Factorization of p-adic Rankin L-series

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Abstract

We prove that the p-adic L-series of the tensor square of a p-ordinary modular form factors as the product of the symmetric square p-adic L-series of the form and a Kubota–Leopoldt p-adic L-series. This establishes a generalization of a conjecture of Citro. Greenberg's exceptional zero conjecture for the adjoint follows as a corollary of our theorem.

Our method of proof follows that of Gross, who proved a factorization result for the Katz p-adic L-series associated to the restriction of a Dirichlet character. Whereas Gross's method is based on comparing circular units with elliptic units, our method is based on comparing these same circular units with a new family of units (called Beilinson–Flach units) that we construct. The Beilinson–Flach units are constructed using Bloch's intersection theory of higher Chow groups applied to products of modular curves. We relate these units to special values of classical and p-adic L-functions using work of Beilinson (as generalized by Lei–Loeffler–Zerbes) in the archimedean case and Bertolini–Darmon–Rotger (as generalized by Kings–Loeffler–Zerbes) in the p-adic case. Central to our method are two compatibility theorems regarding Bloch's intersection pairing and the classical and p-adic Beilinson regulators defined on higher Chow groups.

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1. Introduction

The main result of this paper is a factorization formula for the p-adic L-function associated to the tensor square of a p-ordinary cuspidal eigenform. We introduce some notation to state our result. Let

$$f = \sum_{n=1}^{\infty} a_n q^n \in S_k(\Gamma_1(N_f), \chi_f), \qquad g = \sum_{n=1}^{\infty} b_n q^n \in S_\ell(\Gamma_1(N_g), \chi_g)$$

be two normalized cuspidal eigenforms of weights $k, \ell \geqslant 2$ and nebentype characters χ_f, χ_g, respectively. Let ψ be an auxiliary Dirichlet character of conductor N_ψ, and let $N = N_f N_g N_\psi$. The Rankin L-series of f and g twisted by ψ is defined by

$$D_N(f, g, \psi, s) = L_N(\chi_f \chi_g \psi^2, 2s + 2 - k - \ell) \sum_{n=1}^{\infty} a_n b_n \psi(n) n^{-s},$$

where L_N denotes a Dirichlet L-function with the Euler factors at primes dividing N removed. The Rankin series $D_N(f, g, \psi, s)$ has an Euler product equal to that of the primitive L-series $L(f \otimes g \otimes \psi, s)$ outside of primes dividing N; see §2.3.

Shimura proved that the values of $D_N(f, g, \psi, s)$ normalized by the appropriate period are algebraic when s is critical [Sh]. There exist critical values only when the weights of f and g are unequal; if $k > \ell$, then the critical s are those in the range $\ell \leqslant s \leqslant k - 1$.

Let $p \geqslant 5$ be a prime number. Hida constructed a p-adic L-function interpolating the critical values of $D_N(f, g, \psi, s)$ when f and g are p-ordinary eigenforms. Let $\mathcal{W} = \mathrm{Hom}_{\mathrm{cont}}(\mathbf{Z}_p^*, \mathbf{C}_p^*)$ denote p-adic weight space, which contains $\mathbf{Z}$ via $s \mapsto \nu_s \in \mathcal{W}$, where $\nu_s(z) = z^s$. After modifying Hida's function to preserve primitivity at the bad primes, one obtains a p-adic L-function $L_p(f \otimes g \otimes \psi, \sigma)$ for $\sigma \in \mathcal{W}$ interpolating the algebraic parts of the values $L(f \otimes g \otimes \psi\beta^{-1}, s)$ for characters β of p-power conductor and integers s satisfying $\ell \leqslant s \leqslant k - 1$. Hida extended his result in [Hi3] by allowing f and g to vary in p-adic families. This allows for the definition of $L_p(f \otimes g \otimes \psi, \sigma)$ when $k = \ell$, even though in this case the classical Rankin function $L(f \otimes g \otimes \psi, s)$ has no critical values.

The main theorem of this paper is a factorization of Hida's p-adic Rankin L-series when $f = g$. To motivate this result, we consider the setting for classical L-series. Let ρ_f denote the 2-dimensional ℓ-adic Galois representation attached to the form f, and let ϵ denote the ℓ-adic cyclotomic character. In view of the the decomposition

$$\rho_f \otimes \rho_f \otimes \psi \cong (\mathrm{Sym}^2 \rho_f \otimes \psi) \oplus (\chi\psi\epsilon^{k-1}),$$

the Artin formalism yields an equality of primitive[1] L-series:

$$L(f \otimes f \otimes \psi, s) = L(\mathrm{Sym}^2 f \otimes \psi, s) L(\chi\psi, s - k + 1). \tag{1}$$

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[1] *Primitive* refers to L-series that are defined via the Artin formalism by Euler products given by the characteristic polynomial of Frobenius on the inertia coinvariants of a representation; *imprimitive* L-series are defined explicitly in terms of Fourier coefficients of modular forms. The two differ only by certain Euler factors at the bad primes. Precise definitions are given in §2.2 and §2.3.

Our main theorem is a p-adic analogue of this result. Suppose that $p \nmid N_\psi$ (this is no restriction as p-power conductor twists are incorporated into $\mathcal{W}$). We denote by $L_p(\mathrm{Sym}^2 \otimes \psi, \sigma)$ the Schmidt–Hida p-adic L-function ([Sc]) interpolating the algebraic parts of the classical values $L(\mathrm{Sym}^2 f \otimes \psi\beta^{-1}, s)$ for p-power conductor characters β and integers s satisfying $1 \leqslant s \leqslant k-1$ and $\psi\beta^{-1}(-1) = (-1)^{s+1}$ or $k \leqslant s \leqslant 2k-2$ and $\psi\beta^{-1}(-1) = (-1)^s$.

Theorem 1. *Let $f \in S_k(\Gamma_1(N), \chi)$ be a p-ordinary eigenform. Decompose $\chi = \chi'\chi_p$ into its prime-to-p and p-power parts. Define $\kappa \in \mathcal{W}$ by $\kappa(z) = z^k\chi_p(z)$. If $\sigma(-1) = -\psi(-1)$, we have*

$$L_p(f \otimes f \otimes \psi, \sigma) = L_p(\mathrm{Sym}^2 f \otimes \psi, \sigma) L_p(\chi'\psi, z \cdot \sigma/\kappa). \tag{2}$$

If $p \nmid N$, the same equation holds for $\sigma(-1) = \psi(-1)$ as well.

Here $L_p(\chi'\psi, \sigma)$ denotes a Kubota–Leopoldt p-adic L-function viewed as a function on weight space via a convention described below. As noted above, the interest and difficulty in proving Theorem 1 arises from the fact that $L_p(f \otimes f \otimes \psi, s)$ has no critical values, and hence the proof does not simply arise by p-adically interpolating the classical formula (1). Instead, our factorization formula is proven by generalizing the method of Gross in [Gro].

Remark 1.1. In order to obtain an exact formula such as (2), one must be careful about normalizations. Our conventions are described in Section 3. For now we stress one important point already mentioned above: since Hida's p-adic Rankin L-series interpolates *imprimitive* L-values, certain Euler factors at primes dividing N must be adjusted in defining $L_p(f \otimes f \otimes \psi, \sigma)$ from Hida's function. As a general rule in this paper, imprimitive L-functions are noted with a comma (e.g. $L(\mathrm{Sym}^2 f, \psi)$ or $L(f, g, \psi)$) whereas primitive L-functions are denoted with a tensor symbol (e.g. $L(\mathrm{Sym}^2 f \otimes \psi)$ or $L(f \otimes g \otimes \psi)$). This holds even for p-adic L-functions with the exception of Euler factors at p, where certain factors must always be adjusted for the purpose of interpolation.

Remark 1.2. The reason that we must impose the condition $\sigma(-1) = -\psi(-1)$ when $p \mid N$ is that Hida only defined $L_p(\mathrm{Sym}^2 f \otimes \psi, \sigma)$ on this half of weight space. Under this sign condition, arithmetic weights $\nu_{\alpha,s}$ with $1 \leqslant s \leqslant k-1$ are critical; under the reversed sign condition, weights $\nu_{\alpha,s}$ with $k \leqslant s \leqslant 2k-2$ are critical. Schmidt had earlier defined $L_p(\mathrm{Sym}^2 f \otimes \psi, \sigma)$ for $\sigma(-1) = \psi(-1)$ and showed that it satisfies a functional equation, but only under the assumption $p \nmid N$. It was suggested to us by D. Loeffler that one could *define* $L_p(\mathrm{Sym}^2 f \otimes \psi, \sigma)$ for $\sigma(-1) = \psi(-1)$ even when $p \mid N$ by means of the functional equation, and then prove that it satisfies the desired interpolation property using the classical functional equation; we do not explore this idea here.

We learned after the completion of this project that G. Rosso has generalized the Schmidt–Hida construction and defined a 2-variable p-adic L-function $L_p(\mathrm{Sym}^2 f \otimes \psi, \sigma)$ for $\sigma(-1) = \psi(-1)$, even over arbitrary totally real fields (see [Ros, Theorem A.3] and [Ros2, Theorems A.2 and B.1]). However, the interpolation formula for this function at forms f with level divisible by p is not given explicitly in those articles. We leave open the problem of explicating the interpolation formula in this case and combining it with the methods of this paper to prove equation (2) for $p \mid N$ and $\sigma(-1) = \psi(-1)$.

Remark 1.3. It is common to present p-adic L-functions as functions of a variable $s \in \mathbf{Z}_p$. Let us rephrase our main result in this language. For an equivalence class i mod $(p-1)$, let $L_p^{[i]}(\mathrm{Sym}^2 f \otimes \psi, s)$ denote the branch of Schmidt's p-adic L-function interpolating the algebraic parts of the classical values $L(\mathrm{Sym}^2 f, \omega^{s-i}\psi, s)$ for $1 \leqslant s \leqslant k-1$ when $\psi(-1) = (-1)^{i+1}$ and for $k \leqslant s \leqslant 2k-2$ when $\psi(-1) = (-1)^i$. (Here ω is the Teichmüller character.) In terms of our function defined earlier on weight space, it is given by

$$L_p^{[i]}(\mathrm{Sym}^2 f \otimes \psi, s) = L_p(\mathrm{Sym}^2 f \otimes \psi, \omega^i(z)\langle z\rangle^s), \tag{3}$$

where $\langle z\rangle := z/\omega(z)$. Similarly let

$$L_p^{[i]}(f \otimes f \otimes \psi, s) = L_p(f \otimes f \otimes \psi, \omega^i(z)\langle z\rangle^s).$$

Theorem 1 can then be written as follows:

$$L_p^{[i]}(f \otimes f \otimes \psi, s) = L_p^{[i]}(\mathrm{Sym}^2 f \otimes \psi, s) L_p(\chi^{-1}\psi^{-1}\omega^{i-k+1}, k-s) \tag{4}$$

if $\psi(-1) = (-1)^{i+1}$, and

$$L_p^{[i]}(f \otimes f \otimes \psi, s) = L_p^{[i]}(\mathrm{Sym}^2 f \otimes \psi, s) L_p(\chi\psi\omega^{k-i}, s-k+1) \tag{5}$$

if $\psi(-1) = (-1)^i$.

Here $L_p(\chi, s)$ is the Kubota–Leopoldt p-adic L-function of the even character χ in the usual notation. In comparing (1) and (4), note that the classical values $L(\chi\psi, s-k+1)$ and $L(\chi^{-1}\psi^{-1}, k-s)$ are related by the functional equation for Dirichlet L-series.

Remark 1.4. In the case $\psi = \chi^{-1}$ and $i \equiv k \pmod{p-1}$, equation (5) was conjectured by Citro [Ci] (see Section 1.3 below).

Before outlining the proof of Theorem 1, we recall how $L_p(f \otimes f \otimes \psi)$ is defined and state our application to Greenberg's conjecture.

1.1 Hida families

An integer k and p-power conductor Dirichlet character α give rise to an arithmetic point of weight space $\mathcal{W}$ defined by $\nu_{k,\alpha}(z) = \alpha(z)z^k$. When $\alpha = 1$ we simply write ν_k.

Let F be a Hida family[2] of p-adic cusp forms with tame level N and character χ_F of conductor dividing N. Assume that the family F is N-new. For simplicity in this introduction, we assume that F is parameterized by a connected component

$$\mathcal{W}_{k_0} = \{\kappa \in \mathcal{W} : \kappa(\zeta) = \zeta^{k_0}, \zeta \in \mu_{p-1} \subset \mathbf{Z}_p^*\}, \qquad k_0 \in \mathbf{Z}/(p-1)\mathbf{Z}$$

of weight space.[3] The specialization $F_\kappa = F_{k,\alpha}$ at an arithmetic point $\kappa = \nu_{k,\alpha} \in \mathcal{W}_{k_0}$ with $k \geqslant 2$ is a classical cuspidal p-ordinary eigenform form of weight k and nebentype $\chi_F\alpha$. (Note that χ_F therefore has the same parity as k_0.)

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[2] Details about our definitions and conventions regarding Hida families will be given in §3.4. In particular, in this paper a Hida family corresponds to an irreducible component of Hida's Hecke algebra. This is the usual convention, though for some authors a Hida family corresponds to a connected component of Hida's Hecke algebra.

[3] In general, a Hida family is parameterized by a finite cover of such a connected component; see §3.4.

Given another Hida family G and an auxiliary Dirichlet character ψ, Hida defined a 3-variable p-adic L-function $L_p(F, G, \psi, \kappa, \lambda, \sigma)$ for $(\kappa, \lambda, \sigma) \in \mathcal{W}_{k_0} \times \mathcal{W}_{\ell_0} \times \mathcal{W}$. At arithmetic points $\kappa = \nu_{k,\alpha}$, $\lambda = \nu_{\ell,\beta}$, and $\sigma = \nu_{s,\gamma}$ such that $k-1 \geqslant s \geqslant \ell \geqslant 2$, the function $L_p(F, G)$ interpolates the critical values of the classical Rankin L-series $D_{Np^r}(F_{k,\alpha}, G_{\ell,\beta}, \psi\gamma^{-1}, s)$. Here p^r is the lcm of p and the conductors of α and β.

Now, suppose we are given two p-ordinary cuspidal eigenforms f and g as in the beginning of the introduction. We can find Hida families F and G that interpolate the forms f and g respectively, i.e. such that the specializations F_κ and G_λ at certain arithmetic weights κ, λ are the ordinary p-stabilizations of f and g, respectively (of course, if p already divides the level of f or g, then stabilization is not necessary). For $\sigma \in \mathcal{W}$, one then defines

$$L_p(f, g, \psi, \sigma) := L_p(F, G, \psi, \kappa, \lambda, \sigma). \tag{6}$$

Since $k, \ell \geqslant 2$, the families F, G interpolating f, g are unique by results of Hida, and therefore $L_p(f, g, \psi, \sigma)$ is well-defined.[4]

1.2 Two-variable factorization

Let $L_p(\mathrm{Sym}^2 F, \psi, \kappa, \sigma)$ denote Hida's 2-variable p-adic L-series interpolating the algebraic parts of the classical values $L(\mathrm{Sym}^2 F_\kappa, \psi\beta^{-1}, s)$ in the range $1 \leqslant s \leqslant k-1$ when $\sigma = \nu_{s,\beta}$ and F_κ has weight k. The actual formula we prove is the following.

Theorem 2. *Suppose that $\sigma(-1) = -\psi(-1)$. We have*

$$L_p(F, F, \psi, \kappa, \kappa, \sigma) = \mathcal{E}(\kappa, \sigma) \cdot L_p(\mathrm{Sym}^2 F, \psi, \kappa, \sigma) L_p(\chi_F \psi, z \cdot \sigma/\kappa), \tag{7}$$

where

$$\mathcal{E}(\kappa, \sigma) = \prod_{\ell | N} (1 - \chi_F \psi \kappa \sigma^{-1}(\ell)/\ell).$$

The Euler factor $\mathcal{E}(\kappa, \sigma)$ arises due to imprimitivity issues. The condition $\sigma(-1) = -\psi(-1)$ arises from the fact that Hida only defined the function $L_p(\mathrm{Sym}^2 F, \psi, \kappa, \sigma)$ on this half of weight space (in this case, the integers $1 \leqslant s \leqslant k-1$ are critical). With the proper generalization to the other half, where the integers $k \leqslant s \leqslant 2k-2$ are critical, (7) should continue to hold. These two halves of weight space correspond to the dichotomy between (4) and (5).

