ defines an ℓ-to-1 map from $\Omega(\mathfrak{a}, y, w)$ to $\Omega(\mathfrak{a}\eta, y, w)$ such that $d(q, x, A_w) = d(q, x', A_w)$. This implies the claim that the terms in (76) vanish. Thus we arrive at the formula

$$Z(\mathfrak{a}, y, C, 0) - \ell Z(\mathfrak{a}\eta, y, C, 0) = \sum_{x \in \Omega(\mathfrak{a}, y, w)} \delta(x) - \sum_{x' \in \Omega(\mathfrak{a}\eta, y, w)} \ell \cdot \delta(x'). \quad (77)$$

7 Norm compatibility

Let $H \subset H'$ be two finite abelian extensions of F in which $\mathfrak{p}$ splits completely. Choose a prime $\mathfrak{P}'$ above $\mathfrak{p}$ in H', and let $\mathfrak{P}$ be the prime of H below $\mathfrak{P}'$. The uniqueness of the conjectural unit $u_T(\mathfrak{P}) \in H^\times$ satisfying Stark's Conjecture 2.5 implies the "norm compatibility relation"

$$u_T(\mathfrak{P}) = \mathrm{Norm}_{H'/H}\, u_T(\mathfrak{P}'). \quad (78)$$

As remarked earlier, this justifies our restriction to the case where H' is the largest subfield of a narrow ray class field $H_\mathfrak{f}$ in which $\mathfrak{p}$ splits completely; the Gross–Stark unit for any subfield $H \subset H'$ may be found from the Gross–Stark unit for H', using equation (78).

Furthermore, the norm compatibility relation provides a consistency test for Conjecture 3.21. Let $H_\mathfrak{f} \subset H_{\mathfrak{f}\mathfrak{f}'}$ be two narrow ray class field extensions of F, and let H and H' respectively be the largest subfields in which $\mathfrak{p}$ splits completely. The reciprocity map identifies $\mathrm{Gal}(H'/H)$ with

$$\{\beta \in (\mathcal{O}/\mathfrak{f}\mathfrak{f}')^\times\colon\ \beta \equiv 1 \pmod{\mathfrak{f}}\}/E_\mathfrak{p}(\mathfrak{f}). \quad (79)$$

Let $\mathfrak{b}$ be a fractional ideal of F relatively prime to S and $\mathrm{char}\, T$, and let $\mathcal{D}_\mathfrak{f}$ be a Shintani domain for $E(\mathfrak{f})$. If $\{\gamma\}$ is a set of coset representatives for $E(\mathfrak{f}\mathfrak{f}')$ in $E(\mathfrak{f})$, then

$$\mathcal{D}_{\mathfrak{f}\mathfrak{f}'} := \bigcup_\gamma \gamma \mathcal{D}_\mathfrak{f}$$

is a Shintani domain for $E(\mathfrak{f}\mathfrak{f}')$. Conjecture 3.21 predicts that $u_T(\mathfrak{b}, \mathcal{D}_\mathfrak{f})$ is the Gross–Stark unit for H and the prime $\mathfrak{P}^{\sigma_\mathfrak{b}}$ above $\mathfrak{p}$, and that $u_T(\mathfrak{b}, \mathcal{D}_{\mathfrak{f}\mathfrak{f}'})$ is the Gross–Stark unit for H' and the prime $(\mathfrak{P}')^{\sigma_\mathfrak{b}}$. Furthermore, if G denotes a set of totally positive elements of $\mathcal{O}$ that are relatively prime to S and $\operatorname{char} T$ and whose images in $(\mathcal{O}/\mathfrak{f}\mathfrak{f}')^\times$ are a set of distinct representatives for (79), then the "Shimura Reciprocity Law" in Conjecture 3.21 implies that the conjugates of $u_T(\mathfrak{b}, \mathcal{D}_{\mathfrak{f}\mathfrak{f}'})$ over H are the elements $u_T(\mathfrak{b}(\beta), \beta^{-1}\mathcal{D}_{\mathfrak{f}\mathfrak{f}'})$, for $\beta \in G$. The norm compatibility relation then demands:

