# On the rank one Gross–Stark conjecture for quadratic extensions and the Deligne–Ribet $q$-expansion principle

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**Abstract**

## Contents



# 1 Introduction

Fix a rational prime $p$. In this note, we provide a new proof of the rank 1 Gross–Stark conjecture for a quadratic extension under the assumption that there is only one prime above $p$ in the base field. Let $K$ be a CM field and let $F$ denote the maximal totally real subfield of $K$. We assume that there is only one prime $\mathfrak{p}$ above $p$ in $F$. Denote by $\chi$ the nontrivial character of $G := \mathrm{Gal}(K/F)$. Write $n = [F : \mathbf{Q}]$. To ensure we are in a rank 1 setting, we suppose that $\chi(\mathfrak{p}) = 1$. Fix a prime $\mathfrak{P}$ of $K$ above $\mathfrak{p}$, and suppose $\mathfrak{P}^h = (x)$ for some integer $h$ (e.g. the class number of $K$) and $x \in \mathcal{O}_K$. Let $u = x/\overline{x}$, where here and throughout, $\overline{x}$ denotes the image of $x$ under complex conjugation (i.e. the nontrivial element of $G$).

Let $\omega$ denote the Teichmüller character and $L_p(\chi\omega, s)$ be the usual $p$-adic $L$-function of Deligne–Ribet [5] and Cassou-Noguès [2]. In this setting, the Gross–Stark conjecture states:

**Theorem 1.1.** *We have*

$$\frac{L_p'(\chi\omega, 0)}{L(\chi, 0)} = -\frac{\log_p \mathrm{Norm}_{H_{\mathfrak{P}}/\mathbf{Q}_p}(u)}{\mathrm{ord}_{\mathfrak{P}}(u)}, \tag{1}$$

*where* $\log_p$ *is the Iwasawa branch of the* $p$*-adic logarithm satisfying* $\log_p(p) = 0$.

Theorem 1.1 was proven in a more general setting in the papers [3], [9], and [4]. Our proof here is far simpler in that it does not use the theory of $p$-adic Galois cohomology or the Galois representations associated to $p$-adic modular forms. Instead, we rely on a certain explicit construction using Theta series, congruences, and the $q$-expansion principle of Deligne–Ribet.

Let us briefly comment on our method and our assumptions. In the article [3], a certain $p$-adic family of Hilbert modular cusp forms was constructed specializing in weight 1 to a $p$-stabilization of the Eisenstein series $E_1(1, \chi)$. This family is constructed as a linear combination of products of Eisenstein series, and is in general not an eigenform. However, in the case of a *quadratic* character $\chi$, such a cuspidal eigenfamily can be constructed explicitly using Theta series. Furthermore, viewed as analytic functions of the $p$-adic weight variable $k$ and assuming that there is only one prime above $p$, this Theta series is congruent modulo $(k-1)^2$ to an explicit linear combination of Eisenstein series. This allows us to derive a formula for the first derivative at $k = 1$ for the function $L_p(\chi\omega, 1-k)$ which appears as the constant term of one of these Eisenstein series. This explains both of our assumptions that $\chi$ is quadratic and that the rank is equal to 1. The assumption about only one prime above $p$ is required for the congruence that we construct between Theta series and a linear combination of Eisenstein series. It seems possible that this last assumption can be removed by considering cusps other than the infinity cusp that we consider in this paper (see remark 7.2). It would be very interesting if our argument could be applied in a more general setting where our other assumptions are relaxed or removed. In this regard we remark that in [1] a cuspidal eigenform specialising in weight 1 to a $p$-stabilization of the Eisenstein series $E_1(1, \rho)$ is constructed in the case when $F = \mathbb{Q}$ and $\rho$ is an odd character.

# 2 Group rings and characters

Let $\Gamma = (1+2p\mathbf{Z}_p)^* \cong \mathbf{Z}_p$ and denote by $x \mapsto \langle x \rangle = x/\omega(x)$ the canonical projection $\mathbf{Z}_p^* \to \Gamma$. Let $\Lambda := \mathbf{Z}_p[[\Gamma]] \cong \mathbf{Z}_p[[T]]$. For each positive integer $k$, let $\Gamma_k$ be the quotient of $\Gamma$ of order

$p^k$, i.e.

$$\Gamma_k := (1+2p\mathbf{Z}_p)^*/(1+2p^{k+1}\mathbf{Z}_p)^*.$$

Write $\Lambda_k = \mathbf{Z}_p[\Gamma_k]$. Evidently $\Lambda = \varprojlim_k \Lambda_k$ and there are natural projections

$$\mu_k \colon \Lambda \to \Lambda_k.$$

We view the cyclotomic character of $F$ as a homomorphism on the group $I_F(p)$ of fractional ideals of $F$ relatively prime to $p$, defined by

$$\epsilon \colon I_F(p) \longrightarrow \Lambda^*, \qquad \epsilon(\mathfrak{a}) := \langle \mathrm{N}\mathfrak{a} \rangle \in \Gamma.$$