Let f be a p-ordinary newform and suppose that κ is an arithmetic weight with $F_\kappa = f$ (or the ordinary p-stabilization of f, if p does not divide the level of f). The functions in Theorem 1 are related to those in Theorem 2 by the formulae

$$L_p(\mathrm{Sym}^2 f \otimes \psi, \sigma) := (*) L_p(\mathrm{Sym}^2 F, \psi, \kappa, \sigma) \tag{8}$$

and

$$L_p(f \otimes f \otimes \psi, \sigma) := (*') L_p(F, F, \psi, \kappa, \kappa, \sigma), \tag{9}$$

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[4] It is interesting to consider, however, the uniqueness of this construction if we allow weight $k = 1$, where the Hida family F is not necessarily unique.

where $(*)$ and $(*')$ are certain Euler factors at the bad primes. On the half of weight space satisfying $\sigma(-1) = -\psi(-1)$, Theorem 1 follows from Theorem 2, (8), (9), and careful bookkeeping of Euler factors at the bad primes. Theorem 1 is then deduced for the other half of weight space when $p \nmid N$ by applying a functional equational for $L_p(f \otimes f \otimes \psi, \sigma)$. This functional equation is proven in Section 9 using [LLZ, Prop 5.4.4].

We next state an application of our main theorem.

1.3 Greenberg's exceptional zero conjecture for the adjoint at $s = 0, 1$

Let $f \in S_k(\Gamma_1(N), \chi)$ be a cuspidal newform of level N and weight $k \geqslant 2$. Let p be a prime not dividing N, and fix embeddings $\overline{\mathbf{Q}} \subset \mathbf{C}$ and $\overline{\mathbf{Q}} \subset \mathbf{C}_p$. Suppose that f is ordinary at p.

As we now explain, for each i mod $(p-1)$, there is a p-adic L-function $L_p^{[i]}(\mathrm{ad}\, f, s)$, $s \in \mathbf{Z}_p$, whose interpolation formula has the shape

$$L_p^{[i]}(\mathrm{ad}\, f, s) = (\text{Euler Factor}) \cdot L^{\mathrm{alg}}(\mathrm{ad}\, f \otimes \omega^{s-i}, s) \tag{10}$$

for $2-k \leqslant s \leqslant 0$, i even and $1 \leqslant s \leqslant k-1$, i odd. Here $L^{\mathrm{alg}}(\mathrm{ad}\, f, s)$ is the ratio between the classical value $L(\mathrm{ad}\, f, s)$ and an appropriate period. In view of the relationship

$$\mathrm{ad}\, f \cong \mathrm{Sym}^2 f \otimes \chi^{-1} \epsilon^{1-k},$$

these functions can be defined in terms of the p-adic L-functions considered above by

$$L_p^{[i]}(\mathrm{ad}\, f, s) = L_p^{[i+k-1]}(\mathrm{Sym}^2 f \otimes \chi^{-1}, s+k-1).$$

When $i = 0$ or 1, the Euler factor in (10) vanishes at $s = i$, and hence $L_p^{[i]}(\mathrm{ad}\, f, i)$ is said to have a "trivial" or "exceptional" zero at this point. Greenberg has stated a general conjecture concerning the values of derivatives of p-adic L-functions at exceptional zeroes. To state this conjecture in the current setting, we define the analytic $\mathscr{L}$-invariant of $\mathrm{ad}\, f$ by

$$\mathscr{L}_{\mathrm{an}}(\mathrm{ad}\, f, i) = (-1)^{1-i} \frac{L_p^{[i]\prime}(\mathrm{ad}\, f, i)}{S(f) \cdot L^{\mathrm{alg}}(\mathrm{ad}\, f, i)} \in \mathbf{C}_p, \qquad i = 0, 1, \tag{11}$$

where $S(f) \in \overline{\mathbf{Q}}$ is the nonzero part of the Euler factor in (10); see (35) below.

Greenberg has defined an algebraic counterpart to the $\mathscr{L}$-invariant above. This is an invariant $\mathscr{L}_{\mathrm{alg}}(\mathrm{ad}\, f)$ arising from a certain cohomology class in $H^1(G_{\mathbf{Q}}, \mathrm{ad}\, \rho_f)$, where ρ_f is the 2-dimensional p-adic Galois representation attached to f by Deligne. We do not present the precise definition, referring the reader instead to [Ha] or Greenberg's original work [Gre].

GREENBERG'S CONJECTURE. For $i = 0, 1$, we have

$$\mathscr{L}_{\mathrm{an}}(\mathrm{ad}\, f, i) = \mathscr{L}_{\mathrm{alg}}(\mathrm{ad}\, f). \tag{12}$$

Let F denote the Hida family whose weight k specialization is the ordinary p-stabilization of f. Inspired by the work of Greenberg and Stevens on the Mazur–Tate–Teitelbaum conjecture, one defines the Greenberg–Stevens $\mathscr{L}$-invariant of $\mathrm{ad}\, f$ by

$$\mathscr{L}_{\mathrm{GS}}(\mathrm{ad}\, f) = -\frac{2a_p'(k)}{a_p(k)} \in \mathbf{C}_p,$$

where a_p denotes the analytic function on $\mathcal{W}_k$ giving the U_p-eigenvalue of F.

THEOREM 3 (Hida [Hi4], Harron [Ha]). *We have*

$$\mathscr{L}_{\mathrm{alg}}(\mathrm{ad}\, f) = \mathscr{L}_{\mathrm{GS}}(\mathrm{ad}\, f).$$

The following is a corollary of Theorem 1.

THEOREM 4. *For $i = 0, 1$, we have*

$$\mathscr{L}_{\mathrm{an}}(\mathrm{ad}\, f, i) = \mathscr{L}_{\mathrm{GS}}(\mathrm{ad}\, f),$$

and hence Greenberg's Conjecture holds.

The fact that Theorem 4 follows from Theorem 1 was essentially proved by Citro [Ci], who considered the case $i = 1$. We briefly recall Citro's argument. Applying (4) with $\psi = \chi^{-1}$, $i = k$, and s replaced by $s + k - 1$, we obtain

$$L_p^{[k]}(f \otimes f \otimes \chi^{-1}, s + k - 1) = L_p^{[1]}(\mathrm{ad}\, f, s)\zeta_p(s).$$

Taking leading terms and evaluating at $s = 1$, the fact that the p-adic zeta function has a pole at $s = 1$ with residue $(1 - 1/p)$ yields

$$L_p^{[k]}(f \otimes f \otimes \chi^{-1}, k) = \left(1 - \frac{1}{p}\right) L_p^{[1]\prime}(\mathrm{ad}\, f, 1). \tag{13}$$

The evaluation of $L_p^{[k]}(f \otimes f \otimes \chi^{-1}, k)$ follows from earlier work of Hida [Hi3, Theorem 5.1d']. Suppose that F_k is the ordinary p-stabilization of f. Hida showed that after removal of the Euler factor $(1 - a_p(F_\kappa)/a_p(f))$ in the interpolation property for $L_p(F, F, \chi^{-1}, \kappa, k, \kappa)$, the resulting function of κ has a simple pole at $\kappa = \nu_k$ with residue

$$(1 - 1/p)(*) \cdot L^{\mathrm{alg}}(\mathrm{ad}\, f, 1),$$

where as usual (*) denotes a fudge factor arising from imprimitivity and from factors appearing in the interpolation formula. Note that the removed Euler factor $(1 - a_p(F_\kappa)/a_p(f))$ has a zero at $\kappa = \nu_k$ and its derivative at ν_k is $\frac{1}{2}\mathcal{L}_{\mathrm{GS}}(\mathrm{ad}\, f)$. Taking the limit as $\kappa \to \nu_k$ and combining these results, one finds that

$$L_p^{[k]}(f \otimes f \otimes \chi^{-1}, k) = \mathscr{L}_{\mathrm{GS}}(\mathrm{ad}\, f) \cdot S(f) \left(1 - \frac{1}{p}\right) L^{\mathrm{alg}}(\mathrm{ad}\, f, 1), \tag{14}$$

with $S(f)$ as in (11) and (35). Equations (13) and (14) yield Theorem 4 for $i = 1$. The result for $i = 0$ can be deduced from the case $i = 1$ by means of the functional equations proven in §9. Details of these arguments are provided in §10.

REMARK 1.5. Greenberg's conjecture was proven for arbitrary symmetric powers of p-ordinary CM forms f (when the corresponding L-functions have exceptional zeroes) by R. Harron in [Ha2].

REMARK 1.6. It is also possible for $L_p(\mathrm{Sym}^2 f, s)$ to have a trivial zero at $s = k$ when p divides the level of f. Greenberg's conjecture was proven in this case under certain hypotheses (even over arbitrary totally real fields F) by G. Rosso [Ros2], generalizing unpublished work of Greenberg and Tilouine.

We conclude the introduction by outlining the proof of Theorem 1.

1.4 Outline of proof of Factorization Theorem

Let F be a Hida family, with notation as in §1.1. Let us write χ for χ_F. Let α and β denote Dirichlet characters of p-power conductor such that β has the same parity as ψ and $\nu_{2,\alpha} \in \mathcal{W}_{k_0}$ (so in particular α has the same parity as χ). The set of points (κ, σ) of the form $\kappa = \nu_{2,\alpha}$ and $\sigma = \nu_{1,\beta}$ or $\sigma = \nu_{2,\beta}$ form a dense subset of $\mathcal{W}_{k_0} \times \mathcal{W}$ in the rigid topology, and hence it suffices to prove equality (7) under these specializations. (Note that this remains true if we remove finitely many α and β.) In this paper we consider σ of the first form, $\sigma = \nu_{1,\beta}$, which introduces the condition $\sigma(-1) = -\psi(-1)$ noted above. The formula we would like to prove is then

$$L_p(F, F, \psi, \nu_{2,\alpha}, \nu_{2,\alpha}, \nu_{1,\beta}) = \mathcal{E}(\kappa, \sigma) L_p(\operatorname{Sym}^2 F, \nu_{2,\alpha}, \psi, \nu_{1,\beta}) L_p((\chi\psi)^{-1}, \nu_{1,\alpha\beta^{-1}}). \tag{15}$$

The right side of (15) is easily computed. Let K be the real cyclotomic field cut out by the even character $\eta := \chi\psi\alpha\beta^{-1}$. Leopoldt proved that the value of $L_p((\chi\psi)^{-1}, \nu_{1,\alpha\beta^{-1}})$, which in the classical notation is $L_p(\eta^{-1}, 1)$, is equal (up to an explicit algebraic constant) to $\log_p(u_\eta)$. Here u_η is a *circular unit*

$$u_\eta \in U_\eta := (\mathcal{O}_K^* \otimes \overline{\mathbf{Q}})^{\eta^{-1}}, \tag{16}$$

and $\log_p$ denotes the p-adic logarithm extended by linearity to U_η. In (16), the superscript indicates that the element u_η lies in the η^{-1}-component for the action of Galois. The equivariant form of Dirichlet's unit theorem implies that U_η is a vector space of dimension 1 over $\overline{\mathbf{Q}}$.

Meanwhile $\sigma = \nu_{1,\beta}$ is a critical value for $L_p(\operatorname{Sym}^2 F, \nu_{2,\alpha}, \psi, \sigma)$. Hence the value of this function is equal, up to various interpolation factors, to the algebraic part of the classical L-value $L(\operatorname{Sym}^2 f, \beta^{-1}, 1)$, denoted

$$L^{\mathrm{alg}}(\operatorname{Sym}^2 f, \beta^{-1}, 1) = \frac{L(\operatorname{Sym}^2 f, \beta^{-1}, 1)}{\text{period}}.$$

Here $f = F_{2,\alpha}$, a classical cusp form of weight 2 and character $\chi\alpha$.

The difficulty in proving (15) is in evaluating the left side. In Section 7, we define an element $b_{f,\psi,\beta} \in U_\eta$ using intersection theory of algebraic cycles on the product of two modular curves. Our construction is inspired by the work of Beilinson [Bei] and Flach [Fl] but draws more directly from recent work of Lei, Loeffler, and Zerbes [LLZ] (which in turn was inspired by the work of Bertolini, Darmon, and Rotger [BDR1]); we call $b_{f,\psi,\beta}$ a *Beilinson-Flach unit.*

We prove that

$$L_p(f, f, \psi, \nu_{2,\beta}) = \log_p(b_{f,\psi,\beta}) \tag{17}$$

by combining a formula for $L_p(f, f, \psi, \nu_{2,\beta})$ proven in [KLZ] with a general compatibility result relating the p-adic regulator, cycle class map, and intersection pairing. By the linearity of $\log_p$, the proof of (15) is then reduced to proving that[5]

$$b_{f,\psi,\beta} \approx L^{\mathrm{alg}}(\operatorname{Sym}^2 f, \beta^{-1}, 1) \cdot u_\eta \tag{18}$$

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[5] Our notation here suffers from the usual dilemma when tensoring a multiplicative group with an additive group as in the definition of U_η. Instead of writing $b_{f,\xi} = C \cdot u_\eta$, a more enlightening notation would perhaps be $b_{f,\xi} = u_\eta^C$.

in the $\overline{\mathbf{Q}}$-vector space U_η. Here $\approx$ denotes equality up to multiplication by a specific nonzero algebraic factor that we suppress here for simplicity.

Equation (18) is proved by evaluating the leading terms at $s = 1$ of the factorization of classical L-series (2):

$$L'(f \otimes f \otimes \psi\beta^{-1}, 1) = L(\mathrm{Sym}^2 f \otimes \psi\beta^{-1}, 1)L'(\eta, 0). \tag{19}$$

Dirichlet's class number formula states that $L'(\eta, 0) = -\frac{1}{2}\log_\infty u_\eta$, where $\log_\infty$ is the $\overline{\mathbf{Q}}$-linear extension of the usual logarithm of the complex absolute value on $\mathcal{O}_K^*$. Meanwhile, we prove that

$$\frac{L'(f \otimes f \otimes \psi\beta^{-1}, 1)}{\text{period}} \approx \log_\infty b_{f,\psi,\beta} \tag{20}$$

by combining the Beilinson regulator formula of [LLZ] with a general compatibility result relating the archimedean regulator, cycle class map, and intersection pairing.

Equations (19) and (20) imply that

$$\log_\infty b_{f,\beta} \approx L^{\mathrm{alg}}(\mathrm{Sym}^2 f \otimes \beta^{-1}, 1) \cdot \log_\infty u_\eta. \tag{21}$$

Since U_η is a 1-dimensional $\overline{\mathbf{Q}}$-vector space on which $\log_\infty$ is injective, equation (21) implies (18), and completes the proof of Theorem 1.

Remark 1.7. The debt this article owes to the work of Bertolini, Darmon, and Rotger (in particular the articles [BDR1] and [BDR2]) is clear. We refer the reader to [BCDDPR, §2.4], where this article is placed in the larger context of the Bertolini–Darmon–Rotger program on Euler systems.

2. Classical L-series

We recall various classical L-series that play a role in this paper.

2.1 Dirichlet L-series

Let χ denote a primitive Dirichlet character. Its associated L-series

$$L(\chi, s) = \sum_{n=1}^{\infty} \chi(n)n^{-s} = \prod_p (1 - \chi(p)p^{-s})^{-1}, \qquad \mathrm{Re}(s) > 1,$$

can be analytically continued to the entire complex plane. The function $L(\chi, s)$ is holomorphic unless $\chi = 1$, in which case $L(1, s) = \zeta(s)$ has a simple pole at $s = 1$ with residue 1. We denote by $L_N(\chi, s)$ the L-series obtained from $L(\chi, s)$ by excluding the Euler factors at primes dividing N. (This should not cause confusion with the p-adic L-functions L_p to appear later.)