Theorem 7.1. *We have*

$$u_T(\mathfrak{b}, \mathcal{D}_\mathfrak{f}) = \prod_{\beta \in G} u_T(\mathfrak{b}(\beta), \beta^{-1}\mathcal{D}_{\mathfrak{f}\mathfrak{f}'}). \tag{80}$$

We skip the computational proof of Theorem 7.1; a similar but less complicated calculation is described in detail in Section 8. The choice of Shintani domain $\beta^{-1}\mathcal{D}_{\mathfrak{f}\mathfrak{f}'}$ in the right side of (80) is used to formulate an unconditional statement; the more natural choice $\mathcal{D}_{\mathfrak{f}\mathfrak{f}'}$ can be used if we assume part (1) of Conjecture 3.21 (or if we restrict to special domains and only demand equality up to roots of unity).

8 Real quadratic fields

Let F be a real *quadratic* field of discriminant D. Let $\tau \in F - \mathbf{Q}$, and suppose that τ satisfies the polynomial

$$A\tau^2 + B\tau + C = 0$$

for integers A, B, and C, with $\gcd(A, B, C) = 1$ and $A > 0$. Write $B^2 - 4AC = Df^2$ for an integer $f > 0$. Let ℓ be a prime divisor of A with $\ell \geq 5$, and suppose that D, f, and ℓ are pairwise coprime. Define the formal linear combination of divisors of the prime ℓ:

$$\alpha := \ell[1] - [\ell]. \tag{81}$$

Let p be a rational prime of $\mathbf{Q}$ which is inert in F and write $\mathfrak{p} = (p) \subset F$. Suppose that A and f are each not divisible by p. Denote by H_f^{rng} the narrow ring class field of F of conductor f (see [5, §9] for a definition). Let $U_\mathfrak{p}$ be as in (3) with $H = H_f^{\mathrm{rng}}$. The article [7] defines an element $u(\alpha, \tau) \in F_\mathfrak{p}^\times$ and conjectures that this element lies in $U_\mathfrak{p}$ ([7, Conjecture 2.14]). Furthermore, this conjecture is shown to imply Conjecture 2.4 for the extension H_f^{rng}/F and the sets $S = \{\infty_1, \infty_2, \text{divisors of } f, \mathfrak{p}\}$ and $T = \{\eta\}$, where η is a prime of F lying above ℓ, defined below.

Remark 8.1. In [7], the prime ℓ was replaced by a composite number N, and the element α of (81) was replaced by a formal linear combination of divisors of N:

$$\alpha = \sum_{d|N} n_d[d].$$

However, it was assumed in [7] that $\sum n_d d = 0$ and $\sum n_d = 0$. One can check that this second assumption is in fact irrelevant to the construction, and may be dropped; our choice of α in (81) does not satisfy this condition.

We now review the construction of $u(\alpha, \tau)$ given in [7] and refined in [8]. In the former article, it is shown that the definition of $u(\alpha, \tau)$ depends only on τ modulo the left action of $\Gamma_0(\ell)$ via linear fractional transformations; by translating τ by an appropriate matrix, we may assume that A/ℓ is relatively prime to ℓ and to f. Let $\mathcal{O}_f := \mathbf{Z} + f\mathcal{O}$ denote the order of conductor f in F. Letting $\langle \alpha, \beta \rangle$ denote the $\mathbf{Z}$-lattice generated by α and β, note that $\mathcal{O}_f = \langle 1, A\tau \rangle$. Furthermore, $\langle 1, \ell\tau \rangle$ is an invertible (i.e. proper) fractional ideal of $\mathcal{O}_f$ relatively prime to f and ℓ; we denote its inverse by $\mathfrak{b}_f \subset \mathcal{O}_f$. Let $\mathfrak{b} = \mathfrak{b}_f \mathcal{O}$ be the ideal of $\mathcal{O}$ generated by $\mathfrak{b}_f$. Note that $\mathfrak{b} \cap \mathcal{O}_f = \mathfrak{b}_f$.

Similarly, $\langle 1, \tau \rangle$ is an invertible fractional ideal of $\mathcal{O}_f$, and the quotient

$$\eta_f := \langle \ell, \ell\tau \rangle \cdot \langle 1, \ell\tau \rangle^{-1}$$

is an integral ideal of $\mathcal{O}_f$ of norm ℓ. We denote by η the prime ideal of norm ℓ in $\mathcal{O}$ generated by η_f.