Now we define a certain character $\delta$ on a group of fractional ideals of $K$. Since $\chi(\mathfrak{p}) = 1$ we have a splitting $\mathfrak{p}\mathcal{O}_K = \mathfrak{P}\overline{\mathfrak{P}}$ for the unique prime $\mathfrak{p}$ of $F$ above $p$. We will define $\delta \colon I_K(\mathfrak{P}) \to \Lambda^*$. Let $h$ denote the class number of $K$. For $\mathfrak{a} \in I_K(\mathfrak{P})$, write $\mathfrak{a}^h = (\alpha)$ for some $\alpha \in K$. The generator $\alpha$ is defined up to multiplication by an element of $\mathcal{O}_K^*$, hence $x = \alpha^d$ is well-defined up to multiplication by an element of $\mathcal{O}_F^*$, where $d = [\mathcal{O}_K^* : \mathcal{O}_F^*]$. We then define

$$\delta \colon I_K(\mathfrak{P}) \longrightarrow \Lambda^*, \qquad \delta(\mathfrak{a}) = \langle \mathrm{N}_{\mathcal{O}_{\mathfrak{P}}/\mathbf{Z}_p}(x)^2 \rangle \in \Gamma.$$

It is clear that with respect to the natural inclusion map $I_F(p) \to I_K(\mathfrak{P}), \mathfrak{a} \mapsto \mathfrak{a}\mathcal{O}_K$, the restriction of $\delta$ to $I_F(p)$ is $\epsilon^{2dh}$.

For each positive integer $k$, we denote by $\delta_k$ and $\epsilon_k$ the compositions $\mu_k \circ \delta$ and $\mu_k \circ \epsilon$, respectively.

# 3 $p$-adic $L$-functions

The Deligne–Ribet, Cassou-Noguès $p$-adic $L$-function is the $p$-adic analytic function

$$L_p(\chi\omega, s) \colon \mathbf{Z}_p \longrightarrow \mathbf{Z}_p$$

specified uniquely by the interpolation property

$$L_p(\chi\omega, n) = L^*(\chi\omega^n, n) \text{ for all } n \in \mathbf{Z}^{\leq 0}. \tag{2}$$

Here $L^*$ denotes the classical $L$-value with the Euler factor at the prime $\mathfrak{p}$ removed.

Any $s \in \mathbf{Z}_p$ induces a continuous group homomorphism $\nu_s \colon \Gamma \to \mathbf{Z}_p^*$, $x \mapsto x^s$, which in turn induces a $\mathbf{Z}_p$-algebra homomorphism also denoted $\nu_s \colon \Lambda \to \mathbf{Z}_p$. Deligne–Ribet and Cassou-Noguès showed that there is an element $\mathscr{L}_p(\chi) \in \mathrm{Frac}(\Lambda)$ such that

$$L_p(\chi\omega, -s) = \nu_s(\mathscr{L}_p(\chi)) \tag{3}$$

for all $s \in \mathbf{Z}_p$.

To achieve integrality, let $\mathfrak{c}$ be an integral ideal relatively prime to $\mathfrak{n}$ and $p$, and define

$$\mathscr{L}_{p,\mathfrak{c}}(\chi) = (1 - \chi(\mathfrak{c})\mathrm{N}\mathfrak{c}[\langle \mathrm{N}\mathfrak{c}\rangle])\mathscr{L}_p(\chi).$$

Here $[\langle \mathrm{N}\mathfrak{c}\rangle]$ denotes the group-ring element in $\Lambda$ associated to $\langle \mathrm{N}\mathfrak{c}\rangle \in \Gamma$. For $\mathrm{N}\mathfrak{c}$ large enough, we then have $\mathscr{L}_{p,\mathfrak{c}}(\chi) \in \Lambda$. Since the prime $\mathfrak{p}$ of $F$ above $p$ satisfies $\chi(\mathfrak{p}) = 1$, the Euler factor of $L(\chi, s)$ at $\mathfrak{p}$ vanishes at $s = 0$. It follows from the interpolation property (2) that $L_p(\chi\omega, 0) = 0$. In view of (3), this implies that $\mathscr{L}_{p,\mathfrak{c}}(\chi)$ lies in the kernel of the augmentation map $\nu_0 \colon \Lambda \to \mathbf{Z}_p$. We denote the augmentation ideal by $I_\Lambda = \ker \nu_0$. There is a canonical homomorphism of $\mathbf{Z}_p$-modules given by

$$\mathscr{D} \colon I_\Lambda / I_\Lambda^2 \longrightarrow \mathbf{Z}_p, \qquad [t] - 1 \mapsto \log_p(t).$$

It follows directly from (3) (and the fact that the derivative of $t \mapsto t^s$ at $s = 0$ is $\log_p(t)$ for $t \in \Gamma$) that

$$L_p'(\chi\omega, 0)(1 - \chi(\mathfrak{c})\mathrm{N}\mathfrak{c}) = -\mathscr{D}(\mathscr{L}_{p,\mathfrak{c}}(\chi)).$$