The critical values of $L(\chi, s)$ are the integers $s \leqslant 0$ with $\chi(-1) = (-1)^{s+1}$ and integers $s > 0$ with $\chi(-1) = (-1)^s$. The values of $L(\chi, s)$ for critical $s \leqslant 0$ are algebraic, and in fact live in the field $\mathbf{Q}(\chi)$ generated by the values of the character χ.

2.2 Symmetric square L-series

Let $f = \sum_{n=1}^{\infty} a_n e^{2\pi i n z} \in S_k(\Gamma_1(N), \chi)$ denote a normalized eigenform (so $a_1 = 1$, and f is an eigenform for the operators $T_\ell, \ell \nmid N$ and $U_\ell, \ell \mid N$). For any prime $\ell \nmid N$, let $\rho_{f,\ell} : G_{\mathbf{Q}} \to \mathrm{Aut}(V_\ell)$ denote the 2-dimensional ℓ-adic representation associated to f by Deligne. The symmetric square $\mathrm{Sym}^2 \rho_{f,\ell}$ is a 3-dimensional representation of $G_{\mathbf{Q}}$. Let $\psi : G_{\mathbf{Q}} \longrightarrow \overline{\mathbf{Q}}^{\times}$ be a primitive Dirichlet character of conductor N_ψ. For any prime $q \neq \ell$, we let

$$Z_q(\mathrm{Sym}^2 f \otimes \psi, X) := \det(1 - \mathrm{Frob}_q X | ((\mathrm{Sym}^2 \rho_{f,\ell}) \otimes \psi)_{I_q}),$$

where $I_q \subset G_{\mathbf{Q}}$ is an inertia group at q. The polynomial $Z_q(X)$ has coefficients in $\mathbf{Q}(\chi)$ and is independent of ℓ or the choice of I_q. We define the primitive L-series

$$L(\mathrm{Sym}^2 f \otimes \psi, s) = \prod_q Z_q(\mathrm{Sym}^2 f \otimes \psi, q^{-s})^{-1}. \tag{22}$$

If $q \nmid N$, the Euler factor at q in (22) is

$$[(1 - \alpha_q^2 \psi(q) q^{-s})(1 - \chi\psi(q) q^{k-1-s})(1 - \beta_q^2 \psi(q) q^{-s})]^{-1},$$

where α_q and β_q are the roots of the Hecke polynomial $x^2 - a_q x + \chi(q) q^{k-1}$ of f at q.

We also consider the imprimitive L-series defined by

$$L(\mathrm{Sym}^2 f, \psi, s) := L_{NN_\psi}(2s - 2k + 2, \chi^2\psi^2) \sum_{n=1}^{\infty} \psi(n) a_{n^2} n^{-s}.$$

If we extend the definition of (α_q, β_q) by setting

$$(\alpha_q, \beta_q) = (a_q, 0) \qquad \text{when } q \mid N, \tag{23}$$

then we have

$$L(\mathrm{Sym}^2 f, \psi, s) = \prod_q [(1 - \alpha_q^2 \psi(q) q^{-s})(1 - \alpha_q \beta_q \psi(q) q^{k-1-s})(1 - \beta_q^2 \psi(q) q^{-s})]^{-1}.$$

For primes $q \nmid N$, the Euler factors of $L(\mathrm{Sym}^2 f, \psi, s)$ and $L(\mathrm{Sym}^2 f \otimes \psi, s)$ agree. For $q \mid N$, the Euler factors of $L(\mathrm{Sym}^2 f, \psi, s)$ divide those of $L(\mathrm{Sym}^2 f \otimes \psi, s)$. In other words, we may write

$$L(\mathrm{Sym}^2 f, \psi, s) = L(\mathrm{Sym}^2 f \otimes \psi, s) \cdot P(\mathrm{Sym}^2 f, \psi, s),$$

where

$$P(\mathrm{Sym}^2 f, \psi, s) = \prod_{q \mid N} P_q(\mathrm{Sym}^2 f, \psi, q^{-s}) \tag{24}$$

and $P_q(\mathrm{Sym}^2 f, \psi, x)$ is a polynomial of degree at most 3 in x. For details on the exact evaluation of these polynomials, see [Sc].

2.3 Rankin L-series

Now consider two normalized eigenforms:

$$f = \sum_{n=1}^{\infty} a_n e^{2\pi i n z} \in S_k(\Gamma_1(N_f), \chi_f), \qquad g = \sum_{n=1}^{\infty} b_n e^{2\pi i n z} \in S_\ell(\Gamma_1(N_g), \chi_g).$$

Let ψ be a primitive Dirichlet character of conductor N_ψ. Let $\ell \nmid N := \operatorname{lcm}(N_f, N_g, N_\psi)$; for any prime $q \neq \ell$, put

$$Z_q(f \otimes g \otimes \psi, X) := \det(1 - \operatorname{Frob}_q X | (\rho_{f,\ell} \otimes \rho_{g,\ell} \otimes \psi)_{I_q}).$$

The polynomial $Z_q(f \otimes g \otimes \psi, X)$ is independent of ℓ, and we define the primitive L-series

$$L(f \otimes g \otimes \psi, s) := \prod_q Z_q(f \otimes g \otimes \psi, q^{-s})^{-1}.$$

Meanwhile, the (imprimitive) Rankin L-series of f and g twisted by ψ is defined by

$$D_N(f, g, \psi, s) = L_N(\chi_f \chi_g \psi^2, 2s + 2 - k - \ell) \sum_{n=1}^{\infty} a_n b_n \psi(n) n^{-s},$$

which under the convention (23) has the Euler product

$$\prod_q [(1 - \alpha_q(f)\alpha_q(g)\psi(q)q^{-s})(1 - \alpha_q(f)\beta_q(g)\psi(q)q^{-s})$$
$$(1 - \beta_q(f)\alpha_q(g)\psi(q)q^{-s})(1 - \beta_q(f)\beta_q(g)\psi(q)q^{-s})]^{-1}.$$

For $q \nmid N_f N_g$, the Euler factors of $L(f \otimes g \otimes \psi, s)$ and $D_N(f, g, \psi, s)$ agree, and we may write

$$D_N(f, g, \psi, s) = L(f \otimes g \otimes \psi, s) P(f, g, \psi, s)$$

where

$$P(f, g, \psi, s) = \prod_{q | N_f N_g} P_q(f, g, \psi, q^{-s}) \tag{25}$$

and $P_q(f, g, \psi, x)$ is a polynomial of degree at most 4 in x.

When $f = g$, we have $L(f \otimes g \otimes \psi, s) = L(\operatorname{Sym}^2 f \otimes \psi, s) L(\chi\psi, s - k + 1)$. Furthermore, the Euler factors of the imprimitive $L(\operatorname{Sym}^2 f, \psi, s)$ and $D_N(f, f, \psi, s)$ agree for all $q \mid N$. The error terms defined in (24) and (25) are therefore related by

$$P(f, f, \psi, s) = P(\operatorname{Sym}^2 f, \psi, s) \prod_{q|N} (1 - \chi\psi(q) q^{k-1-s}),$$

with the understanding that $\chi\psi(q)$ denotes the primitive character associated to $\chi\psi$ evaluated at q (hence we may have $\chi\psi(q) \neq 0$ even if $\chi(q) = 0$ or $\psi(q) = 0$).

3. p-adic L-series

For clarity we present p-adic L-functions both as functions on weight space $\mathcal{W}$ and more classically as functions of a variable $s \in \mathbf{Z}_p$.

3.1 Kubota-Leopoldt p-adic L-series

Let p be a prime number, and fix once and for all embeddings $\overline{\mathbf{Q}} \hookrightarrow \mathbf{C}$ and $\overline{\mathbf{Q}} \hookrightarrow \mathbf{C}_p$. Suppose that the Dirichlet character χ is even. Kubota and Leopoldt proved the existence of a unique p-adic meromorphic function

$$L_p(\chi, s) \colon \mathbf{Z}_p \longrightarrow \mathbf{Q}_p(\chi)$$

such that

$$L_p(\chi, n) = L(\chi\omega^{n-1}, n)\mathcal{E}(\chi\omega^{n-1}, n) \qquad \text{for integer } n \leqslant 0, \tag{26}$$

where $\omega : (\mathbf{Z}/p\mathbf{Z})^* \longrightarrow \mu_{p-1}$ is the Teichmuller character (or $\omega : (\mathbf{Z}/4\mathbf{Z})^* \longrightarrow \{\pm 1\}$ if $p = 2$), and $\mathcal{E}(\chi\omega^{n-1}, n) = 1 - \chi\omega^{n-1}(p)p^{-n}$ is the Euler factor at p of the given L-function. The p-adic L-series $L_p(\chi, s)$ is analytic unless $\chi = 1$, in which case $L_p(1, s) = \zeta_p(s)$ has a simple pole at $s = 1$ with residue $(1 - p^{-1})$.

Let us now describe how the Kubotda–Leopoldt p-adic L-series may be viewed as a function on weight space. We suppose that χ has conductor prime to p. (This is no restriction, since the p-part of the character may be absorbed into the weight.) We no longer insist that χ is even and define $a_\chi \in \{0, 1\}$ by $(-1)^{a_\chi} = \chi(-1)$. There is a unique meromorphic function $L_p(\chi, \sigma)\colon \mathcal{W} \longrightarrow \mathbf{C}_p$ such that for arithmetic weights $\sigma = \nu_{s,\alpha}$, we have

$$L_p(\chi, \sigma) = L(\chi\alpha^{-1}, s)\mathcal{E}(\chi\alpha^{-1}, s) \tag{27}$$

if $\chi\alpha(-1) = (-1)^{s+1}$ and $s \leqslant 0$ and

$$L_p(\chi, \sigma) = L(\chi\alpha^{-1}, s) \cdot \frac{2\Gamma(s)i^{a_\chi}}{(2\pi i)^s} \cdot \frac{\chi(N_\alpha)\tau(\alpha^{-1})}{N_\alpha^s}\mathcal{E}(\chi^{-1}\alpha, 1 - s) \tag{28}$$

if $\chi\alpha(-1) = (-1)^s$ and $s \geqslant 1$. Here $\tau(\alpha)$ is the Gauss sum

$$\tau(\alpha) = \sum_{n=1}^{N_\alpha} \alpha(n) \exp(2\pi i n/N_\alpha). \tag{29}$$

The function L_p is analytic unless $\chi = 1$, in which case $L_p(1, \sigma) = \zeta_p(\sigma)$ has a simple poles at $\sigma(z) = z$ and $\sigma(z) = 1$. Comparing (26) and (27), we see that for any p-power conductor Dirichlet character α such that $\chi\alpha$ is odd, we have

$$L_p(\chi, \alpha\langle z\rangle^s) = L_p(\chi\alpha^{-1}\omega, s), \qquad s \in \mathbf{Z}_p. \tag{30}$$

Remark 3.1. For most of the paper, we will be concerned with the half of weight space on which $\chi(-1) = -\sigma(-1)$, where the formulae (27) and (30) hold. As explained in the next section, the constants in (28) have been chosen so that there is a functional equation relating $L_p(\chi, \sigma)$ and $L_p(\chi^{-1}, \nu_1/\sigma)$ mirroring the classical functional equation relating $L(\chi, s)$ and $L(\chi^{-1}, 1 - s)$. Note that our normalization differs slightly from that of Colmez in [Col]; the functional equation of our $L_p(\chi, \sigma)$ involves the same epsilon factor as the classical $L(\chi, s)$.

3.2 Functional Equation

Recall the standard notation

$$\Gamma_{\mathbf{R}}(s) = \pi^{-s/2}\Gamma(s/2), \qquad \Gamma_{\mathbf{C}}(s) = 2(2\pi)^{-s}\Gamma(s).$$

Let χ be a Dirichlet character of conductor N_χ. Let $a_\chi \in \{0, 1\}$ such that $\chi(-1) = (-1)^{a_\chi}$. Define

$$\Lambda(\chi, s) = \Gamma_{\mathbf{R}}(s + a_\chi)L(\chi, s).$$

The following functional equational is well-known.

Theorem 3.2. *Let*

$$\epsilon(\chi, s) = \frac{i^{a_\chi}}{\tau(\chi^{-1})} N_\chi^s.$$

We have

$$\Lambda(\chi, 1-s) = \epsilon(\chi, s)\Lambda(\chi^{-1}, s).$$

Suppose that $p \nmid N_\chi$. Denote by

$$\widetilde{\epsilon}(\chi, \sigma) = \frac{i^{a_\chi}}{\tau(\chi^{-1})}\sigma(N_\chi)$$

the unique analytic function on $\mathcal{W}$ such that $\widetilde{\epsilon}(\chi, \nu_s) = \epsilon(\chi, s)$ for $s \in \mathbf{Z}$.

Theorem 3.3. *We have*

$$L_p(\chi, \nu_1/\sigma) = \widetilde{\epsilon}(\chi, \sigma) L_p(\chi^{-1}, \sigma).$$

Proof. To prove the result on the half of weight space such that $\sigma(-1) = -\chi(-1)$, it suffices to consider the dense set of points of the form $\sigma = \nu_s$ where $\chi(-1) = (-1)^{s+1}$ and $s \leqslant 0$. By (27) and (28) we see that the right hand side equals $\epsilon(\chi, s)L(\chi^{-1}, s)\mathcal{E}(\chi^{-1}, s)$ and the left hand side equals $L(\chi, 1-s)\mathcal{E}(\chi^{-1}, s) \cdot 2(-s)! i^{a_\chi}/(2\pi i)^{1-s}$. These are equal by Theorem 3.2 and standard formulae for the Gamma function.

To deduce the result on the half of weight space such that $\sigma(-1) = -\chi(-1)$, one enacts the change of variables $\sigma \mapsto \nu_1/\sigma$, noting that $\widetilde{\epsilon}(\chi, \sigma)\widetilde{\epsilon}(\chi^{-1}, \nu_1/\sigma) = 1$. □

3.3 Schmidt's p-adic Symmetric Square L-series

Theorem 3.4 (Schmidt, [Sc], Theorem 5.5). *Let f be a p-ordinary cuspidal newform with weight $k \geqslant 2$, character χ, and level N_f. Let ψ be a Dirichlet character of conductor N_ψ. Suppose that $p \nmid N_f N_\psi$. There exists a unique meromorphic function $L_p(\mathrm{Sym}^2 f \otimes \psi, \sigma)\colon \mathcal{W} \to \mathbf{C}_p$ such that for all but finitely many characters β of conductor p^r, $r > 0$, we have*

$$L_p(\mathrm{Sym}^2 f \otimes \psi, \nu_{\beta,s}) = \frac{(-1)^{s-k+1}\beta(-1)\Gamma(s)}{i^{a_{\chi\psi}}2^{2k}}(\alpha_p^{-2}\psi^{-1}(p)p^{s-1})^r \frac{\tau(\beta)}{(2\pi i)^{s-k+1}} \frac{L(\mathrm{Sym}^2 f \otimes \psi\beta^{-1}, s)}{\pi^{k-1}\langle f, f\rangle} \tag{31}$$

if $1 \leqslant s \leqslant k-1$ and $\psi\beta(-1) = (-1)^{s+1}$, and

$$L_p(\mathrm{Sym}^2 f \otimes \psi, \nu_{\beta,s}) = \frac{\Gamma(s-k+1)\Gamma(s)}{2^{2s+1}}(\alpha_p^{-2}\chi^{-1}\psi^{-2}(p)p^{s-1})^r \beta(N_{\psi\chi})^2\tau(\beta)^2 \frac{L(\mathrm{Sym}^2 f \otimes \psi\beta^{-1}, s)}{\pi^{2s-k+1}\langle f, f\rangle} \tag{32}$$

if $k \leqslant s \leqslant 2k-2$ and $\psi\beta(-1) = (-1)^s$.