Fix the real embedding of F in which τ is greater than its conjugate. Let ϵ be the fundamental totally positive unit of $\mathcal{O}_f^\times$ with $0 < \epsilon < 1$. Write $\epsilon = c\tau - a$ for integers a and c, with $A|c$. Note that $\ell \nmid a$. Let $\mathbf{X} = (\mathbf{Z}_p \times \mathbf{Z}_p)'$ denote the set of "primitive vectors" in $\mathbf{Z}_p \times \mathbf{Z}_p$, i.e. those vectors not divisible by p. Define a $\mathbf{Z}$-valued measure ξ on $\mathbf{X}$ by the rule

$$\xi(U_{u,v,s}) = 2 \sum_{h \pmod{c}} \tilde{B}_1\left(\frac{a}{c}\left(h + \frac{v}{p^s}\right) - \frac{u}{p^s}\right) \left[\tilde{B}_1\left(\frac{\ell}{c}\left(h + \frac{v}{p^s}\right)\right) - \ell \tilde{B}_1\left(\frac{1}{c}\left(h + \frac{v}{p^s}\right)\right)\right] \quad (82)$$

on the basis of compact open sets given by

$$U_{u,v,s} = (u + p^s\mathbf{Z}_p) \times (v + p^s\mathbf{Z}_p),$$

where u, v, s are integers with u or v not divisible by p, and $s \geq 0$. In equation (82), the "periodic Bernoulli polynomial" $\tilde{B}_1$ is defined by

$$\tilde{B}_1(x) := \begin{cases} 0 & \text{if } x \in \mathbf{Z} \\ x - \lfloor x \rfloor - \frac{1}{2} & \text{otherwise.} \end{cases}$$

We then have the definition

$$u(\alpha, \tau) = p^{2 \cdot \zeta_{R,T}(H_f^{\mathrm{rng}}/F, \mathfrak{b}, 0)} \times\!\!\!\!\!\!\int_{\mathbf{X}} (x - y\tau) d\xi(x, y) \in F_{\mathfrak{p}}^\times. \quad (83)$$

Remark 8.2. In [7], the exponent of p in the definition of $u(\alpha, \tau)$ is given to be a certain explicit Dedekind sum, namely that which is obtained by setting $u = v = 0$ in the right side of (82). It is proven in [7, Theorem 3.1] that this sum is equal to $2 \cdot \zeta_{S,T}(H_f^{\mathrm{rng}}/H, \mathfrak{b}, 0)$. Also, the constant 2 in the definition of ξ is replaced by 12 in [7] to ensure the integrality of the measure in the cases $\ell = 2, 3$. Indeed, the assumption $\ell \geq 5$ can be eliminated in the discussion to follow if we multiply the measures ξ and ν by 4 or 3, respectively, when $\ell = 2, 3$. For ease of notation, we will simply retain the assumption $\ell \geq 5$.

The goal of this section is to provide a concrete formula relating $u(\alpha, \tau)$ to the constructions of the current article. Following the notations set out earlier, let H_f denote the narrow *ray* class field of conductor f, and let H denote the largest subextension of H_f/F in which the prime $\mathfrak{p}$ splits completely. We then have $H_f^{\text{rng}} \subset H \subset H_f$. Note that the image of ϵ in $(\mathcal{O}/f\mathcal{O})^\times$ lies in $(\mathbf{Z}/f\mathbf{Z})^\times$. Class field theory provides a canonical isomorphism

$$\text{rec} : (\mathbf{Z}/f\mathbf{Z})^\times/\langle p, \epsilon\rangle \cong \text{Gal}(H/H_f^{\text{rng}}). \tag{84}$$