For each $\ell \in \mathbf{Z}_p$, the map $\Gamma \to \Gamma, t \mapsto t^\ell$, induces a $\mathbf{Z}_p$-algebra homomorphism $\tilde{\nu}_\ell : \Lambda \to \Lambda$. It is clear that if $x \in I_\Lambda$, then $\tilde{\nu}_\ell(x)$ lies in $I_\Lambda$ as well, and we have $\mathscr{D}(\tilde{\nu}_\ell(x)) = \ell\mathscr{D}(x)$. For notational simplicity let $\mathscr{L}_{p,\mathfrak{c}}^\ell(\chi) := \tilde{\nu}_\ell(\mathscr{L}_{p,\mathfrak{c}}(\chi))$. The following is the form of Theorem 1.1 that we will prove.

**Proposition 3.1.** *Let $\ell = dh$ with notation as in §2. Write*

$$L_\mathfrak{c}(\chi, 0) = (1 - \chi(\mathfrak{c})\mathrm{N}\mathfrak{c})L(\chi, 0) \in \mathbf{Z}_p$$

*with $\mathfrak{c}$ as above. Theorem 1.1 is equivalent to the assertion*

$$\mathscr{L}_{p,\mathfrak{c}}^\ell(\chi) \equiv 2L_\mathfrak{c}(\chi, 0)(1 - \delta(\overline{\mathfrak{P}})) \pmod{I_\Lambda^2}.$$

*Proof.* Suppose the stated congruence holds. Applying $\mathscr{D}$, we obtain

$$-\ell L_p'(\chi\omega, 0)(1 - \chi(\mathfrak{c})\mathrm{N}\mathfrak{c}) = -2L_\mathfrak{c}(\chi, 0)\log_p(\delta(\overline{\mathfrak{P}})).$$

This simplifies to

$$\frac{L_p'(\chi\omega, 0)}{L(\chi, 0)} = \frac{\log_p(\delta(\overline{\mathfrak{P}}))}{\ell/2}. \tag{4}$$

Write $\overline{\mathfrak{P}}^h = (\alpha)$ and $\alpha^d = x$. Then we may take $u = \overline{x}/x$ in the statement of Theorem 1.1 with

$$\mathrm{ord}_\mathfrak{P}(u) = dh = \ell/2. \tag{5}$$

Meanwhile

$$\begin{aligned} \log_p(\delta(\overline{\mathfrak{P}})) &= 2\log_p \mathrm{N}_{K_\mathfrak{P}/\mathbf{Q}_p} x \\ &= \log_p \mathrm{N}_{K_\mathfrak{P}/\mathbf{Q}_p} x - \log_p \mathrm{N}_{K_{\overline{\mathfrak{P}}}/\mathbf{Q}_p} x \qquad (6) \\ &= -\log_p \mathrm{N}_{K_\mathfrak{P}/\mathbf{Q}_p} u. \qquad (7) \end{aligned}$$

Note that (6) follows since $\log_p \mathrm{N}_{K_{\mathfrak{P}}/\mathbf{Q}_p}(x) + \log_p \mathrm{N}_{K_{\overline{\mathfrak{P}}}/\mathbf{Q}_p}(x) = \log_p \mathrm{N}_{K/\mathbf{Q}}(x) = 0$, as $\mathrm{N}_{K/\mathbf{Q}}(x)$ is a power of $p$ up to sign. Combining (7) with (4) and (5) gives the desired result (1). These steps are reversible. □

We conclude this section by noting that the $p$-adic $L$-function $\mathscr{L}_p(\chi)$ satisfies another interpolation property in addition to (3). For each positive integer $k$, the image of $\mathscr{L}_p(\chi)$ in $\Lambda_k$ actually lies in $\mathbf{Q}[\Gamma_k]$. For any character $\psi\colon \Gamma_k \to \mathbf{C}$, it then makes sense to consider $\psi(\mathscr{L}_p(\chi)) \in \mathbf{C}$. The interpolation property we are interested in is:

$$\psi(\mathscr{L}_p(\chi)) = L^*(\chi\epsilon_\psi, 0), \tag{8}$$

where $\epsilon_\psi = \psi \circ \epsilon$.

# 4 Hilbert modular forms

We follow Shimura for the definition of Hilbert modular forms (see [8] and [3, §4]). This definition is slightly more robust than that of Deligne–Ribet, in that it allows for an action of Hecke operators. Our HMFs can be viewed as an $h_F$-tuple of functions on $\mathcal{H}^n$, where $h_F$ is the narrow class number of $F$. The HMFs of Deligne–Ribet are single functions on $\mathcal{H}^n$, and can be viewed as the first component of our tuples.