Proof. In the notation of [Sc, Theorem 5.5], our function $L_p(\mathrm{Sym}^2 f \otimes \psi, \sigma)$ is given by

$$L_p(\mathrm{Sym}^2 f \otimes \psi, \sigma) = \frac{\mathfrak{C}_{\psi^{-1}\chi^{-1}}(\sigma/\nu_{k-1}) \cdot N_{\psi\chi}^{k-1}}{i^{a_{\chi\psi}}2^{2k}\tau(\psi^{-1}\chi^{-1})\sigma(-N_{\psi\chi})} \tag{33}$$

if $\sigma(-1) = -\psi(-1)$ and

$$L_p(\mathrm{Sym}^2 f \otimes \psi, \sigma) = \frac{\mathfrak{C}_{\psi^{-1}\chi^{-1}}(\sigma/\nu_{k-1}) \cdot N_{\psi\chi}^{2k-1}}{\tau(\psi^{-1}\chi^{-1})^2 2^{2k}\sigma(N_{\psi\chi}^2)(-1)^{k-1}} \tag{34}$$

if $\sigma(-1) = \psi(-1)$. The interpolation properties (31) and (32) then follow directly from [Sc, Theorem 5.5(ii)], keeping in mind the sign error in the definition of $Q_{m,\lambda}$ for $1 \leqslant m \leqslant k-1$ pointed out in [Hi2, pg. 134] (and the corresponding correction that must be made for $k \leqslant m \leqslant 2k-1$, multiplying $Q_{m,\lambda}$ by $(-1)^{m-k}$ for m in that range). ☐

As we discuss in §3.5, Schmidt's result was generalized by Hida to allow f to vary in a p-ordinary family. By specializing to general forms in the family, one obtains a definition of $L_p(\mathrm{Sym}^2 f \otimes \psi, \sigma)$ even in the case $p \mid N_f$. For most of the paper, where we work on the half of weight space satisfying $\sigma(-1) = -\psi(-1)$, we use Hida's more general construction. However in Section 9 we will appeal to Schmidt's functional equation for $L_p(\mathrm{Sym}^2 f \otimes \psi, \sigma)$ in order to deduce our factorization theorem on the other half of weight space when $p \nmid N_f$.

Remark 3.5. The reason that we have scaled Schmidt's p-adic L-function by the factors in (33) and (34) is that our function is more closely aligned with the standard classical completed L-function $\Lambda(\mathrm{Sym}^2 f \otimes \psi, s)$. For example, the epsilon factor appearing in the functional equation of our $L_p(\mathrm{Sym}^2 f \otimes \psi, \sigma)$ is exactly the p-adic function that interpolates the classical epsilon factor for integer s (see §9). Retaining this convention for all our p-adic L-functions is quite natural and will simplify later proofs.

3.4 Hida Families

Let $p \geqslant 5$ be prime, and let N be a positive integer such that $p \nmid N$. Let $\mathcal{O}$ denote the ring of integers in a finite extension of $\mathbf{Q}_p$, and let

$$\Lambda = \mathcal{O}[[(1 + p\mathbf{Z}_p)^*]] \cong \mathcal{O}[[T]].$$

Let $\mathrm{Spec}\,\widetilde{R}$ denote an irreducible component of Hida's ordinary Hecke algebra of tame level N defined over $\mathcal{O}$, and let R denote the integral closure of $\widetilde{R}$ in its quotient field. Thus R is a finite flat extension of Λ such that $R \cap \overline{\mathbf{Q}}_p = \mathcal{O}$.

The *Hida family* associated to R is the formal q-expansion $F = \sum_{n=1}^{\infty} a_n q^n$ where $a_n \in R$ is the image in R of the Hecke operator T_n; in particular $a_1 = 1$. There exists an even Dirichlet character ψ_F with modulus Np such that the Hida family F satisfies the following interpolation property. Let $\kappa \in \mathrm{Hom}_{\mathcal{O}-\mathrm{alg}}(R, \mathbf{C}_p)$ be such that the restriction to group-like elements $[x] \in \Lambda$ for $x \in (1 + p\mathbf{Z}_p)^*$ has the form $x \mapsto \alpha(x)x^k$, where $k \geqslant 2$ is an integer and α has p-power conductor and order p^{r-1}. Then the values $\kappa(a_n)$ lie in $\overline{\mathbf{Q}}$ and

$$F_\kappa := \sum_{n=1}^{\infty} \kappa(a_n) q^n \in \mathbf{C}[[q]]$$

is the q-expansion of a p-ordinary eigenform in $S_k(Np^r, \psi_F \alpha \omega^{-k})$.

Let $\psi_F = \chi_F \omega^{k_0}$ denote the factorization of ψ_F into characters of modulus N and p, where k_0 is an integer determined modulo $(p-1)$. An $\mathcal{O}$-algebra homomorphism κ as above is called an *arithmetic point* of R. The associated element $\nu(\kappa) = \nu_{k, \alpha\omega^{k_0 - k}} \in \mathcal{W}$ is called the *weight* of κ and lies in the connected component $\mathcal{W}_{k_0} \subset \mathcal{W}$ containing the integer k_0. More generally, any $\kappa \in \mathrm{Hom}_{\mathcal{O}-\mathrm{alg}}(R, \mathbf{C}_p)$ defines an element of $\mathcal{W}_{k_0}$, denoted $\nu(\kappa)$, via $x \mapsto \omega^{k_0}(x)\kappa(\langle x \rangle)$ for $x \in \mathbf{Z}_p^*$.

In this way, $\mathrm{Hom}_{\mathcal{O}-\mathrm{alg}}(R, \mathbf{C}_p)$ defines (the set of $\mathbf{C}_p$-points of) a rigid analytic space $\mathcal{W}_F$ that is a finite cover of $\mathcal{W}_{k_0}$. In the introduction, we assumed that $\mathcal{W}_F = \mathcal{W}_{k_0}$ for simplicity.

3.5 Hida's p-adic Symmetric Square L-series

Let F be a Hida family and ψ an auxiliary Dirichlet character of conductor N_ψ ($p \nmid N_\psi$). In [Hi2], Hida defined a p-adic L-function $L_p(\mathrm{Sym}^2 F, \psi, \kappa, \sigma)$ for $\kappa \in \mathcal{W}_F$ and $\sigma \in \mathcal{W}$ interpolating the classical critical L-values $L(\mathrm{Sym}^2 F_{k,\alpha}, \psi\beta^{-1}, s)$ when $\nu(\kappa) = \nu_{k,\alpha}, \sigma = \nu_{s,\beta}$ such that $1 \leqslant s \leqslant k-1$. These values are critical when $\sigma(-1) = -\psi(-1)$.

In this paper, we scale Hida's function in [Hi2] by

$$\frac{-\psi(-1)N_\psi(W'(F_\kappa)N^{k/2})}{4i^{a_{\chi_F \psi}}\sigma(N_\psi)\tau(\psi^{-1})}$$

to align more closely with our normalization. Here $W'(f)$ denotes the prime-to-p part of the root number of f (see [Hi3, pp.38–39]); the function $W'(F_\kappa)N^{k/2}$ extends to an analytic function on $\mathcal{W}_F$.

We state the precise interpolation formula when $\mathrm{cond}(\beta) = p^w$ with $w > 1$. Let $\mathrm{cond}(\alpha) = p^r$. Let $\xi = \psi\beta^{-1}$. Let $f = F_\kappa$. If f is a not a newform, we write $f^\#$ for the associated newform (so f is the ordinary p-stabilization of $f^\#$); if f is a newform, we write $f^\# = f$. Define $S(f) = 1$ if $\alpha \neq 1$, $S(f) = -1$ if $\alpha = 1$ and $f^\# = f$ (this can only happen if $k = 2$), and

$$S(f) := (1 - \chi_F(p)\alpha_p(f)^{-2}p^{k-1})(1 - \chi_F(p)\alpha_p(f)^{-2}p^{k-2}) \tag{35}$$

if $\alpha = 1$ and $f \neq f^\#$. We then have:

Theorem 3.6 (Hida, [Hi2], Theorem 5.1d). *There is a unique p-adic meromorphic function $L_p(\mathrm{Sym}^2 F, \psi, \kappa, \sigma)$ on $\mathcal{W}_F \times \mathcal{W}$ such that for (κ, σ) satisfying the conditions above, we have:*

$$L_p(\mathrm{Sym}^2 F, \psi, \kappa, \sigma) = \frac{-\psi(-1)}{4i^{a_{\chi_F\psi}}} \cdot \frac{\Gamma(s)p^{w(s-1)}a_p(f)^{2r-2w}\tau(\beta)\psi(p)^{-w}}{\chi_F(p)^r\tau(\alpha)S(f)} \cdot \frac{L(\mathrm{Sym}^2 f^\#, \xi, s)}{(2i)^{s+k-1}\pi^s\langle f^\#, f^\#\rangle} \in \overline{\mathbf{Q}}. \tag{36}$$

We will essentially define the p-adic L-function of f to be the specialization of Hida's 2-variable function at κ, with two adjustments: (1) we would like to interpolate the *primitive* values $L(\mathrm{Sym}^2 f)$, so an adjustment must be made at primes dividing N_f; (2) we scale the function according to our prior conventions, so that in particular in the case where $p \nmid N_f$, we recover the function defined in Theorem 3.4.

To this end, we define the Euler factor

$$P(\mathrm{Sym}^2 f, \psi, \sigma) = \prod_{q \mid N_f} P_q(\mathrm{Sym}^2 f, \psi, \sigma(q)^{-1})$$

with notation as in (24). For $\sigma \in \mathcal{W}$ such that $\sigma(-1) = -\psi(-1)$, we define (for a newform f such that $F_\kappa = f$ or $F_\kappa =$ ordinary p-stabilization of f)

$$L_p(\mathrm{Sym}^2 f \otimes \psi, \sigma) = S(f) \cdot \frac{L_p(\mathrm{Sym}^2 F, \psi, \kappa, \sigma)}{P(\mathrm{Sym}^2 f, \psi, \sigma)}. \tag{37}$$

It is a straightforward calculation to verify that this definition of $L_p(\mathrm{Sym}^2 f \otimes \psi, \sigma)$ agrees with that of Schmidt from Theorem 3.4 when $p \nmid N_f$ (in which case we have in particular $\alpha = 1$).

3.6 p-adic Rankin L-series

Let F and G be Hida families of tame level N_F, N_G, respectively. Let $N = \mathrm{lcm}(N_F, N_G)$. Hida has defined a 3-variable p-adic L-function $L_p(F, G, \kappa, \lambda, \sigma)$ interpolating the algebraic parts of the critical values $D_{Np^r}(F_\kappa, G_\lambda, \sigma)$, for appropriately chosen r. Let F_κ and G_λ be specializations at arithmetic points of weight $k > \ell \geqslant 2$, respectively, such that F_κ and G_λ have trivial character at p. Then F_κ and G_λ are the ordinary p-stabilizations of forms f and g of level N_F and N_G, respectively (with the possible exception of g if $\ell = 2$; we exclude this case in what follows). Then for $\ell \leqslant s \leqslant k-1$, Hida's function satisfies the following interpolation formula (see [KLZ, Theorem 2.6.2]).

Theorem 3.7 (Hida, [Hi3]). *There exists a unique p-adic meromorphic function $L_p(F, G, \kappa, \lambda, \nu)$ on $\mathcal{W}_F \times \mathcal{W}_G \times \mathcal{W}$ such that for (κ, λ, ν_s) satisfying the conditions above, we have:*

$$L_p(F, G, \kappa, \lambda, \nu_s) = \frac{\mathcal{E}(f,g,s)}{S(f)} \cdot \frac{\Gamma(s)\Gamma(s-\ell+1)}{\pi^{2s-\ell+1}(-i)^{k-\ell}2^{2s+k-\ell}\langle f, f\rangle_N} D_N(f,g,s) \tag{38}$$

where

$$\mathcal{E}(f,g,s) = \left(1 - \frac{p^{s-1}}{\alpha_f \alpha_g}\right)\left(1 - \frac{p^{s-1}}{\alpha_f \beta_g}\right)\left(1 - \frac{\beta_f \alpha_g}{p^s}\right)\left(1 - \frac{\beta_f \beta_g}{p^s}\right).$$

Here $\alpha_f = a_p(F_\kappa)$ denotes the p-adic unit root of the Hecke polynomial of f at p, and $\beta_f = \chi_F(p)p^{k-1}/\alpha_f$ denotes the other root; similarly for α_g, β_g. Equation (38) specifies L_p on a dense collection of points in $\mathcal{W}_F \times \mathcal{W}_G \times \mathcal{W}$.

Now let ψ be an auxiliary Dirichlet character of conductor N_ψ, with $p \nmid N_\psi$. $f = F_\kappa$ and $g = G_\lambda$ be specializations of F, G at arithmetic points (not necessarily satisfying the condition above of having trivial character at p). Denote by G_ψ the twist of the Hida family G by the character ψ. We define

$$L_p(f \otimes g \otimes \psi, \sigma) := S(f) \cdot \frac{L_p(F, G_\psi, \kappa, \lambda, \sigma)}{P(f, g, \psi, \sigma)} \tag{39}$$

with

$$P(f, g, \psi, \sigma) := \prod_{q \mid N_F N_G} P_q(f, g, \psi, \sigma(q)^{-1}),$$

where P_q is as in (25) and $S(f)$ as in (35). Note that unlike the setting of the symmetric square, in the case $k \leqslant \ell$ the function $L_p(f \otimes g \otimes \psi, \sigma)$ is not characterized by any interpolation property in the variable σ, as there are no critical values; it may only be defined by specializing the 3-variable function $L_p(F, G)$. Note also the asymmetry between F and G in the definition of $L_p(F, G)$, implying that the functions $L_p(f \otimes g \otimes \psi)$ and $L_p(g \otimes f \otimes \psi)$ may not be equal.

4. Circular units

We recall special value formulae for classical and p-adic Dirichlet L-series in terms of circular units.

4.1 Definition of circular units

Let χ be a nontrivial primitive even Dirichlet character with conductor N. Let $K = \mathbf{Q}(\mu_N)$, and define

$$U_\chi := \{x \in \mathcal{O}_K^* \otimes_{\mathbf{Q}} \overline{\mathbf{Q}} : \sigma(x) = \chi^{-1}(\sigma) \cdot x \text{ for } \sigma \in \mathrm{Gal}(K/\mathbf{Q})\}.$$

This is a $\overline{\mathbf{Q}}$-vector space of dimension 1.

Fix the primitive Nth root of unity $\zeta \in \mu_N$ given by $\zeta = e^{2\pi i/N}$. (Recall that we have chosen an embedding $\overline{\mathbf{Q}} \subset \mathbf{C}$, so we may view ζ as a well-defined element of K.) Define the circular unit

$$u_\chi := \prod_{1 \leqslant a \leqslant N, (a,N)=1} \left(\frac{1 - \zeta^a}{1 - \zeta} \right) \otimes \chi(a) \in U_\chi.$$

4.2 Dirichlet's formula

Consider the group homomorphism $\log_\infty \colon \mathcal{O}_K^* \longrightarrow \mathbf{R}$ obtained by composing the embedding $\mathcal{O}_K^* \subset \mathbf{C}$ with the map $x \mapsto \log|x|$, where $|x|$ denotes the usual complex absolute value. The map $\log_\infty$ can be extended by linearity to a $\overline{\mathbf{Q}}$-linear map

$$\log_\infty : \mathcal{O}_K^* \otimes \overline{\mathbf{Q}} \longrightarrow \mathbf{C}.$$

The following is a special case of Dirichlet's celebrated class number formula:

$$L'(\chi, 0) = -\frac{1}{2} \log_\infty(u_\chi), \qquad L(\chi^{-1}, 1) = -\frac{\tau(\chi^{-1})}{N} \log_\infty(u_\chi). \tag{40}$$

Here $\tau(\chi)$ is the Gauss sum defined in (29).