Let $\mathcal{D}$ be any Shintani domain for the action of $E(f)$ on Q for which T is good (note that $\ell > n + 2 = 4$). Since $n = 2$, part (1) of Conjecture 3.21 holds unconditionally (see the comments following Theorem 5.3). Thus $u_T(\mathfrak{b}, \mathcal{D})$ does not depend on $\mathcal{D}$, and in fact only depends on $\sigma_{\mathfrak{b}} \in \text{Gal}(H/F)$. Conjecture 3.21 predicts that $u_T(\mathfrak{b}, \mathcal{D})$ is the Gross–Stark unit for $(S, T, H/F)$. Furthermore, if we let G be a set of positive integers relatively prime to ℓ, f, and p, whose images in $(\mathbf{Z}/f\mathbf{Z})^\times$ form a set of distinct coset representatives of the subgroup $\langle p, \epsilon\rangle$, then Conjecture 3.21 implies that the norm of $u_T(\mathfrak{b}, \mathcal{D})$ from H to H_f^{rng} is

$$\prod_{a \in G} u_T(\mathfrak{b}(a), \mathcal{D}).$$

The norm compatibility of Gross–Stark units implies that this element should be the Gross-Stark unit for $(S, T, H_f^{\text{rng}}/F)$. The compatibility of the conjectures in [7] with those in this article therefore requires:

Theorem 8.3. *With the notation as above, we have*

$$u(\alpha, \tau) = \prod_{a \in G} u_T(\mathfrak{b}(a), \mathcal{D})^2. \tag{85}$$

8.1 The case $f = 1$

We first prove Theorem 8.3 in the case $f = 1$. Then

$$H_f^{\text{rng}} = H = H_f = \text{narrow Hilbert class field of } F.$$

Consider the Shintani domain $\mathcal{D} = C(1, \epsilon) \cup C(1)$. Since $\pi = p \in \mathbf{Q}$, the ϵ-factor in the definition (25) of $u_T(\mathfrak{b}, \mathcal{D})$ is trivial. Furthermore, the powers of p in the definitions of $u(\alpha, \tau)$ and $u_T(\mathfrak{b}, \mathcal{D})^2$ are equal. Thus it suffices to prove that

$$⨍_{\mathbf{X}} (x - y\tau) d\xi(x, y) = ⨍_{\mathcal{O}_{\mathfrak{p}}^\times} z^2 \, d\nu(\mathfrak{b}, \mathcal{D}, z).$$

The map $(x, y) \mapsto z = (-\ell)(x - y\tau)$ identifies $\mathbf{X}$ with $\mathcal{O}_{\mathfrak{p}}^\times$. Let

$$V = V_{\ell(v\tau - u), s} := \{x \in \mathcal{O}_{\mathfrak{p}}^\times : x \equiv \ell(v\tau - u) \pmod{\mathfrak{p}^s}\}.$$

We will show that

$$\xi(U_{u,v,s}) = 2\nu(\mathfrak{b}, C(1, \epsilon), V) - \nu(\mathfrak{b}, C(1), V) - \nu(\mathfrak{b}, C(\epsilon), V). \tag{86}$$

Now the measures $\nu(\mathfrak{b}, C(1))$ and $\nu(\mathfrak{b}, C(\epsilon))$ are easy to calculate explicitly. For example, if $\mathfrak{b}^{-1} \cap \mathbf{Q}$ is generated by the positive rational b, then $\zeta_R(\mathfrak{b}, C(1), x + \mathfrak{p}^m \mathcal{O}_\mathfrak{p}, s)$ can be written in terms of the Hurwitz zeta-function as

$$(\mathrm{N}\mathfrak{b} \cdot b^2 p^2)^{-s} \cdot \zeta(2s, x'/p^m)$$

where x' is the unique integer such that $x' \equiv xb^{-1} \pmod{p^m}$ and $0 < x' < p^m$. (Note that $\zeta_R(\mathfrak{b}, C(1), x + \mathfrak{p}^m \mathcal{O}_\mathfrak{p}, s)$ is identically 0 if x is not congruent to an integer in $\mathcal{O}/\mathfrak{p}^m$.) Since it is well known that

$$\zeta\left(0, \frac{x'}{p^m}\right) = \frac{1}{2} - \frac{x'}{p^m},$$

it follows that

$$\nu(\mathfrak{b}, C(1), -U) = -\nu(\mathfrak{b}, C(1), U).$$

The same is true with $C(\epsilon)$ replacing $C(1)$, and we conclude that

$$⨰_{\mathcal{O}_\mathfrak{p}^\times} z \, d\nu(\mathfrak{b}, C(\epsilon), z) = ⨰_{\mathcal{O}_\mathfrak{p}^\times} z \, d\nu(\mathfrak{b}, C(1), z) = \pm 1.$$