For a ring $R$, integer $k$, and ideal $\mathfrak{n} \subset \mathcal{O}_F$, we denote by $M_k(\mathfrak{n}, R) = M_k(\mathfrak{n}, \mathbf{Z}) \otimes R$ the $R$-module of Hilbert modular forms over $F$ of weight $k$ and level $\mathfrak{n}$ over the ring $R$. A modular form $f$ in this space is determined by its $q$-expansion "at infinity", which we describe conveniently using the notation of Shimura and Wiles as a collection of elements

$$c_\lambda(f, 0) \in R, \quad \lambda = 1, 2, \ldots, h_F, \qquad c(f, \mathfrak{m}) \in R, \quad \mathfrak{m} \subset \mathcal{O}_F \text{ a nonzero ideal.}$$

One knows that if the $q$-expansion coefficients of a form $f$ live in a subring $R_0 \subset R$, then $f$ lies in $M_k(\mathfrak{n}, R_0)$ (see [5, 5.13]).

The simplest example of Hilbert modular forms are the Eisenstein series. Let $\mathfrak{n}$ denote the conductor of the quadratic character $\chi$. We fix fractional ideals $\mathfrak{t}_1, \ldots, \mathfrak{t}_{h_F}$ representing the $h_F$ distinct classes in the narrow class group of $F$.

**Proposition 4.1.** *There is an element $E_1(1, \chi) \in M_k(\mathfrak{n}, \mathbf{Q})$ whose $q$-expansion coefficients are given by*

$$c_\lambda(E_1(1, \chi), 0) = \begin{cases} 2^{-n} L(\chi, 0) & \text{if } \mathfrak{n} \neq 1 \\ 2^{-n}(L(\chi, 0) + \chi(\mathfrak{t}_\lambda) L(\chi, 0)) & \text{if } \mathfrak{n} = 1. \end{cases} \tag{9}$$

$$c(E_1(1, \chi), \mathfrak{m}) = \sum_{\mathfrak{a} \mid \mathfrak{m}} \chi(\mathfrak{a}). \tag{10}$$

**Remark 4.2.** In (10), $\chi(\mathfrak{a})$ is understood to be 0 if $\mathfrak{a}$ is divisible by a prime ramified in $K$.

For a proof of this proposition, see [3, Proposition 2.1].

# 5 Group-ring valued modular forms

We introduce the following definition:

**Definition 5.1.** For fixed integer $k_0$, define $\mathcal{M}_{k_0}(\mathfrak{n},\Lambda)$ to be the $\Lambda$-module of collections $f$ of elements

$$c_\lambda(f,0)\in \mathrm{Frac}(\Lambda),\quad \lambda=1,2,\dots,h_F,\qquad c(f,\mathfrak{m})\in\Lambda,\quad \mathfrak{m}\subset\mathcal{O}_F \text{ a nonzero ideal}$$

such that for each $k$ large enough, the images of $c_\lambda(f,0), c(f,\mathfrak{m})$ in $\mathrm{Frac}(\Lambda_k)$ are the $q$-expansion coefficients of an element of $M_{k_0}(\mathfrak{n}p^{k+1},\mathrm{Frac}(\Lambda_k))$.

Simple explicit examples of such modular forms are given by the Eisenstein series.

**Proposition 5.2.** *For every integer $\ell$, there are Eisenstein series*

$$E_1(\chi,\epsilon^\ell), E_1(1,\chi\epsilon^\ell)\in\mathcal{M}_1(\mathfrak{n},\Lambda)$$

*given by*

$$c_\lambda(E_1(\chi,\epsilon^\ell),0)=\begin{cases}0 & \text{if } \mathfrak{n}\neq 1\\ 2^{-n}\chi(\mathfrak{t}_\lambda)\mathscr{L}_p^\ell(\chi) & \text{if } \mathfrak{n}=1\end{cases},\ c(E_1(\chi,\epsilon^\ell),\mathfrak{m})=\sum_{\mathfrak{a}\mid\mathfrak{m}}\chi(\mathfrak{a})\epsilon^\ell(\mathfrak{m}/\mathfrak{a}) \tag{11}$$

$$c_\lambda(E_1(1,\chi\epsilon^\ell),0)=2^{-n}\mathscr{L}_p^\ell(\chi),\qquad c(E_1(1,\chi\epsilon^\ell),\mathfrak{m})=\sum_{\mathfrak{a}\mid\mathfrak{m}}\chi(\mathfrak{a})\epsilon^\ell(\mathfrak{a}). \tag{12}$$

**Remark 5.3.** In (11) and (12), $\chi(\mathfrak{a})$ is understood to be zero if $\mathfrak{a}$ is divisible by a prime ramified in $K$, and $\epsilon(\mathfrak{a})$ is understood to be zero if $\mathfrak{a}$ is divisible by a prime above $p$.