4.3 Leopoldt's formula

Let $\mathcal{O}_p \subset \overline{\mathbf{Q}}_p$ denote the ring of p-adic integers, and let $\log_p \colon \mathcal{O}_p^* \longrightarrow \mathcal{O}_p$ denote the p-adic logarithm. The group homomorphism $\log_p$ can be extended by linearity to a $\overline{\mathbf{Q}}$-linear map

$$\log_p : \mathcal{O}_K^* \otimes \overline{\mathbf{Q}} \longrightarrow \overline{\mathbf{Q}}_p.$$

Leopoldt proved the following p-adic analogue of Dirichlet's class number formula:

$$L_p(\chi^{-1}, 1) = -\frac{\tau(\chi^{-1})}{N} \left(1 - \frac{\chi^{-1}(p)}{p}\right) \log_p(u_\chi). \tag{41}$$

In particular, if the conductor N of χ is divisible by p, we obtain

$$L_p(\chi^{-1}, 1) = -\frac{\tau(\chi^{-1})}{N} \log_p(u_\chi). \tag{42}$$

5. Chow groups

The Beilinson–Flach units that we will construct in Section 7 and relate to classical and p-adic Rankin L-series will be defined using Bloch's intersection theory on higher Chow groups. In this section we recall the basics of this theory that we will require, and prove compatibility results relating an intersection pairing and regulator maps. Our discussion of Chow groups is by no means complete, and we refer the reader to [Bl1] and [Le] for more details.

5.1 Definition of Chow groups

Let S be a smooth projective variety of dimension d over a field K. We recall the definitions of $\mathrm{CH}^r(S)$ and $\mathrm{CH}^r(S,1)$. Let $z^r(S)$ denote the free abelian group on the set of irreducible varieties $Z \subset S$ of codimension r. Let $z^r_{\mathrm{rat}}(S) \subset z^r(S)$ denote the subgroup of cycles rationally equivalent to zero (i.e. for which there exists a subvariety $Y \subset S$ and a rational function on Y whose divisor is the given cycle). We have $\mathrm{CH}^r(S) := z^r(S)/z^r_{\mathrm{rat}}(S)$.

For a nonnegative integer r, let S^r denote the set of points of S of codimension r. The higher Chow group $\mathrm{CH}^{r+1}(S,1)$ is isomorphic to the cohomology of the Gersten complex

$$\bigoplus_{x\in S^{r-1}} K_2(k(x)) \longrightarrow \bigoplus_{x\in S^r} k(x)^* \longrightarrow \bigoplus_{x\in S^{r+1}} \mathbf{Z}. \tag{43}$$

(see e.g. [La, Theorem 2.5]).

Here the second map simply sends a rational function to its divisor. The first map sends a symbol $\{f,g\} \in K_2(k(x))$ associated to pair of functions $f, g \in k(x)^*$ with $x \in S^{r-1}$ to the tame symbol

$$T(\{f,g\}) = (u_Z)_{Z\in S^r}, \qquad u_Z = (-1)^{\nu_Z(f)\nu_Z(g)} \frac{f^{\nu_Z(g)}}{g^{\nu_Z(f)}}. \tag{44}$$

5.2 An intersection pairing

There is a natural pairing $\langle\ ,\ \rangle : \mathrm{CH}^{r+1}(S,1) \times \mathrm{CH}^{d-r}(S) \to K^*$ defined by

$$\langle [(u_Z)_Z], [Y]\rangle = \prod u_Z(Z \cap Y), \tag{45}$$

where the representatives are chosen so that Y intersects each Z properly and avoids the zeros and poles of the u_Z. The fact that this pairing is well-defined is easily verified using Weil reciprocity.

We now give another description of this map that was first explained to us by G. Kings. It is known that there is an isomorphism between Bloch's higher Chow groups and the motivic cohomology groups:

$$\mathrm{CH}^d(S,r) \cong H^{2d-r}_{\mathrm{mot}}(S,d).$$

Let $\pi\colon S \to \operatorname{Spec} K$ be the structure map. Our pairing is the composition

$$H^{2r+1}_{\mathrm{mot}}(S,r+1) \cup H^{2d-2r}_{\mathrm{mot}}(S,d-r) \to H^{2d+1}_{\mathrm{mot}}(S,d+1) \xrightarrow{\pi_*} H^1_{\mathrm{mot}}(\operatorname{Spec} K,1) \cong K^*,$$

where the first arrow is the cup product on motivic cohomology and the second arrow is the push forward under π.

5.3 The Beilinson Regulator

Now let us specialize to $K = \mathbf{C}$ and $d = \dim S = 2$. Beilinson defined a regulator map

$$\mathrm{reg}_{\mathbf{C}} \colon \mathrm{CH}^2(S,1) \longrightarrow H^{1,1}(S,\mathbf{R})^{\vee}$$

by

$$\mathrm{reg}_{\mathbf{C}}\left([(u_Z)]\right)(\omega) = \frac{1}{2\pi i}\sum \int_Z \log|u_Z| \cdot \omega. \tag{46}$$

(Here and throughout, integration on complex manifolds is taken in the i-oriented sense for a fixed choice of $i = \sqrt{-1} \in \mathbf{C}$, so that the right side of (46) becomes independent of this choice.) It is an elementary and pleasant calculation using Stokes' Theorem to verify that the definition of $\mathrm{reg}_{\mathbf{C}}$ depends only on the image of ω in $H^{1,1}(S,\mathbf{R})$ and that the regulator is well-defined on $\mathrm{CH}^2(S,1)$ on the left (see [Le]).

5.4 The cycle class map and a compatibility result

Denote by

$$\mathrm{cl}_{\mathbf{C}} : \mathrm{CH}^1(S) \to H^{1,1}(S,\mathbf{R})$$

the complex de Rham cycle class map, which sends a cycle to the class of two-forms associated by Poincaré duality. In other words, $\mathrm{cl}_{\mathbf{C}}(Y)$ is specified by the property

$$-\frac{1}{2\pi i}\int_Z \mathrm{cl}_{\mathbf{C}}(Y) = \#(Z \cap Y) \tag{47}$$

for each $[Z] \in H_2(S,\mathbf{Z})$ represented by a cycle Z that intersects Y properly.

Remark 5.1. It is known (see, for instance [Con]) that the period $-1/2\pi i$ must be included when using i-oriented integration in the definition of the Poincaré pairing:

$$(\omega_1,\omega_2)_{\mathrm{dR},\mathbf{C}} = -\frac{1}{2\pi i}\int_X \omega_1 \wedge \omega_2$$

in order to be compatible with the algebraic Poincaré pairing defined on $H^1_{\mathrm{dR}}(X)$ by cup product and the de Rham trace: $\mathrm{tr}_{\mathrm{dR}} \colon H^2_{\mathrm{dR}}(X) \to \mathbf{C}$. Since we will use this algebraic manifestation of the Poincaré pairing in §7 in conjunction with the present computaiton, it is essential to include the factor $-1/2\pi i$ in (47) above.

The following theorem will allow us to reinterpret regulator formulae for Rankin L-series of tensor squares in terms of logarithms of units.

Theorem 5.2. *The following diagram commutes.*

$$\begin{tikzcd}
\mathrm{CH}^2(S,1) \times \mathrm{CH}^1(S) \arrow[r, "\langle\,,\rangle"] \arrow[d, equal] \arrow[d, shift left=12ex, "\mathrm{cl}_{\mathbf{C}}"] & \mathbf{C}^* \arrow[d, "-\log|\,|"] \\
\mathrm{CH}^2(S,1) \times H^{1,1}(S,\mathbf{R}) \arrow[r, "\mathrm{reg}_{\mathbf{C}}"] & \mathbf{R}
\end{tikzcd}$$

Proof. It is possible to give a direct proof of this result using the explicit description of the regulator map given above. For this, the key fact is that the pairing $\mathrm{reg}_{\mathbf{C}}$ depends only on the class of the form ω in $H^{1,1}$. Therefore we can choose a form ω representing the class $\mathrm{cl}_{\mathbf{C}}(Y)$ for which the desired formula becomes "obvious"; more precisely, we can choose for each $\epsilon > 0$ a representing class ω_ϵ such that the value of $\mathrm{reg}_{\mathbf{C}}$ is easily seen to be within ϵ of the desired one.

However, with a view towards generalizations and future applications, rather than explain the details of this explicit argument we give here a more abstract proof suggested by the referee. Our result expresses nothing but the compatibility of the Beiliinson regulator with cup product and push-forward. Namely, for each d and r there is a regulator map

$$r_{\mathcal{D}} \colon H^d_{\mathrm{mot}}(S, r) \to H^d_{\mathcal{D}}(S, r),$$

where $H^d_{\mathcal{D}}(S, r)$ denotes Deligne cohomology (see [Bl1] or [DS, §2]). For $(d, r) = (3, 2)$, there is a canonical map

$$H^3_{\mathcal{D}}(S, 2) \to H^{1,1}(S, \mathbf{R}) \cong H^{1,1}(S, \mathbf{R})^\vee,$$

where the last isomorphism is Poincaré duality, and the composition of this map with the regulator $r_{\mathcal{D}}$ is precisely the map we have denoted $\mathrm{reg}_{\mathbf{C}}$. For $(d, r) = (2, 1)$, there is a canonical map $H^2_{\mathcal{D}}(S, 1) \to H^{1,1}(S, \mathbf{R})$, and the composition of this map with $r_{\mathcal{D}}$ is the cycle class map (see [EV, §7]). For $(d, r) = (5, 3)$, we have a canonical map $H^5_{\mathcal{D}}(S, 3) \to H^{2,2}(S, \mathbf{R})$, and the cup product $H^3_{\mathcal{D}}(S, 2) \times H^2_{\mathcal{D}}(S, 1) \to H^5_{\mathcal{D}}(S, 3)$ is compatible with the usual cup product on differential forms. Finally, we have $H^1_{\mathcal{D}}(\operatorname{Spec} \mathbf{C}, 1) = \mathbf{R}$, and the push-forward map $H^5_{\mathcal{D}}(S, 3) \to \mathbf{R}$ is integration over $S(\mathbf{C})$.

In other words, the commutativity of our diagram follows from that of:

$$\begin{tikzcd}
H^3_{\mathrm{mot}}(S,2) \arrow[d, "r_{\mathcal{D}}"] & \times & H^2_{\mathrm{mot}}(S,1) \arrow[r, "\cup"] \arrow[d, "r_{\mathcal{D}}"] & H^5_{\mathrm{mot}}(S,3) \arrow[r, "\pi_*"] \arrow[d, "r_{\mathcal{D}}"] & H^1_{\mathrm{mot}}(\operatorname{Spec}\mathbf{C},1) = \mathbf{C}^* \arrow[d, "r_{\mathcal{D}} = -\log|\ |"] \\
H^3_{\mathcal{D}}(S,2) \arrow[d] & \times & H^2_{\mathcal{D}}(S,1) \arrow[r, "\cup"] \arrow[d] & H^5_{\mathcal{D}}(S,3) \arrow[r, "\pi_*"] \arrow[d] & H^1_{\mathcal{D}}(\operatorname{Spec}\mathbf{C},1) = \mathbf{R} \arrow[d, equal] \\
H^{1,1}(S,\mathbf{R}) & \times & H^{1,1}(S,\mathbf{R}) \arrow[r, "\cup"] & H^{2,2}(S,\mathbf{R}) \arrow[r, "\pi_* = \int"] & \mathbf{R}.
\end{tikzcd} \tag{48}$$

The commutativity of the above diagram—i.e., the compatibility of the Beilinson regulator $r_{\mathcal{D}}$ with cup product and proper push-forward—is well-known; for instance, for the cup product see [Nek].

□

5.5 The étale regulator

Let us retain the assumption that S is a smooth projective surface over a field K. We now consider the setting where K is a finite extension of $\mathbf{Q}_p$.

Recall that there is an étale cycle class map

$$\mathrm{cl}_{\text{ét}} \colon \mathrm{CH}^r(S) \longrightarrow H^{2r}_{\text{ét}}(S, \mathbf{Q}_p(r)) \longrightarrow H^{2r}_{\text{ét}}(\overline{S}, \mathbf{Q}_p(r))^{G_K}.$$

The étale regulator is defined in terms of Bloch's generalization of the étale cycle class map to higher Chow groups (see [Bl1], [Bl2])

$$\mathrm{cl}_{\text{ét}} : \mathrm{CH}^r(S,1) \longrightarrow H^{2r-1}_{\text{ét}}(S, \mathbf{Q}_p(r+1)),$$

as follows. Let $N^1\,\mathrm{CH}^r(S,1) \subset \mathrm{CH}^r(S,1)$ denote the subspace of elements whose image under the cycle class map $\mathrm{cl}_{\text{ét}}$ lies in

$$N^1 H^{2r-1}_{\text{ét}}(S, \mathbf{Q}_p(r)) := \ker(H^{2r-1}_{\text{ét}}(S, \mathbf{Q}_p(r)) \longrightarrow H^{2r-1}_{\text{ét}}(\overline{S}, \mathbf{Q}_p(r))^{G_K}).$$

For a smooth proper surface S over a p-adic field K we have $H^3_{\text{ét}}(\overline{S}, \mathbf{Q}_p(2))^{G_K} = 0$ and hence $N^1\,\mathrm{CH}^2(S,1) = \mathrm{CH}^2(S,1)$ (see for instance the proof of [CTR, Theorem 6.1] or [SS, §3.1]).

Consider the Leray spectral sequence

$$E_2^{p,q} = H^p(K, H^q_{\text{ét}}(\overline{S}, \mathbf{Q}_p(r))) \Longrightarrow H^{p+q}_{\text{ét}}(S, \mathbf{Q}_p(r)). \tag{49}$$

From this spectral sequence, we extract a map

$$\delta_r \colon N^1 H^{2r-1}_{\text{ét}}(S, \mathbf{Q}_p(r)) \longrightarrow E_\infty^{1,2r-2} \subset E_2^{1,2r-2} = H^1(K, H^{2r-2}_{\text{ét}}(\overline{S}, \mathbf{Q}_p(r))).$$

The étale regulator is defined by

$$\mathrm{reg}_{\text{ét}} = \delta_r \circ \mathrm{cl}_{\text{ét}} : N^1\,\mathrm{CH}^r(S,1) \longrightarrow H^1(K, H^{2r-2}_{\text{ét}}(\overline{S}, \mathbf{Q}_p(r))).$$

The following is the analog of (48) in the étale context.