Thus (86) implies that

$$\begin{aligned} ⨰_{\mathbf{X}} (x - y\tau) d\xi(x, y) &= ⨰_{\mathcal{O}_\mathfrak{p}^\times} (-z/\ell)^2 \, d\nu(\mathfrak{b}, \mathcal{D}, z) \\ &= ⨰_{\mathcal{O}_\mathfrak{p}^\times} z^2 \, d\nu(\mathfrak{b}, \mathcal{D}, z), \end{aligned}$$

since $\nu(\mathfrak{b}, \mathcal{D}, \mathcal{O}_\mathfrak{p}^\times) = 0$. It remains to prove (86).

In the notation of (69),

$$\nu(\mathfrak{b}, \mathcal{D}, V) = Z(\mathfrak{b}^{-1}\mathfrak{p}^s, \ell(v\tau - u), \mathcal{D}, 0) - \ell Z(\mathfrak{b}^{-1}\mathfrak{p}^s \eta, \ell(v\tau - u), \mathcal{D}, 0).$$

Let $C = C(1, \epsilon)$, and let $(w_1, w_2) = (\ell p^s, \ell p^s \epsilon)$ as in section 6.2; note that $w_i/\ell \in \mathfrak{b}^{-1}\mathfrak{p}^s$ and $w_i/\ell \notin \mathfrak{b}^{-1}\mathfrak{p}^s\eta$, as required. From (77), we obtain

$$\nu(\mathfrak{b}, C, V) = \sum_{x \in \Omega(\mathfrak{b}^{-1}\mathfrak{p}^s, \ell(v\tau - u), w)} B_1(x_1) B_1(x_2) \quad - \sum_{x' \in \Omega(\mathfrak{b}^{-1}\mathfrak{p}^s\eta, \ell(v\tau - u), w)} \ell B_1(x_1') B_1(x_2'). \tag{87}$$

The fractional ideal $\mathfrak{b}^{-1}\mathfrak{p}^s$ is the set of elements of the form

$$h \cdot p^s \ell \tau + j \cdot p^s$$

for $h, j \in \mathbf{Z}$. Solving the equation

$$hp^s \ell \tau + jp^s + \ell(v\tau - u) = x_1(\ell p^s) + x_2(\ell p^s \epsilon)$$

yields

$$\begin{aligned} x_1 &= \frac{a}{c}\left(h+\frac{v}{p^s}\right)-\frac{u}{p^s}+\frac{j}{\ell} \\ x_2 &= \frac{1}{c}\left(h+\frac{v}{p^s}\right). \end{aligned}$$

For each equivalence class modulo c, there exists a unique integer h in that class such that $0 < x_2 \le 1$. For this fixed h, and each possible equivalence class modulo ℓ, there exists a unique integer j in that class such that $0 < x_1 \le 1$. Thus the first sum in (87) equals

$$\sum_{h \pmod c} \tilde{B}_1\left(\frac{1}{c}\left(h+\frac{v}{p^s}\right)\right) \sum_{j \pmod \ell} \tilde{B}_1\left(\frac{a}{c}\left(h+\frac{v}{p^s}\right)-\frac{u}{p^s}+\frac{j}{\ell}\right) + \mathrm{Err}_1, \tag{88}$$

where Err_1 is an explicit error term arising from the discrepancy between $\tilde{B}_1(1) = 0$ and $B_1(1) = 1/2$. The sum in (88) may be shown to equal

$$\sum_{h \pmod c} \tilde{B}_1\left(\frac{1}{c/\ell}\left(h+\frac{v}{p^s}\right)\right) \tilde{B}_1\left(\frac{a}{c}\left(h+\frac{v}{p^s}\right)-\frac{u}{p^s}\right) \tag{89}$$

using the distribution relation

$$\sum_{j \,(\mathrm{mod}\ \ell)} \tilde{B}_1\left(x+\frac{j}{\ell}\right) = \tilde{B}_1(\ell x).$$