*Proof.* Fix a positive integer $k$ large enough to ensure that $\chi\epsilon_k$ does not have trivial conductor (usually this will be true for all $k$). We need to show that there exist Eisenstein series in

$$M_{k_0}(\mathfrak{n}p^{k+1},\mathrm{Frac}(\Lambda_k))$$

whose $q$-expansion coefficients are the images of the expressions in (11) and (12). Yet these images in fact lie in the subring $\mathbf{Q}[\Gamma_k]$. By Deligne–Ribet it suffices to show these are modular forms over

$$\mathbf{C}[\Gamma_k]\cong\prod_\psi\mathbf{C},$$

where the product ranges over all $p^k$ characters $\psi\colon\Gamma_k\to\mathbf{C}^*$ and the isomorphism is induced by $[d]\to(\psi(d))_\psi$.

For $E_1(\chi,\epsilon^\ell)$, we therefore need to show that for each character $\psi$ of $\Gamma_k$, there is a modular form $f\in M_1(\mathfrak{n}p^{k+1},\mathbf{C})$ such that

$$c_\lambda(f,0)=\begin{cases}0 & \text{if } \mathfrak{n}\neq 1\\ 2^{-n}\chi(\mathfrak{t}_\lambda)L^*(\chi\epsilon_\psi^*,0) & \text{if } \mathfrak{n}=1\end{cases},\qquad c(f,\mathfrak{m})=\sum_{\mathfrak{a}\mid\mathfrak{m}}\chi(\mathfrak{a})(\epsilon_\psi^*)^\ell(\mathfrak{m}/\mathfrak{a}).$$

Here $\epsilon_\psi^*$ is the character $\epsilon_\psi$ viewed as having modulus divisible by all primes above $p$. Yet this is just the classical form $E_1(\chi,(\epsilon_\psi^*)^\ell)$ (see for instance [3, Proposition 2.1]). For the form $E_1(1,\chi\epsilon^\ell)$, the image under $\psi$ is the classical form $E_1(1,\chi(\epsilon_\psi^*)^\ell)$. For the constant terms, the verification of this follows from (8). ∎

# 6 Theta series

Another class of modular forms that can be written down explicitly are the Theta series.

**Proposition 6.1.** *There is an element $\theta_\delta \in \mathcal{M}_1(\mathfrak{n}, \Lambda)$ given by*

$$c_\lambda(\theta_\delta, 0) = 0, \qquad c(\theta_\delta, \mathfrak{m}) = \sum_{\mathfrak{a} \subset \mathcal{O}_K, (\mathfrak{a}, \mathfrak{P})=1, \mathfrak{a}\bar{\mathfrak{a}}=\mathfrak{m}} \delta(\mathfrak{a}).$$

*Proof.* As in the proof of Proposition 5.2, we need to show that for each character $\psi$ of $\Gamma_k$ there is a modular form $f \in M_1(\mathfrak{n}p^{k+1}, \mathbf{C})$ whose $q$-expansion is given by

$$c_\lambda(f, 0) = 0, \qquad c(f, \mathfrak{m}) = \sum_{\mathfrak{a} \subset \mathcal{O}_K, (\mathfrak{a}, \mathfrak{P})=1, \mathfrak{a}\bar{\mathfrak{a}}=\mathfrak{m}} \delta_\psi(\mathfrak{a}),$$

where $\delta_\psi := \psi \circ \delta$. This the usual theta series associated to the ray class character $\delta_\psi$; see for example [7, Theorem 2.72]. □

# 7 A congruence of modular forms

As in §2 and §3, write $\ell = 2dh$. We define two elements $\mathcal{F}, \mathcal{G} \in \mathcal{M}_1(\mathfrak{n}, \Lambda)$:

$$\mathcal{F} = \delta(\overline{\mathfrak{P}})\theta_\delta + (1 - \delta(\overline{\mathfrak{P}}))E_1(1, \chi), \tag{13}$$

$$\mathcal{G} = (E_1(\chi, \epsilon^\ell) + E_1(1, \chi\epsilon^\ell))/2. \tag{14}$$

**Proposition 7.1.** *The non-constant q-expansion coefficients of $\mathcal{F}$ and $\mathcal{G}$ lie in $\Lambda$ and satisfy the congruence*

$$c(\mathcal{F}, \mathfrak{m}) \equiv c(\mathcal{G}, \mathfrak{m}) \pmod{I_\Lambda^2} \quad \textit{for all nonzero ideals } \mathfrak{m}.$$

*Proof.* First we verify the result for $\mathfrak{m}$ of the form $\mathfrak{p}^m, m \geq 0$. We have:

$$\begin{aligned} c(\theta_\delta, \mathfrak{p}^m) &= \delta(\overline{\mathfrak{P}})^m, \\ c(E_1(1, \chi), \mathfrak{p}^m) &= m + 1, \\ c(E_1(\chi, \epsilon^\ell), \mathfrak{p}^m) &= c(E_1(1, \chi\epsilon^\ell), \mathfrak{p}^m) = 1. \end{aligned} \tag{15}$$