Proposition 5.3. *Let S be a smooth proper surface over a finite extension $K/\mathbf{Q}_p$. The following diagram commutes:*

$$\begin{tikzcd}
\mathrm{CH}^2(S,1) \arrow[d, "\mathrm{reg}_{\text{ét}}"] & \times & \mathrm{CH}^1(S) \arrow[d, "\mathrm{cl}_{\text{ét}}"] \arrow[r, "\cup"] & \mathrm{CH}^3(S,1) \arrow[r, "\pi_*"] \arrow[d, "\mathrm{reg}_{\text{ét}}"] & \mathrm{CH}^1(\operatorname{Spec} K, 1) = K^* \arrow[d, "\mathrm{reg}_{\text{ét}} = \delta_{\mathrm{Kum}}"] \\
H^1(K, H^2_{\text{ét}}(\overline{S}, \mathbf{Q}_p(2))) & \times & H^2_{\text{ét}}(\overline{S}, \mathbf{Q}_p(1))^{G_K} \arrow[r, "\cup"] & H^1(K, H^4_{\text{ét}}(\overline{S}, \mathbf{Q}_p(3))) \arrow[r, "\pi_* = (\mathrm{tr}_{\text{ét}})_*"] & H^1(K, \mathbf{Q}_p(1)),
\end{tikzcd} \tag{50}$$

where

$$\delta_{\mathrm{Kum}} \colon (\varprojlim K^*/(K^*)^{p^n}) \otimes \mathbf{Q} \longrightarrow H^1(K, \mathbf{Q}_p(1))$$

is the usual connecting homomorphism in Kummer Theory and $(\mathrm{tr}_{\text{ét}})_*$ *is the map induced by the étale trace*

$$\mathrm{tr}_{\text{ét}} \colon H^4_{\text{ét}}(\overline{S}, \mathbf{Q}_p(2)) \xrightarrow{\sim} \mathbf{Q}_p.$$

Proof. The commutativity of the left square expresses the compatibility of the étale regulator with cup product, which we now briefly explain.[6] If $w \in \mathrm{CH}^r(S,1)$ and $z \in \mathrm{CH}^m(S)$, then by [GL, Proposition 4.7] we have

$$\mathrm{cl}_{\text{ét}}(w \cup z) = \mathrm{cl}_{\text{ét}}(w) \cup \mathrm{cl}_{\text{ét}}(z). \tag{51}$$

Given elements $a \in N^1 H^{2r-1}_{\text{ét}}(S, \mathbf{Q}_p(r))$ and $b \in H^{2m}_{\text{ét}}(S, \mathbf{Q}_p(m))$ we have

$$\delta_r(a) \cup b' = \delta_{r+m}(a \cup b), \tag{52}$$

---

[6] This commutativity is well-known to the experts and appears in various forms in the literature (see for instance [Hu, Corollary 2.3.4]), but we offer a short explanation here for the benefit of the reader.

where b' denotes the image of b in $H^{2m}_{\text{ét}}(\overline{S}, \mathbf{Q}_p(m))^{G_K}$. This follows from the fact that cup product with b induces a morphism in the category of spectral sequences between the Leray spectral sequences (49) for r and $r+m$ (with a shift in degree of $2m$), which on the E_2 page is realized by cup product with b'. The commutativity of the left square of (50) follows by combining (51) and (52).

The commutativity of the right square of (50) expresses the compatibility of the étale regulator with push-forward, which in this context is elementary. Elements of $\mathrm{CH}^3(S_{\overline{K}}, 1)$ are generated by those of the form $w \cup z$ where $w \in L^* = \mathrm{CH}^1(S_L, 1)$ and $z \in S(L) \subset \mathrm{CH}^2(S_L)$ for some finite extension L/K. By functoriality with respect to field extension, it suffices to prove the commutativity for such an element (with K replaced by L). It follows from the definitions that

$$\mathrm{reg}_{\text{ét}}(w) = \delta_1(\mathrm{cl}_{\text{ét}}(w)) = \delta_{\mathrm{Kum}}(w) \in H^1(L, \mathbf{Q}_p(1)).$$

Hence by (51) and (52) we have

$$\delta_3(\mathrm{cl}_{\text{ét}}(w \cup z)) = \delta_1(\mathrm{cl}_{\text{ét}}(w)) \cup \mathrm{cl}(z)' = \delta_{\mathrm{Kum}}(w) \cup \mathrm{cl}(z)'.$$

By definition, $\mathrm{tr}_{\text{ét}}(\mathrm{cl}(z)') = 1$ and hence $\mathrm{tr}_{\text{ét}}(\delta_3(\mathrm{cl}_{\text{ét}}(w \cap z))) = \delta_{\mathrm{Kum}}(w)$ as desired. ☐

5.6 The syntomic regulator and a compatibility result

We now prove a p-adic analogue of Theorem 5.2 by connecting the above étale picture to p-adic (or "syntomic") regulators on our motivic cohomology groups. The theory of syntomic regulators has a long history, including the works of Fontaine–Messing [FM], Niziol [N], Besser [Bes], Besser–Loeffler–Zerbes [BLZ], and Nekovár–Niziol [NN]. Rather than survey this deep and important theory, however, we give an ad hoc definition of the p-adic regulator that suffices for our applications.

Recall that Bloch and Kato [BK, Def. 3.10] have defined an "exponential" map

$$\exp\colon H^2_{\mathrm{dR}}(S/K)/F^2 \longrightarrow H^1(K, H^2_{\text{ét}}(\overline{S}, \mathbf{Q}_p(2))). \tag{53}$$

Let $\mathrm{CH}^2(S,1)_{\text{ét}}$ denote the subspace of classes $\Xi \in \mathrm{CH}^2(S,1)$ such that $\mathrm{reg}_{\text{ét}}(\Xi)$ lies in the image of the Bloch–Kato exponential. We define the p-adic regulator

$$\mathrm{reg}_p\colon\ \mathrm{CH}^2(S,1)_{\text{ét}} \longrightarrow (F^1 H^2_{\mathrm{dR}}(S/K))^{\vee}$$

as the composition of the étale regulator, the inverse of the Bloch–Kato exponential[7] and the Poincaré duality isomorphism

$$H^2_{\mathrm{dR}}(S/K)/F^2 \cong (F^1 H^2_{\mathrm{dR}}(S/K))^{\vee}.$$

Next denote by

$$\mathrm{cl}_p : \mathrm{CH}^1(S) \longrightarrow F^1 H^2_{\mathrm{dR}}(S/K)$$

the de Rham cycle class map. The p-adic analog of Theorem 5.2 is the following.

---

[7] Strictly speaking, reg_p as defined here is only well-defined up to the kernel of exp. However, in view of (55), classes in the kernel of exp do not contribute to our pairing and can be safely ignored in our applications.

Theorem 5.4. *The pairing* $\langle\,,\,\rangle$ *restricted to* $\mathrm{CH}^2(S,1)_{\text{ét}} \times \mathrm{CH}^1(S)$ *takes values in* $\mathcal{O}_K^*$, *and we have a commutative diagram*

$$\begin{tikzcd}
\mathrm{CH}^2(S,1)_{\text{ét}} \times \mathrm{CH}^1(S) \arrow[r, "\langle\,,\,\rangle"] \arrow[d, equal] \arrow[d, "\mathrm{cl}_p"] & \mathcal{O}_K^* \arrow[d, "\log_p"] \\
\mathrm{CH}^2(S,1)_{\text{ét}} \times F^1H^2_{\mathrm{dR}}(S/K) \arrow[r, "\mathrm{reg}_p"] & K,
\end{tikzcd} \tag{54}$$

where $\log_p\colon \mathcal{O}_K^* \to K$ *denotes the* p*-adic logarithm.*

Proof. We can connect the bottom row of (50) to de Rham cohomology groups via the following commutative diagram:

$$\begin{tikzcd}
H^1(K, H^2_{\text{ét}}(\overline{S}, \mathbf{Q}_p(2))) \times H^2_{\text{ét}}(\overline{S}, \mathbf{Q}_p(1))^{G_K} \arrow[r, "\cup"] \arrow[d, "\alpha"] & H^1(K, H^4_{\text{ét}}(\overline{S}, \mathbf{Q}_p(3))) \arrow[r, "\pi_*=(\mathrm{tr}_{\text{ét}})_*"] & H^1(K, \mathbf{Q}_p(1)) \\
H^2_{\mathrm{dR}}(S/K)/F^2 \times F^1H^2_{\mathrm{dR}}(S/K) \arrow[u, "\exp"] \arrow[r, "\cup"] & H^4_{\mathrm{dR}}(S/K) \arrow[u, "\exp"] \arrow[r, "\mathrm{tr}_{\mathrm{dR}}"] & K. \arrow[u, "\exp"]
\end{tikzcd} \tag{55}$$

The map α is the composition of the inclusion

$$H^2_{\text{ét}}(\overline{S}, \mathbf{Q}_p(1))^{G_K} \subset D_{\mathrm{dR}}(H^2_{\text{ét}}(\overline{S}, \mathbf{Q}_p(1))) = (H^2_{\text{ét}}(\overline{S}, \mathbf{Q}_p(1)) \otimes_{\mathbf{Q}_p} B_{\mathrm{dR}})^{G_K}$$

given by $x \mapsto x \otimes 1$ with the étale-to-de Rham comparison isomorphism

$$D_{\mathrm{dR}}(H^2_{\text{ét}}(\overline{S}, \mathbf{Q}_p(1))) \xrightarrow{\sim} H^2_{\mathrm{dR}}(S/K)(1),$$

where the twist on the right represents a shift in the filtration. The image of α clearly lands in $F^1H^2_{\mathrm{dR}}(S/K)$. The commutativity of the square on the left in (55) follows directly from the definition of exp, granting the compatibility of the comparison isomorphism with cup product (which is proven in [Ts, Theorem A1]). The commutativity of the right square expresses the compatibility of trace maps on top degree de Rham and étale cohomology, proven in *loc. cit.*

Diagrams (50) and (55) together show that the pairing $\langle\,,\,\rangle$ restricted to $\mathrm{CH}^2(S,1)_{\text{ét}} \times \mathrm{CH}^1(S)$ takes values in $K^* \cap \delta_{\mathrm{Kum}}^{-1}(\exp(K))$, which by [BK, Page 359] is equal to $\mathcal{O}_K^*$.

The commutativity of (54) also follows from (50) and (55). On $\mathrm{CH}^1(S)$, the p-adic cycle class map cl_p is given by the composition of the étale cycle class map and the map α (see [Fa, Theorem 8.1] or [Ts, Theorem A1]). Furthermore, Poincaré duality is given by cup product and the trace map on top degree cohomology (on both the étale and de Rham side). The theorem then follows from the fact that for $\mathbf{Q}_p(1)$, the Bloch–Kato exponential is an inverse to the p-adic logarithm, i.e. the composition

$$\mathcal{O}_K^* \xrightarrow{\log_p} K \xrightarrow{\exp} H^1(K, \mathbf{Q}_p(1))$$

is equal to δ_{Kum}. This last fact is proven in [BK, Page 358]. □

Theorem 5.4 is the key technical result that will allow us to relate the special values of p-adic Rankin L-series to the p-adic logarithms of units.

6. Regulator Formulae for Rankin L-series

6.1 Beilinson–Flach elements, after Lei–Loeffler–Zerbes

Let m and M be positive integers and let $X = X_1(M)$. In our applications, m will be a power of p and M will have the form Np^r. In [LLZ], Lei, Loeffler, and Zerbes define an element

$$\widetilde{\Xi}_{m,M,1} \in \mathrm{CH}^2(X^2 \otimes \mathbf{Q}(\mu_m), 1)$$

related to Rankin L-series of weight 2 modular forms. Let β be a primitive Dirichlet character with conductor m. Define

$$\Xi_{\beta,M} = \sum_{a \in (\mathbf{Z}/m\mathbf{Z})^*} \sigma_a(\widetilde{\Xi}_{m,M,1}) \otimes \beta^{-1}(a) \in \mathrm{CH}^2(X^2 \otimes \mathbf{Q}(\mu_m), 1) \otimes \overline{\mathbf{Q}},$$

where $\sigma_a \in \mathrm{Gal}(\mathbf{Q}(\mu_m)/\mathbf{Q})$ denotes the automorphism $\zeta_m \mapsto \zeta_m^a$. The class $\Xi_{\beta,M}$ is called the *Beilinson–Flach element* associated to β and M.

Let $f, g \in S_2(\Gamma_1(M))$ such that f and g are eigenforms for the Hecke operators away from M. Let $f^* = \overline{f(-\overline{z})}$ denote the modular form in $S_2(\Gamma_1(M))$ whose Fourier coefficients are the complex conjugates of those of f. Define $\omega_f, \eta_f^{\mathrm{ah}} \in H^1_{\mathrm{dR}}(X)$ by

$$\omega_f = 2\pi i f(z)dz, \qquad \eta_f^{\mathrm{ah}} = \frac{\overline{f^*(z)}d\overline{z}}{\langle f^*, f^*\rangle} = \frac{f(-\overline{z})d\overline{z}}{\langle f^*, f^*\rangle},$$

and similarly for g. Here we are employing the usual identification $X(\mathbf{C}) = \Gamma_1(M)\backslash(\mathcal{H} \cup \mathbf{P}^1(\mathbf{Q}))$. Via the Kunneth decomposition, we can consider $H^1_{\mathrm{dR}}(X)^{\otimes 2} \subset H^2_{\mathrm{dR}}(X^2)$. The following is Theorem 4.3.7 of [LLZ], which generalizes a formula of Beilinson in the case $\beta = 1$ and is based on Proposition 4.1 of [BDR1].

Theorem 6.1 (Lei, Loeffler, Zerbes). *We have*

$$\mathrm{reg}_{\mathbf{C}}(\Xi_{\beta,M})(\eta_f^{\mathrm{ah}} \otimes \omega_g) = -\frac{L'_{mM}(f, g, \beta^{-1}, 1)}{4\pi i \langle f^*, f^*\rangle} A(f, g, \beta^{-1}, 1), \tag{56}$$

where

$$A(f, g, \beta^{-1}, s) := \tau(\beta) \prod_{\substack{\ell \mid M \\ \ell \nmid m}} \frac{1}{1 - \beta^{-1}(\ell) a_\ell(f) a_\ell(g) \ell^{-s}}. \tag{57}$$

Now let ψ be an auxiliary Dirichlet character with conductor relatively prime to m, and suppose that M is large enough so that $f_\psi, g_\psi \in S_2(\Gamma_1(M))$. Since $\langle f, f\rangle_M = \langle f_\psi, f_\psi\rangle_M$, the right side of (56) is equal for the pairs (f_ψ, g) and (f, g_ψ). Now, the involution on $X_1(M)^2$ given by switching factors acts on $\Xi_{\beta,M}$ by $\beta(-1)$. On the other hand, the induced action of this involution on $H^1_{\mathrm{dR}}(X)^{\otimes 2} \subset H^2_{\mathrm{dR}}(X^2)$ is given by $x \otimes y \mapsto -y \otimes x$ because of the anti-commutativity of the cup product in degree 1. This implies that

$$\mathrm{reg}_{\mathbf{C}}(\Xi_{\beta,M})(\eta_f^{\mathrm{ah}} \otimes \omega_g) = -\beta(-1)\, \mathrm{reg}_{\mathbf{C}}(\Xi_{\beta,M})(\omega_g \otimes \eta_f^{\mathrm{ah}}).$$

Combining these observations, we obtain from Theorem 6.1:

Corollary 6.2. *With notation as above, we have:*

$$\mathrm{reg}_{\mathbf{C}}(\Xi_{\beta,M})(\eta_{f_\psi}^{\mathrm{ah}} \otimes \omega_g - \beta(-1)\omega_{g_\psi} \otimes \eta_f^{\mathrm{ah}}) = -\frac{L'_{mM}(f,g,\psi\beta^{-1},1)}{2\pi i\langle f^*, f^*\rangle_M} A(f_\psi, g, \beta^{-1}, 1).$$

6.2 The classes η_f^{ur} and ω_g

Before stating the p-adic analogues of Theorem 6.1 and Corollary 6.2, we must introduce some notation. Continuing with the notation of the previous section, we assume that the prime p divides M (the level of the weight 2 newforms f and g) and m (the modulus of the character β). Over any field K containing $\mathbf{Q}(\mu_M)$, we have the short exact sequence

$$0 \longrightarrow H^0(X_K, \Omega^1_X) \longrightarrow H^1_{\mathrm{dR}}(X_K) \longrightarrow H^1(X_K, \mathcal{O}_X) \longrightarrow 0. \tag{58}$$

The image of the class $\eta_f^{\mathrm{ah}} \in H^1_{\mathrm{dR}}(X_{\mathbf{C}})$ in $H^1(X_{\mathbf{C}}, \mathcal{O}_X)$ is actually defined over the number field K_f generated over $\mathbf{Q}(\mu_M)$ by the Fourier coefficients of f, in the sense that it arises from an element $\eta_f \in H^1(X_{K_f}, \mathcal{O}_X)$ by base extension ([DR, Corollary 2.13]). If K is a finite extension of $\mathbf{Q}_p$ containing K_f, we can then view η_f as an element of $H^1(X_K, \mathcal{O}_X)$. Now, for such a K, the space $H^1_{\mathrm{dR}}(X_K)$ is endowed with an action of Frobenius. On the 2-dimensional f-isotypic subspace of $H^1_{\mathrm{dR}}(X_K)$ (i.e. the subspace on which the Hecke operators away from p act via the eigenvalues of f), there is a canonicial 1-dimensional subspace on which Frobenius acts by multiplication by $\alpha_f = a_p(f) =$ the U_p-eigenvalue of f. This subspace maps isomorphically via (58) to $H^1(X_K, \mathcal{O}_X)^f$. The lift of $\eta_f \in H^1(X_K, \mathcal{O}_X)^f$ via this isomorphism is denoted

$$\eta_f^{\mathrm{ur}} \in H^1_{\mathrm{dR}}(X_K).$$

Meanwhile, the differential $\omega_g \in H^0(X_{\mathbf{C}}, \Omega^1_X)$ is well-known to represent a class $H^1_{\mathrm{dR}}(X_{\mathbf{C}})$ defined over the number field K_g, and may therefore be viewed as an element of $H^1_{\mathrm{dR}}(X_K)$ for any p-adic field K containing K_g.