A similar argument evaluates the second sum in (87):

$$\sum_{x' \in \Omega(\mathfrak{b}^{-1}\mathfrak{p}^s\eta, \ell(v\tau-u), w)} \ell \tilde{B}_1(x_1')\tilde{B}_1(x_2') = \ell \sum_{h \pmod c} \tilde{B}_1\left(\frac{1}{c}\left(h+\frac{v}{p^s}\right)\right) \tilde{B}_1\left(\frac{a}{c}\left(h+\frac{v}{p^s}\right)-\frac{u}{p^s}\right) + \mathrm{Err}_2. \tag{90}$$

Combining (82), (87), (89), and (90), we find

$$\xi(U_{u,v,s}) = 2\nu(\mathfrak{b}, C(1,\epsilon), V) - 2(\mathrm{Err}_1 + \mathrm{Err}_2).$$

One similarly calculates that $\nu(\mathfrak{b}, C(1), V) + \nu(\mathfrak{b}, C(\epsilon), V)$ is equal to the sum $2(\mathrm{Err}_1 + \mathrm{Err}_2)$, giving the desired equality (86).

8.2 General f

We now prove Theorem 8.3 for general $f \ge 1$. Denote by e the order of $\mathfrak{p}$ in G_f; this is the smallest power of p which is congruent to a power ϵ^g of ϵ in $(\mathcal{O}/f\mathcal{O})^\times$. Let $\pi = p^e \cdot \epsilon^{-g} \equiv 1 \pmod f$. Finally, let $\mathcal{D}$ be a Shintani domain for the action of $E(f)$ on Q. Note that $E(f) = \langle \epsilon^h \rangle$, where h is the order of ϵ in $(\mathbf{Z}/f\mathbf{Z})^\times$. Thus if $\mathcal{D}_f$ is a Shintani domain for the action of $\langle \epsilon \rangle$ on Q, then we may take

$$\mathcal{D} = \bigcup_{i=0}^{h-1} \epsilon^i \mathcal{D}_f.$$

To reduce Theorem 8.3 to the calculation we have already done for $f = 1$, we will show that the right side of (85) can be written as in the case $f = 1$, with the ideal $\mathfrak{b} \subset \mathcal{O}$ replaced by $\mathfrak{b}_f \subset \mathcal{O}_f$. To be precise, define a zeta-function attached to the ring $\mathcal{O}_f$ by

$$\zeta^f(\mathfrak{b}_f, \mathcal{D}_f, U, s) = \mathrm{N}\mathfrak{b}_f^{-s} \sum_{\substack{\alpha \in \mathfrak{b}_f^{-1} \cap \mathcal{D}_f \\ \alpha \in U}} \mathrm{N}\alpha^{-s},$$

and define ζ_T^f from ζ^f as usual. Define ν_f by

$$\nu_f(\mathfrak{b}_f, \mathcal{D}_f, U) := \zeta_T^f(\mathfrak{b}_f, \mathcal{D}_f, U, 0). \tag{91}$$

Recall that G is a set of positive integers relatively prime to p and ℓ which forms a set of distinct representatives for the cosets of $\langle p, \epsilon \rangle$ in $(\mathbf{Z}/f\mathbf{Z})^\times$. Then we will prove that

$$\prod_{a \in G} u_T(\mathfrak{b}(a), \mathcal{D}) = p^{\zeta_{R,T}(H_f^{\mathrm{rng}}/F, \mathfrak{b}, 0)} \times\!\!\!\!\!\!\int_{\mathcal{O}_\mathfrak{p}^\times} x \, d\nu_f(\mathfrak{b}_f, \mathcal{D}_f, x). \tag{92}$$

The proof that the square of the right side of (92) is equal to $u(\alpha, \tau)$ then proceeds as in the case $f = 1$, simply by replacing $\mathfrak{b} \subset \mathcal{O}$ by $\mathfrak{b}_f \subset \mathcal{O}_f$.