Then our desired congruence reads

$$\delta(\overline{\mathfrak{P}})^{m+1} + (1 - \delta(\overline{\mathfrak{P}}))(m + 1) \equiv 1 \pmod{I_\Lambda^2}.$$

This indeed holds, since $\delta(\overline{\mathfrak{P}}) - 1 \in I_\Lambda$, hence

$$\begin{aligned} \delta(\overline{\mathfrak{P}})^{m+1} - 1 &= (\delta(\overline{\mathfrak{P}}) - 1)(1 + \delta(\overline{\mathfrak{P}}) + \cdots + \delta(\overline{\mathfrak{P}})^m) \\ &\equiv (\delta(\overline{\mathfrak{P}}) - 1)(m + 1) \pmod{I_\Lambda^2}. \end{aligned}$$

For the remaining coefficients, we can argue inductively using the Hecke operators. Break up $\mathcal{F}$ and $\mathcal{G}$ as sums $\mathcal{F} = \mathcal{F}_1 + \mathcal{F}_2$, $\mathcal{G} = \mathcal{G}_1 + \mathcal{G}_2$, where

$$\begin{aligned}
\mathcal{F}_1 &= \delta(\overline{\mathfrak{P}})\theta_\delta, \\
\mathcal{F}_2 &= (1-\delta(\overline{\mathfrak{P}}))E_1(1,\chi), \\
\mathcal{G}_1 &= \frac{1}{2}E_1(\chi,\epsilon^\ell), \\
\mathcal{G}_2 &= \frac{1}{2}E_1(1,\chi\epsilon^\ell).
\end{aligned}$$

Each of these forms $\mathcal{F}_1, \mathcal{F}_2, \mathcal{G}_1, \mathcal{G}_2$ is an eigenform for the Hecke operators at all primes $\mathfrak{l}$ of $F$ not equal to $\mathfrak{p}$, i.e. for the operators $U_\mathfrak{l}$ for $\mathfrak{l} \mid \mathfrak{n}$, and $T_\mathfrak{l}$ for $\mathfrak{l} \nmid \mathfrak{n}\mathfrak{p}$. Furthermore, we will show that the following congruences hold for the corresponding Hecke eigenvalues:

(A) $\lambda_\mathfrak{l}(\mathcal{F}_1) \equiv \lambda_\mathfrak{l}(\mathcal{F}_2) \equiv \lambda_\mathfrak{l}(\mathcal{G}_1) \equiv \lambda_\mathfrak{l}(\mathcal{G}_2) \pmod{I_\Lambda}$.

(B) $2\lambda_\mathfrak{l}(\mathcal{F}_1) \equiv \lambda_\mathfrak{l}(\mathcal{G}_1) + \lambda_\mathfrak{l}(\mathcal{G}_2) \pmod{I_\Lambda^2}$.

We also have

(C) $c(\mathcal{F}_2, \mathfrak{m}) \in I_\Lambda$.

(D) $c(\mathcal{G}_1, \mathfrak{m}) \equiv c(\mathcal{G}_2, \mathfrak{m}) \pmod{I_\Lambda}$.

Let us now explain why these congruences give the desired result. Suppose that we have the congruence

$$c(\mathcal{F}_1, \mathfrak{m}) + c(\mathcal{F}_2, \mathfrak{m}) \equiv c(\mathcal{G}_1, \mathfrak{m}) + c(\mathcal{G}_2, \mathfrak{m}) \pmod{I_\Lambda^2}, \tag{16}$$

as well as, if $\mathfrak{l} \mid \mathfrak{m}$, the congruence

$$c(\mathcal{F}_1, \mathfrak{m}/\mathfrak{l}) + c(\mathcal{F}_2, \mathfrak{m}/\mathfrak{l}) \equiv c(\mathcal{G}_1, \mathfrak{m}/\mathfrak{l}) + c(\mathcal{G}_2, \mathfrak{m}/\mathfrak{l}) \pmod{I_\Lambda^2}.$$

Using the formulae for the action of the Hecke operator $U_\mathfrak{l}$ or $T_\mathfrak{l}$ in the cases $\mathfrak{l} \mid \mathfrak{n}$ and $\mathfrak{l} \nmid \mathfrak{n}\mathfrak{p}$, respectively, the same congruence (16) will hold for $\mathfrak{m}$ replaced by $\mathfrak{m}\mathfrak{l}$ if we can show that

$$\lambda_\mathfrak{l}(\mathcal{F}_1)c(\mathcal{F}_1, \mathfrak{m}) + \lambda_\mathfrak{l}(\mathcal{F}_2)c(\mathcal{F}_2, \mathfrak{m}) \equiv \lambda_\mathfrak{l}(\mathcal{G}_1)c(\mathcal{G}_1, \mathfrak{m}) + \lambda_\mathfrak{l}(\mathcal{G}_2)c(\mathcal{G}_2, \mathfrak{m}) \pmod{I_\Lambda^2}. \tag{17}$$