6.3 p-adic Rankin L-series

In this section, we state p-adic analogues of Theorem 6.1 and Corollary 6.2.

Let F and G be Hida families. Let $\kappa \in \mathcal{W}_F, \kappa' \in \mathcal{W}_G$ be arithmetic weights such that $\nu(\kappa) = \nu_{2,\alpha}, \nu(\kappa') = \nu_{2,\alpha'}$ for some Dirichlet characters α, α' of p-power conductor. Note that α has the same parity as χ_F and α' has the same parity as χ_G. The specializations $f := F_\kappa$ and $g := G_{\kappa'}$ are p-ordinary forms of weight 2 and characters $\chi_F\alpha, \chi_G\alpha'$, respectively. Let M be the lcm of the levels of f and g. Let β be a Dirichlet character of conductor $p^w, w > 0$. Let the conductors of α and $\beta^{-1}\alpha'$ be $p^r, p^{r'}$, respectively, and assume that $r, r' > 0$. Define

$$\Lambda_p(f,g,\beta) := L_p(F,G,\kappa,\kappa',\nu_{1,\beta}) \times \frac{2i \cdot \alpha(-1)\tau(\alpha)\tau(\beta^{-1}\alpha')}{a_p(f)^{-w-r'+2r}a_p(g)^{-w+r'}\chi_F(p)^{-r}\chi_G(p)^{-r'}}. \tag{59}$$

In [KLZ, Theorem 10.2.2], the following p-adic analogue of Theorem 6.1 is proven.

Theorem 6.3 (Kings, Loeffler, Zerbes. *Let $M = N_f N_g N_\psi^2$, and let $S = X_1(M) \times X_1(M)$. We have*

$$\mathrm{reg}_p(\Xi_{\beta,M})(\eta_f^{\mathrm{ur}} \otimes \omega_g) = \Lambda_p(f,g,\beta).$$

Theorem 6.3 can be deduced by specializing Theorem 10.2.2 of [KLZ] to our setting. The specialization of the class ${}_c\mathcal{BF}^{\mathbf{f},\mathbf{g}}$ at the point $(\kappa, \kappa', \nu_{1,\beta})$ is equal to the étale regulator of the projection of the class we have denoted $\Xi_{\beta,M}$ onto the (f,g)-isotypic component of $\mathrm{CH}^2(S,1)$, up to the factor depending on c on the right side of [KLZ, Theorem 10.2.2], and a factor $(a_p(f)a_p(g))^{-w}$. The function denoted $\mathcal{L}$ is a logarithm, i.e. in our context an inverse to the Bloch–Kato exponential, and hence the pairing indicated is exactly our p-adic regulator reg_p. The specialization of $\eta_{\mathbf{a}} \otimes \omega_{\mathbf{g}}$ at our point is $\eta_f^{\mathrm{ur}} \otimes \omega_g$ up to a factor of $a_p(f)^{-u}$, where u is the power of p dividing N_f. Comparing the normalizations for the p-adic L-functions in [KLZ] and in this paper explains the constants in (59).

REMARK 6.4. Theorem 6.3 is a generalization of Theorems 4.2 and Corollary 4.2 of [BDR1], which give the version of this result in the case $\alpha = \alpha' = \beta = 1$, and Theorem 3.9 of [BDR2], which handles the case when f is fixed but g moves in a p-adic Hida family G.

Now let ψ be a Dirichlet character with $p \nmid N_\psi$. It is easy to see from the interpolation property that $L_p(F_\psi, G) = L_p(F, G_\psi)$. Arguing as before with the signs of the "swapping" involution on $X_1(M)^2$, we obtain:

COROLLARY 6.5. *Let* $M = N_f N_g N_\psi^2$, *and let* $S = X_1(M) \times X_1(M)$. *There is a functional* $\mathscr{L}_p \in F^1 H^2_{\mathrm{dR}}(S/K))^\vee$ *such that* $\exp_\alpha(\mathrm{pd}(\mathscr{L}_p)) = \mathrm{reg}_{\text{ét}}(\Xi_{\beta,M})$ *and*

$$\mathscr{L}_p(\eta_{f_\psi}^{\mathrm{ur}} \otimes \omega_g - \beta(-1)\omega_{g_\psi} \otimes \eta_f^{\mathrm{ur}}) = 2\Lambda_p(f_\psi, g, \beta).$$

7. Beilinson–Flach units

In this section, we prove that the regulators appearing in Corollaries 6.2 and 6.5 can be interpreted as the logarithms (archimedian and p-adic) of an algebraic unit in a cyclotomic field. As a first step, we show that the de Rham cohomology classes appearing in these regulators have a common algebraic source, namely a certain element of $\mathrm{CH}^1(X^2)$. This element will be constructed out of Hecke operators.

7.1 Algebraic cycles attached to f

Let $X = X_1(M)$, $J = J_1(M) = \mathrm{Jac}(X)$ and $S = X_1(M) \times X_1(M)$. As in [DRS, pg. 19, eq. (65)], we can view the graph of an element $T \in \mathrm{End}(J)$ as an element $\mathrm{gr}(T) \in \mathrm{CH}^1(S)$, well defined up to vertical and horizontal components (i.e. up to an element of $(\pi_1)^* \mathrm{CH}^1(X) + (\pi_2)^* \mathrm{CH}^1(X)$, where $\pi_i : X \times X \to X$ are the projection maps). For example, a Hecke operator T_ℓ can be described as a correspondence associated to a pair of morphisms $f_1, f_2 : X_1(M\ell) \to X_1(M)$. These induce a morphism $f_1 \times f_2 : X_1(M\ell) \to S$, and $\mathrm{gr}(T_\ell) \in \mathrm{CH}^1(S)$ is the class of the image of $f_1 \times f_2$.

We now describe the various homomorphisms that we need:

- Let $f \in S_2(\Gamma_1(N_f), \chi_f)$ be a newform of level N_f. Let $\mathbf{T} \subset \mathrm{End}(J_1(N_f))$ denote the Hecke algebra of $J_1(N_f)$ generated by operators T_ℓ for $\ell \nmid N_f$, U_ℓ for $\ell \mid N_f$, and the diamond operators $\langle d\rangle$ for $d \in (\mathbf{Z}/N_f\mathbf{Z})^*$. Let K denote a field containing K_f. Let $T_{f^*} \in \mathbf{T}\otimes K$ denote the idempotent defining projection onto the f^*-isotypic component of $\mathbf{T}$.

- Let $W_{N_f} \in \mathrm{End}(J_1(N_f))$ denote the Atkin–Lehner involution. The root number of f is the algebraic number of complex absolute value one satisfying $f|_{W_{N_f}} = W(f)f^*$. We suppose that $W(f) \in K$ and we define $\widetilde{W}_{N_f} := W(f)\cdot W_{N_f} \in \mathrm{End}(J_1(N_f))\otimes K$.
- Let ψ denote a Dirichlet character and let $M = N_fN_\psi^2$. Suppose that the field K contains the values of the character ψ and the N_ψth roots of unity. There is a twisting map
$$\mathrm{tw}_\psi \in \mathrm{Hom}(J_1(M), J_1(N_f))\otimes K$$
described analytically as follows. For $z \in \Gamma_1(M)\backslash\mathcal{H} = Y_1(M)(\mathbf{C})$, let
$$\mathrm{tw}_\psi([z]) = \frac{1}{\tau(\psi^{-1})}\sum_{a=1}^{N_\psi}\psi^{-1}(a)[z + a/N_\psi] \in (\mathrm{Div}\, Y_1(N_f)(\mathbf{C}))\otimes K$$
as a map on divisors.
- Let $\mathrm{tr} : J_1(N_f) \to J_1(M)$ be the trace map induced $[z] \mapsto \sum_{x\in\pi^{-1}z}[x]$ where π is the usual projection $X_1(M) \to X_1(N_f)$. The map tr is the standard Albanese morphism attached to the map π. We let $\widetilde{\mathrm{tr}} := \deg(\pi)^{-1}\,\mathrm{tr}$.

Let $\epsilon = \psi(-1) = \pm 1$. Define the endomorphism $T_{f,\psi} \in \mathrm{End}(J)\otimes K$ as the composition

$$\begin{tikzcd} J_1(M) \arrow[r, "\epsilon\cdot\mathrm{tw}_\psi"] & J_1(N_f) \arrow[r, "\widetilde{W}_{N_f}"] & J_1(N_f) \arrow[r, "T_{f^*}"] & J_1(N_f) \arrow[r, "\widetilde{\mathrm{tr}}"] & J_1(M) \end{tikzcd}$$

As mentioned above, $\mathrm{gr}(T_{f,\psi}) \in \mathrm{CH}^1(S)\otimes K$ is well-defined only up to vertical and horizontal components. In order to handle this ambiguity, we choose a rational base point $\infty \in X(\mathbf{Q})$—for example, we may take the usual point corresponding to the cusp ∞—and, following [DDLR, §2.3], define a projector ϵ_∞ on $\mathrm{CH}^1(X\times X)$ as follows:
$$\epsilon_\infty(Z) = Z - (i_1)_*(\pi_1)_*Z - (i_2)_*(\pi_2)_*Z.$$
Here $i_1, i_2 : X \to X\times X$ are the inclusions of vertical and horizontal components over the base point ∞. We define
$$Z_{f,\psi} := \frac{1}{2i}\epsilon_\infty(\mathrm{gr}(T_{f,\psi})) \in \mathrm{CH}^1(X\times X)\otimes K. \tag{60}$$
This element still has an ambiguity in its vertical and horizontal components (due to the original ambiguity of $T_{f,\psi}$ and the choice of base point ∞), but any ambiguity in these components is now algebraically equivalent to zero. Let $\mathrm{CH}^1(S)_{\mathrm{HV}}$ denote the quotient of $\mathrm{CH}^1(X\times X)$ by the subroup of horizontal and vertical classes algebraically equivalent to zero; the image of $Z_{f,\psi}$ in $\mathrm{CH}^1(S)_{\mathrm{HV}}\otimes K$ is well-defined.

Proposition 7.1. *Classes in* $\mathrm{CH}^1(S)$ *represented by vertical or horizontal components algebraically equivalent to zero lie in the kernel of the intersection pairing* $\langle\ ,\ \rangle$ *defined in Section 5.2. Therefore the intersection pairing descends to a pairing on* $\mathrm{CH}^1(S)_{\mathrm{HV}}$.

Proof. We must show that for fixed $A \in \mathrm{CH}^1(S)$, the function $\langle A, \pi_1^* B\rangle$ is constant for $B \in X$. Yet this map defines a morphism $X \to \mathbf{G}_{m/K}$; since X is a smooth projective curve, such a morphism must be constant. ☐

The following proposition shows that the cycle $Z_{f,\psi}$ is a common algebraic source of the differentials appearing the regulator formulae of Corollaries 6.2 and 6.5.

PROPOSITION 7.2. *Let f, ψ, and K be as in the beginning of this section. Let $\widetilde{\eta}_f \in H^1_{dR}(X_1(N)_K)^f$ be any lift of the class $\eta_f \in H^1(X_1(N)_K, \mathcal{O}_X)$ defined in Section 6.3. Let $\epsilon = \psi(-1)$ and define $\widetilde{\eta}_{f_\psi} = \epsilon \cdot \mathrm{tw}_\psi^* \widetilde{\eta}_f$. We have*

$$\mathrm{cl}_{\mathrm{dR}}(Z_{f,\psi}) = \widetilde{\eta}_{f_\psi} \otimes \omega_f - \epsilon \cdot \omega_{f_\psi} \otimes \widetilde{\eta}_f. \tag{61}$$

In particular over $\mathbf{C}$ we have

$$\mathrm{cl}_{\mathbf{C}}(Z_{f,\psi}) = \eta_{f_\psi}^{\mathrm{ah}} \otimes \omega_f - \epsilon \cdot \omega_{f_\psi} \otimes \eta_f^{\mathrm{ah}} \tag{62}$$

and over any finite extension of $\mathbf{Q}_p$ containing K_f we have

$$\mathrm{cl}_p(Z_{f,\psi}) = \eta_{f_\psi}^{\mathrm{ur}} \otimes \omega_f - \epsilon \cdot \omega_{f_\psi} \otimes \eta_f^{\mathrm{ur}}. \tag{63}$$

REMARK 7.3. Note that $\omega_{f_\psi} = \mathrm{tw}_\psi^* \omega_f$ and that, as the notation suggests, $\widetilde{\eta}_{f_\psi}$ is a lift of the class $\eta_{f_\psi} \in H^1(X_1(N)_K, \mathcal{O}_X)$ defined in Section 6.3. Note that $\widetilde{\eta}_f$ is well-defined up to the addition of a multiple $C \cdot \omega_f$, and such an addition changes $\widetilde{\eta}_{f_\psi}$ by $C\epsilon \cdot \omega_{f_\psi}$. Therefore, the right side of (61) is well-defined.

Proof. The effect of ϵ_∞ is to project onto the Kunneth $(1,1)$-component of $H^2_{\mathrm{dR}}(X_K \times X_K)$. The class $\mathrm{cl}_{\mathrm{dR}}(Z_{f,\psi}) \in H^1_{\mathrm{dR}}(X_K)^{\otimes 2}$ is equal to the image of the endomorphism of $H^1_{\mathrm{dR}}(X_K)$ induced by $T_{f,\psi}$ under the identifications

$$\mathrm{End}_K(H^1_{\mathrm{dR}}(X_K)) \cong H^1_{\mathrm{dR}}(X_K) \otimes_K H^1_{\mathrm{dR}}(X_K)^\vee \cong H^1_{\mathrm{dR}}(X_K) \otimes_K H^1_{\mathrm{dR}}(X_K).$$

Here the last isomorphism is induced by the Poincaré duality pairing: $H^1_{\mathrm{dR}}(X_K) \cong H^1_{\mathrm{dR}}(X_K)^\vee$ via $\omega \mapsto (\omega, -)_{\mathrm{pd}}$.

Let us now compute the endomorphism of $H^1_{\mathrm{dR}}(X_K)$ induced by pulling back $T_{f,\psi}$. The map $\widetilde{\mathrm{tr}}^*$ projects onto the subspace of forms arising from $H^1_{\mathrm{dR}}(X_1(N_f)_K)$. The map $T_{f^*}^*$ then projects onto the f^*-isotypic component of $H^1_{\mathrm{dR}}(X_1(N_f)_K)$, which is spanned by ω_{f^*} and $\widetilde{\eta}_{f^*}$. The map $\widetilde{W}_{N_f}^*$ maps ω_{f^*} to ω_f and $\widetilde{\eta}_{f^*}$ to $\widetilde{\eta}_f$. Finally, $\epsilon \cdot \mathrm{tw}_\psi^*$ maps $\omega_f \mapsto \epsilon \cdot \omega_{f_\psi}$ and $\widetilde{\eta}_f \mapsto \widetilde{\eta}_{f_\psi}$. Our composition therefore sends $\omega_{f^*} \mapsto \epsilon \cdot \omega_{f_\psi}$ and $\widetilde{\eta}_{f^*} \mapsto \widetilde{\eta}_{f_\psi}$.