It thus remains to prove (92). To proceed, we will require some extra notation; for any subset A of equivalence classes in $(\mathcal{O}/f\mathcal{O})^\times$, let $\nu_A(\mathfrak{b}, \mathcal{D}, U) = \zeta_T^A(\mathfrak{b}, \mathcal{D}, U, 0)$ where ζ^A is the zeta-function:

$$\zeta^A(\mathfrak{b}, \mathcal{D}, U, s) = \mathrm{N}\mathfrak{b}^{-s} \sum_{\substack{\alpha \in \mathfrak{b}^{-1} \cap \mathcal{D} \\ \alpha \in U, \alpha \in A}} \mathrm{N}\alpha^{-s}, \tag{93}$$

and ζ_T^A is obtained from ζ^A as usual. This generalizes $\zeta_T(\mathfrak{b}, \mathcal{D}, U, s) = \zeta_T^{\{1\}}(\mathfrak{b}, \mathcal{D}, U, s)$ and $\nu(\mathfrak{b}, \mathcal{D}, U) = \nu_{\{1\}}(\mathfrak{b}, \mathcal{D}, U)$.

Since the definition of the ϵ-factor given in (24) is unchanged by multiplying the third argument by a rational factor, we have

$$u_T(\mathfrak{b}, \mathcal{D}) = \frac{p^{e \cdot \zeta_{R,T}(H_f/F, \mathfrak{b}, 0)} \epsilon(\mathfrak{b}, \mathcal{D}, \epsilon^{-g})}{\epsilon^{g \cdot \zeta_{R,T}(H_f/F, \mathfrak{b}, 0)}} \times\!\!\!\!\!\!\int_{\mathbf{O}} x \, d\nu(\mathfrak{b}, \mathcal{D}, x) \tag{94}$$

Note that

$$\mathbf{O} = \bigcup_{i=0}^{e-1} p^i \mathcal{O}_\mathfrak{p}^\times.$$

A direct calculation from the definitions shows that for $U \subset \mathcal{O}_\mathfrak{p}^\times$, we have

$$\nu(\mathfrak{b}, \mathcal{D}, p^i U) = \nu_{\{p^{-i}\}}(\mathfrak{b}, \mathcal{D}, U).$$

Thus the integral in (94) may be written:

$$p^{\sum_{i=0}^{e-1} i \cdot \nu(\mathfrak{b}, \mathcal{D}, p^i \mathcal{O}_\mathfrak{p}^\times)} \times\!\!\!\!\!\!\int_{\mathcal{O}_\mathfrak{p}^\times} x \, d\nu_A(\mathfrak{b}, \mathcal{D}, x) = p^{\zeta_{R,T}(H/F, \mathfrak{b}, 0) - e \cdot \zeta_{R,T}(H_f/F, \mathfrak{b}, 0)} \times\!\!\!\!\!\!\int_{\mathcal{O}_\mathfrak{p}^\times} x \, d\nu_A(\mathfrak{b}, \mathcal{D}, x),$$

where $A = \{1, p^{-1}, \cdots, p^{-(e-1)}\}$. It follows from the definition (93) that

$$\nu_A(\mathfrak{b}, \epsilon^i \mathcal{D}_f, U) = \nu_{\epsilon^{-i}A}(\mathfrak{b}, \mathcal{D}_f, \epsilon^{-i}U),$$

from which we deduce

$$\begin{aligned}
\times\!\!\!\!\!\!\int_{\mathcal{O}_\mathfrak{p}^\times} x \, d\nu_A(\mathfrak{b}, \mathcal{D}, x) &= \prod_{i=0}^{h-1} \times\!\!\!\!\!\!\int_{\mathcal{O}_\mathfrak{p}^\times} x \, d\nu_A(\mathfrak{b}, \epsilon^i \mathcal{D}_f, x) \\
&= \prod_{i=0}^{h-1} \left[\epsilon^{i \cdot \nu_{\epsilon^{-i}A}(\mathfrak{b}, \mathcal{D}_f, \mathcal{O}_\mathfrak{p}^\times)} \times\!\!\!\!\!\!\int_{\mathcal{O}_\mathfrak{p}^\times} x \, d\nu_{\epsilon^{-i}A}(\mathfrak{b}, \mathcal{D}_f, x) \right] \\
&= \epsilon^{\sum_{i=0}^{h-1} i \cdot \nu_{\epsilon^{-i}A}(\mathfrak{b}, \mathcal{D}_f, \mathcal{O}_\mathfrak{p}^\times)} \times\!\!\!\!\!\!\int_{\mathcal{O}_\mathfrak{p}^\times} x \, d\nu_{\langle \epsilon, p \rangle}(\mathfrak{b}, \mathcal{D}_f, x),
\end{aligned}$$