In view of (16), the congruence (17) is equivalent to

$$(\lambda_\mathfrak{l}(\mathcal{F}_2) - \lambda_\mathfrak{l}(\mathcal{F}_1))c(\mathcal{F}_2, \mathfrak{m}) \equiv (\lambda_\mathfrak{l}(\mathcal{G}_1) - \lambda_\mathfrak{l}(\mathcal{F}_1))c(\mathcal{G}_1, \mathfrak{m}) + (\lambda_\mathfrak{l}(\mathcal{G}_2) - \lambda_\mathfrak{l}(\mathcal{F}_1))c(\mathcal{G}_2, \mathfrak{m}) \pmod{I_\Lambda^2}. \tag{18}$$

Applying (A) and (C), the left side of (18) lies in $I_\Lambda^2$. Meanwhile, in view of (B), the right side is congruent modulo $I_\Lambda^2$ to

$$(\lambda_\mathfrak{l}(\mathcal{G}_1) - \lambda_\mathfrak{l}(\mathcal{F}_1))(c(\mathcal{G}_1, \mathfrak{m}) - c(\mathcal{G}_2, \mathfrak{m})). \tag{19}$$

Applying (A) and (D), the expression (19) lies in $I_\Lambda^2$ as well. This proves the congruence (16) for $\mathfrak{m}$ replaced by $\mathfrak{m}\mathfrak{l}$, and hence concludes the proof of the theorem up to verifying the congruences (A)–(D).

The Hecke eigenvalues of the forms are given below.

| Form | $U_\mathfrak{l},\ \mathfrak{l}\mathcal{O}_K = \mathfrak{q}^2$ | $T_\mathfrak{l},\ \mathfrak{l}\mathcal{O}_K = \mathfrak{q}\overline{\mathfrak{q}}$ | $T_\mathfrak{l},\ \mathfrak{l}\mathcal{O}_K = \mathfrak{q}$ |
|---|---|---|---|
| $\mathcal{F}_1 \sim \theta_\delta$ | $[\delta(\mathfrak{q})]$ | $[\delta(\mathfrak{q})] + [\delta(\overline{\mathfrak{q}})]$ | 0 |
| $\mathcal{F}_2 \sim E_1(1,\chi)$ | 1 | 2 | 0 |
| $\mathcal{G}_1 \sim E_1(\chi,\epsilon^\ell)$ | $[\epsilon_\ell(\mathfrak{l})]$ | $1 + [\epsilon^\ell(\mathfrak{l})]$ | $-1 + [\epsilon^\ell(\mathfrak{l})]$ |
| $\mathcal{G}_2 \sim E_1(1,\chi\epsilon^\ell)$ | 1 | $1 + [\epsilon^\ell(\mathfrak{l})]$ | $1 - [\epsilon^\ell(\mathfrak{l})]$ |

The verification of (A) from the table is trivial. For (B), one uses the fact that $\delta(\mathfrak{l}\mathcal{O}_K) = \epsilon^\ell(\mathfrak{l})$. For example, for $\mathfrak{l}\mathcal{O}_K = \mathfrak{q}^2$, we calculate

$$\begin{aligned} 2\lambda_\mathfrak{l}(\mathcal{F}_1) - \lambda_\mathfrak{l}(\mathcal{G}_1) - \lambda_\mathfrak{l}(\mathcal{G}_2) &= 2[\delta(\mathfrak{q})] - [\epsilon^\ell(\mathfrak{l})] - 1 \\ &= 2[\delta(\mathfrak{q})] - [\delta(\mathfrak{q}^2)] - 1 \\ &= -(1 - [\delta(\mathfrak{q})])^2 \in I_\Lambda^2. \end{aligned}$$

The other cases are similar.

(C) is immediate since $\mathcal{F}_2$ contains a factor of $(1 - \delta(\overline{\mathfrak{P}})) \in I_\Lambda$. (D) follows immediately from the last congruence in (A), as long as we verify the base cases $\mathfrak{m} = \mathfrak{p}^m$. But this was already observed in (15). This concludes the proof. □

**Remark 7.2.** Here we remark about the case when there are primes in $F$ above $p$ other than $\mathfrak{p}$. To ensure that we are in rank 1 situation we still assume that $\mathfrak{p}$ is the only prime above $p$ such that $\chi(\mathfrak{p}) = 1$. If there are primes other than $\mathfrak{p}$ over $p$ in $F$, then the congruence in the above theorem does not hold at these primes (the definition of $\delta$ must be modified to even make sense of this statement). The congruence can be "reinstated" if all the Eisenstein series in the theorem are "$\mathfrak{q}$-depleted" for all primes $\mathfrak{q} \mid p$ and $\mathfrak{q} \neq \mathfrak{p}$. However, the constant terms of these Eisenstein series at infinity are zero. It may be possible to establish the main theorem by looking at $q$-expansions at different cusps. We leave this as an open problem.

# 8 The Deligne–Ribet $q$-expansion principle

We may now prove Theorem 1.1 by applying the following $q$-expansion principle of Deligne–Ribet. Let $x$ be a finite idele of $F$. If $\mathcal{F}$ is a modular form, then we denote by $\mathcal{F}_x$ the form $\mathcal{F}\Big|\begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}$ (for explanation see [5, page 262 and proposition 5.8]). In view of [5, proposition 5.8] the $q$-expansion of $\mathcal{F}$ at "cusps determined by $x$" is the $q$-expansion of $\mathcal{F}_x$ at infinity.

**Proposition 8.1.** *Let $R$ be a $p$-adic ring, i.e. such that the natural map $R \to \varprojlim R/p^n$ is an isomorphism. Suppose that $\mathcal{F} \in M_k(\mathfrak{n}, R \otimes \mathbf{Q}_p)$ has non-constant $q$-expansion coefficients $c(\mathfrak{m}, \mathcal{F})$ lying in an ideal $I \subset R$. Then for every $\lambda$ the difference $c_\lambda(\mathcal{F}, 0) - c_\lambda(\mathcal{F}_x, 0)$ also lies in $I$.*

*Proof.* The proof follows that of [5, Corollary 5.14]. The form $\mathcal{F} - (c_\lambda(\mathcal{F}, 0))_\lambda$ can be viewed as a $p$-adic modular form with constant terms 0 at infinity and other $q$-expansion coefficients lying in $I \subset R$. In particular $\mathcal{F} - (c_\lambda(\mathcal{F}, 0))_\lambda$ has all Fourier coefficients in $R$ and hence can be viewed as a $p$-adic modular form over $R$. Furthermore, its image in $R/I$ is 0. Then by [5, (5.13)], it follows that the $q$-expansion of $\mathcal{F} - (c_\lambda(\mathcal{F}, 0))_\lambda$ at cusps determined by $x$ has all coefficients lying in $I$. The $q$-expansion of the weight zero form $(c_\lambda(\mathcal{F}, 0))_\lambda$ is the constant tuple $(c_\lambda(\mathcal{F}, 0))_\lambda$. Therefore the difference $c_\lambda(\mathcal{F}, 0) - c_\lambda(\mathcal{F}_x, 0)$ lies in $I$, for all $\lambda$. □

We apply this to the difference between the forms $\mathcal{F}$ and $\mathcal{G}$ constructed in the previous section, or more precisely to their specializations $\mathcal{F}_k, \mathcal{G}_k \in M_1(\mathfrak{n}, \Lambda_k)$ for each positive integer $k$. Let $x$ be a finite idele of $F$. It determines an ideal $\mathfrak{c}$ of $F$. We assume that $\mathfrak{c}$ is relatively prime to $\mathfrak{n}p$. It follows from Proposition 7.1 that the difference in the constant terms of $\mathcal{F}_k - \mathcal{G}_k$ at infinity and at cusps determined by $x$ lies in $I_k^2$, where $I_k$ is the augmentation ideal of $\Lambda_k$. Passing to the inverse limit, it follows that the difference in the constant terms of $\mathcal{F} - \mathcal{G}$ at infinity and at cusps determined by $x$ lies in $I_\Lambda^2$.

Now the form $\theta_\delta$ is a cusp form, hence the constant terms of $\mathcal{F}$ at infinity are

$$c_\lambda(\mathcal{F}, 0) = \begin{cases} 2^{-n} L(\chi, 0) & \text{if } \mathfrak{n} \neq 1 \\ 2^{-n}(L(\chi, 0) + \chi(\mathfrak{t}_\lambda) L(\chi, 0)) & \text{if } \mathfrak{n} = 1. \end{cases}$$

The constant terms of $\mathcal{G}$ at infinity are

$$c_\lambda(\mathcal{G}, 0) = \begin{cases} 2^{-(n+1)} \mathscr{L}_p^\ell(\chi) & \text{if } \mathfrak{n} \neq 1 \\ 2^{-(n+1)}(\mathscr{L}_p^\ell(\chi) + \chi(\mathfrak{t}_\lambda) \mathscr{L}_p^\ell(\chi)) & \text{if } \mathfrak{n} = 1. \end{cases}$$

Now, as calculated in [5, (6.1)], the constant terms of $\mathcal{F}$ and $\mathcal{G}$ at cusps determined by $x$ are the above constant terms at infinity multiplied by $\chi(\mathfrak{c})\mathrm{N}\mathfrak{c}$ and $\chi(\mathfrak{c})\mathrm{N}\mathfrak{c}[\langle \mathrm{N}\mathfrak{c} \rangle]$, respectively. We therefore conclude that

$$\mathscr{L}_{p,\mathfrak{c}}^\ell(\chi)/2 \equiv L_\mathfrak{c}(\chi, 0)(1 - \delta(\overline{\mathfrak{P}})) \pmod{I_\Lambda^2}.$$

In view of Proposition 3.1, this concludes the proof of Theorem 1.1.

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