where $\langle \epsilon, p \rangle$ is the subgroup of $(\mathbf{Z}/f\mathbf{Z})^\times \subset (\mathcal{O}/f\mathcal{O})^\times$ generated by ϵ and p. A simple combinatorial argument shows that

$$\sum_{i=0}^{h-1} i \cdot \nu_{\epsilon^{-i}A}(\mathfrak{b}, \mathcal{D}_f, \mathcal{O}_\mathfrak{p}^\times) = g \cdot \nu_{\langle \epsilon \rangle}(\mathfrak{b}, \mathcal{D}_f, \mathcal{O}_\mathfrak{p}) - h \cdot \nu_{\{1, \epsilon^{-1}, \ldots, \epsilon^{-(g-1)}\}}(\mathfrak{b}, \mathcal{D}_f, \mathcal{O}_\mathfrak{p}).$$

One also computes

$$\epsilon(\mathfrak{b}, \mathcal{D}, \epsilon^{-g}) = \epsilon^{h \cdot \nu_{\{1, \epsilon^{-1}, \ldots, \epsilon^{-(g-1)}\}}(\mathfrak{b}, \mathcal{D}_f, \mathcal{O}_\mathfrak{p})}$$

and

$$\zeta_{R,T}(H_f/F, \mathfrak{b}, 0) = \nu_{\langle \epsilon \rangle}(\mathfrak{b}, \mathcal{D}_f, \mathcal{O}_\mathfrak{p}).$$

Combining these calculations, all of the terms involving ϵ cancel out, and we obtain

$$u_T(\mathfrak{b}, \mathcal{D}) = p^{\zeta_{R,T}(H/F, \mathfrak{b}, 0)} \times\!\!\!\!\!\!\int_{\mathcal{O}_\mathfrak{p}^\times} x \, d\nu_{\langle \epsilon, p \rangle}(\mathfrak{b}, \mathcal{D}_f, x).$$

It follows that

$$\begin{aligned}
\prod_{a \in G} u_T(\mathfrak{b}(a), \mathcal{D}) &= \prod_{a \in G} \left[p^{\zeta_{R,T}(H/F, \mathfrak{b}(a), 0)} \times\!\!\!\!\!\!\int_{\mathcal{O}_\mathfrak{p}^\times} x \, d\nu_{\langle \epsilon, p \rangle}(\mathfrak{b}(a), \mathcal{D}_f, x) \right] \\
&= p^{\zeta_{R,T}(H_f^{\mathrm{rng}}/F, \mathfrak{b}, 0)} \prod_{a \in G} \times\!\!\!\!\!\!\int_{\mathcal{O}_\mathfrak{p}^\times} x \, d\nu_{a\langle \epsilon, p \rangle}(\mathfrak{b}, \mathcal{D}_f, ax) \\
&= p^{\zeta_{R,T}(H_f^{\mathrm{rng}}/F, \mathfrak{b}, 0)} \times\!\!\!\!\!\!\int_{\mathcal{O}_\mathfrak{p}^\times} x \, d\nu_{(\mathbf{Z}/f\mathbf{Z})^\times}(\mathfrak{b}, \mathcal{D}_f, x),
\end{aligned}$$

since $\nu_{a\langle \epsilon, p \rangle}(\mathfrak{b}, \mathcal{D}_f, \mathcal{O}_\mathfrak{p}^\times) = 0$ for each $a \in G$. It follows directly from the definitions that

$$\nu_{(\mathbf{Z}/f\mathbf{Z})^\times}(\mathfrak{b}, \mathcal{D}_f) = \nu_f(\mathfrak{b}_f, \mathcal{D}_f),$$

where the right side is defined in (91). This completes the proof of (92), and hence of Theorem 8.3.

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