Under our normalization for Poincaré duality we have $(\omega_f, \widetilde{\eta}_{f^*})_{\mathrm{pd}} = 2i$, as can be calculated from the complex realization (see Remark 5.1):

$$-\frac{1}{2\pi i}\int_X \omega_f \wedge \eta_{f^*}^{\mathrm{ah}} = 2i,$$

with the factor of $2i$ coming from $dz \wedge d\overline{z} = (2i) dx \wedge dy$. This explains the factor of $2i$ in (60), and we obtain

$$\mathrm{cl}_{\mathrm{dR}}(Z_f) = \widetilde{\eta}_{f_\psi} \otimes \omega_f - \epsilon \cdot \omega_{f_\psi} \otimes \widetilde{\eta}_f$$

as desired. ☐

7.2 Definition of Beilinson–Flach units

Let $f \in S_2(\Gamma_1(N_f), \chi_f)$ be a newform. Let ψ and β be Dirichlet characters with the same parity and coprime conductors N_ψ, N_β. Let $M = N_f N_\psi^2$. We define the *Beilinson–Flach unit* $b_{f,\psi,\beta}$ via the intersection pairing (45):

$$b_{f,\psi,\beta} := \langle \Xi_{\beta,M}, Z_{f,\psi} \rangle \in \overline{\mathbf{Q}}^* \otimes \overline{\mathbf{Q}}. \tag{64}$$

This intersection is taking place on the surface $S = X_1(M) \times X_1(M)$. The following proposition explains why the element $b_{f,\psi,\beta}$ is called a unit.

Proposition 7.4. *Let $K = \mathbf{Q}(\zeta_{N_f N_\psi N_\beta})$. Let $\eta = (\chi_f \psi)^{-1}\beta$ and assume that $\eta \neq 1$. We have*

$$b_{f,\psi,\beta} \in (\mathcal{O}_K^* \otimes \overline{\mathbf{Q}})^\eta.$$

Proof. We view points on $Y = Y_1(M)$ in the usual way as parameterizing pairs $x = (E, P)$ where E is an elliptic curve and $P \in E$ is a point of exact order M. We first show that for every point $(x_1, x_2) \in Y^2$ appearing in the intersection of $Z_{f,\psi}$ and the cycles occuring in the definition of $\Xi_{\beta,M}$, the corresponding elliptic curves E_1 and E_2 have complex multiplication.

Since f is new, multiplicity one implies that f is distinguished in $S_2(\Gamma_1(M))$ by its Hecke eigenvalues at primes not dividing any fixed integer A. Therefore, if we complete $f_1 = f$ to a basis $\{f_1, \ldots, f_r\}$ of eigenforms in $S_2(\Gamma_1(M))$ and we choose for each $i \geqslant 2$ a prime $\ell_i \nmid A$ such that $a_{\ell_i}(f_i) \neq a_{\ell_i}(f)$, we can write

$$T_f = \prod_{i=2}^{r} \frac{T_\ell - a_{\ell_i}(f_i)}{a_{\ell_i}(f) - a_{\ell_i}(f_i)}.$$

Choosing $A = N_f N_\beta N_\psi$, we see that T_f can be written as a linear combination of operators T_n with $\gcd(n, N_f N_\beta N_\psi) = 1$. By definition, for any point $(x_1, x_2) \in \mathrm{gr}(T_n)$, the elliptic curves E_1 and E_2 are related by a cyclic n-isogeny. Now the remaining maps occuring in the homomorphism $Z_{f,\psi}$ also send elliptic curves to isogenous elliptic curves; furthermore, the primes occuring the in the degrees of these isogenies all divide $N_f N_\psi$, and in particular are relatively prime to the n occuring above.

Now by definition (see [LLZ, §2.7]), the curves arising in the definition of $\Xi_{\beta,M}$ are the images C_j of the maps $Y_1(N_\beta^2 M) \to Y_1(M)^2$ given by $z \mapsto (z, z + j/N_\beta)$ with $\gcd(j, N_\beta) = 1$ under the usual complex analytic isomorphism $Y_1(M)(\mathbf{C}) = \Gamma_1(M)\backslash\mathcal{H}$. For any $z \in \mathcal{H}$, the elliptic curves $E_1 = \mathbf{C}/\langle 1, z\rangle$ and $E_2 = \mathbf{C}/\langle 1, z + j/N_\beta\rangle$ are related by a cyclic N_β^2-isogeny.

Suppose that $(x_1, x_2) \in \mathrm{gr}(Z_{f,\psi})$ and that (x_1, x_2) lies on some C_j. The elliptic curves E_1 and E_2 underlying the points x_1, x_2 are related on the one hand by a cyclic isogeny of degree divisible by n, with $\gcd(n, N_\beta) = 1$, and on the other hand by a cyclic isogeny of degree N_β^2. It follows (by composing one of these isogenies with the dual of the other) that E_1 is related to itself by a nontrivial cyclic isogeny. (This argument does not quite work if $n = 1$, but it may be easily fixed, e.g. by replacing the constant 1 by an appropriate linear combination of T_n with $n > 1$; we leave the necessary modifications to the reader.) The only elliptic curves related to themselves by cyclic isogenies are those with complex multiplication. This proves the claim that E_1 (and hence the isogenous curve E_2) have complex multiplication.

Now, the functions on the curves C_j used in the definition of $\Xi_{\beta,M}$ are Siegel units. It is well-known from the theory of complex multiplication that Siegel units of level $N_\beta^2 M$ evaluated at CM points are units in abelian extensions of quadratic imaginary fields, unless $N_\beta^2 M$ is a power of a prime ℓ, in which case one obtains ℓ-units (see e.g. [Ram]). The same is true for the values of Siegel units at cusps at which the units are regular (one obtains circular units in abelian extensions of $\mathbf{Q}$). In conclusion, we find that $b_{f,\psi,\beta} := \langle \Xi_{\beta,M}, Z_{f,\psi}\rangle$ lies in $\mathcal{O}_H^* \otimes \mathbf{Q}$ for some number field H (or $\mathcal{O}_H[1/\ell]^* \otimes \mathbf{Q}$ if $N_\beta^2 M$ is a power of a prime ℓ).

To conclude the proof, we must show that Galois acts on $b_{f,\psi,\beta}$ via the character $\eta = \chi_f^{-1}\psi^{-1}\beta$. This finishes the proof even in the case that $N_\beta^2 M$ is a power of a prime ℓ, since the inclusion $(\mathcal{O}_K^* \otimes \overline{\mathbf{Q}})^\eta \subset (\mathcal{O}_K[1/\ell]^* \otimes \overline{\mathbf{Q}})^\eta$ is an isomorphism when $\eta \neq 1$ as we have assumed.

The intersection pairing $\langle \ , \ \rangle$ is Galois equivariant. Galois clearly acts on $\Xi_{\beta,M}$ via β. The Hecke operators, and hence also the idempotent T_f, are defined over $\mathbf{Q}$. Galois acts on tw_ψ via ψ^{-1} (see [LLZ, Prop. 2.7.5(3)]). The Atkin-Lehner involution W_{N_f} is defined over $\mathbf{Q}(\mu_{N_f})$, and for a prime $q \nmid N_f$, Frobenius at q acts on W_{N_f} via $\sigma_q(W_{N_f}) = W_{N_f} \circ \langle q \rangle$. The diamond operators act on f^* through the character χ_f^{-1}. Combining these observations, it follows that Galois acts on $Z_{f,\psi}$ by $\chi_f^{-1}\psi^{-1}$. The desired result follows. ☐

Theorem 7.5. *Let $M = N_f N_\psi^2$ as above. Suppose that β and ψ have the same parity. We have*

$$\log_\infty(b_{f,\psi,\beta}) = \frac{L'_{N_\beta M}(f,f,\psi\beta^{-1},1)}{2\pi i \langle f^*, f^*\rangle_M} A(f_\psi, f, \beta^{-1}, 1)$$

where $A(f_\psi, f, \beta^{-1}, s)$ is as in (57) and $\log_\infty$ is as defined in §4.2.

Proof. In view of the definition of $b_{f,\psi,\beta}$ given in (64), the result follows by combining Theorem 5.2, Corollary 6.2 , and Proposition 7.2 with $g = f$. ☐

Similarly combining Theorem 5.4, Corollary 6.5, and Proposition 7.2 we obtain:

Theorem 7.6. *Let $\kappa \in \mathcal{W}_F$ with $\nu(\kappa) = \nu_{2,\alpha}$. Suppose that β and ψ have the same parity. With notation and assumptions as in Section 6.3, we have*

$$L_p(F, F_\psi, \kappa, \kappa, \nu_{1,\beta}) = \frac{\alpha(-1) a_p(f)^{-2w+2r} \chi_F(p)^{-r-r'} \psi(p)^{-w-r'}}{4i\tau(\alpha)\tau(\beta^{-1}\alpha)} \log_p(b_{f,\psi,\beta}).$$

8. Factorization on half of weight space

8.1 Two-variable factorization

We are now in a position to prove Theorem 2 from the Introduction, which states:

$$L_p(F,F,\psi,\kappa,\kappa,\sigma) = \mathcal{E}(\kappa,\sigma) \cdot L_p(\mathrm{Sym}^2 F, \psi, \kappa, \sigma) L_p(\chi_F \psi, z \cdot \sigma/\kappa)$$

for $\kappa \in \mathcal{W}_F$ and $\sigma \in \mathcal{W}$ such that $\sigma(-1) = -\psi(-1)$.

Proof of Theorem 2. By continuity, it suffices to prove the result on the dense set of points (κ, σ) in $\mathcal{W}_F \times \mathcal{W}$ such that $\nu(\kappa) = \nu_{2,\alpha}$ and $\sigma = \nu_{1,\beta}$ where β has the same parity as ψ. The equation we want to prove is:

$$L_p(F,F,\psi,\kappa,\kappa,\nu_{1,\beta}) = \mathcal{E}(\kappa,\sigma)L_p(\mathrm{Sym}^2 F,\psi,\kappa,\nu_{1,\beta})L_p(\chi_F\psi,\nu_{0,\beta\alpha^{-1}}). \tag{65}$$

Theorem 7.6, equation (36), and equation (42) give formulae for the three p-adic L-functions in this equation. We recall the notation $\xi = \psi\beta^{-1}$, $\eta = \chi_F\psi\alpha\beta^{-1}$, $\mathrm{cond}(\alpha\beta^{-1}) = p^{r'}$, $\mathrm{cond}(\alpha) = p^r$. Note that, relative to the classical notation, we have

$$\begin{aligned} L_p(\chi_F\psi,\nu_{0,\beta\alpha^{-1}}) &= \widetilde{\epsilon}(\chi_F\psi,\nu_{1,\alpha\beta^{-1}})L_p(\chi_F^{-1}\psi^{-1}\beta\alpha^{-1},1) \\ &= \frac{-\chi_F^{-1}\psi^{-1}(-p^{r'})i^{a_{\chi_F\psi}}}{\tau(\beta^{-1}\alpha)}\log_p(u_\eta). \end{aligned}$$

The second equation here uses (42) along with the well-known formula $\tau(\chi)\tau(\chi^{-1}) = \chi(-1)N_\chi$ and the fact that $\beta\alpha^{-1}$ has the same parity as $\chi_F\psi$. We have from (36):

$$L_p(\mathrm{Sym}^2 F,\psi,\kappa,\sigma) = \frac{-\psi(-1)}{4i^{a_{\chi_F\psi}}}\cdot\frac{a_p(f)^{2r-2w}\tau(\beta)\psi(p)^{-w}}{\chi_F(p)^r\tau(\alpha)W'(f)}\cdot\frac{L(\mathrm{Sym}^2 f,\xi,1)}{-4\pi\langle f,f\rangle}.$$

and from Theorem 7.6:

$$L_p(F,F_\psi,\kappa,\kappa,\nu_{1,\beta}) = \frac{\alpha(-1)a_p(f)^{-2w+2r}\chi_F(p)^{-r-r'}\psi(p)^{-w-r'}}{4i\tau(\alpha)\tau(\beta^{-1}\alpha)}\log_p(b_{f,\psi,\beta}).$$

Cancelling common terms, our desired result may therefore be written:

$$\log_p(b_{f,\psi,\beta}) = \tau(\beta)\frac{L(\mathrm{Sym}^2 f,\xi,1)}{-4\pi i\langle f,f\rangle}\prod_{\ell|N}(1-\eta(\ell))\log_p(u_\eta). \tag{66}$$

Meanwhile, we consider the classical factorization formula

$$L'_M(f,f,\xi,1) = L_M(\mathrm{Sym}^2 f,\xi,1)L'_M(\chi_F\alpha\xi,0),$$

where in each instance L_M indicates that Euler factors at primes dividing M have been removed. Combining Theorem 7.5 and equation (40), we obtain

$$\log_\infty(b_{f,\psi,\beta})\cdot\frac{2\pi i\langle f^*,f^*\rangle}{\tau(\beta)\prod_{\ell|M}(1-\xi(\ell)a_\ell(f)^2\ell^{-1})^{-1}} = -\frac{1}{2}L_M(\mathrm{Sym}^2 f,\xi,1)\prod_{\ell|N}(1-\eta(\ell))\log_\infty(u_\eta).$$

Now, the Euler product on the left represents exactly the missing terms between $L_M(\mathrm{Sym}^2 f,\xi,1)$ and the imprimitive L-value $L(\mathrm{Sym}^2 f,\xi,1)$. Also, since $\langle f^*,f^*\rangle = \langle f,f\rangle$, our formula reads:

$$\log_\infty(b_{f,\psi,\beta}) = \tau(\beta)\frac{L(\mathrm{Sym}^2 f,\xi,1)}{-4\pi i\langle f,f\rangle}\prod_{\ell|N}(1-\eta(\ell))\log_\infty(u_\eta). \tag{67}$$

Since $b_{f,\xi}$ and u_η are both elements in the 1-dimensional $\overline{\mathbf{Q}}$-vector space U_η on which the linear map $\log_\infty$ is injective, it follows from (67) that we have

$$b_{f,\xi} = \tau(\xi^{-1})\frac{L(\mathrm{Sym}^2 f,\xi,1)}{-4\pi i\langle f,f\rangle}\prod_{\ell|N}(1-\eta(\ell))u_\eta$$

in U_η. Applying $\log_p$, we obtain exactly the desired result (66). □

8.2 One variable factorization

We now prove Theorem 1 on half of weight space. Recall that this theorem states that for a p-ordinary cuspidal newform f of weight k and nebentype character χ, we have

$$L_p(f \otimes f \otimes \psi, \sigma) = L_p(\mathrm{Sym}^2 f \otimes \psi, \sigma) L_p(\chi'\psi, z\sigma\nu_{k,\chi_p}^{-1}), \qquad \sigma \in \mathcal{W},$$

where $\chi = \chi_p\chi'$ is the decomposition of χ into p-power and prime-to-p parts. We prove the result when $\sigma(-1) = -\psi(-1)$.

Proof of Theorem 1 when $\sigma(-1) = -\psi(-1)$. Let N be the prime-to-p part of the level of f. Let F be the Hida family of tame level N such that $F_\kappa = f$ (or $F_\kappa =$ the ordinary p-stabilization of f, if p does not divide the level of f) for some arithmetic weight κ with $\nu(\kappa) = \nu_{k,\alpha}$ (so $\chi' = \chi_F$ and $\chi_p = \alpha$). Recall equations (37) and (39):

$$L_p(\mathrm{Sym}^2 f \otimes \psi, \sigma) = S(f) \frac{L_p(\mathrm{Sym}^2 F, \kappa, \sigma)}{P(\mathrm{Sym}^2 f, \psi, \sigma) S(f, \psi)},$$
$$L_p(f \otimes f \otimes \psi, \sigma) = S(f) \frac{L_p(F, F, \psi, \kappa, \kappa, \sigma)}{P(f, f, \psi, \sigma) S(f, \psi)}.$$

In view of Theorem 2 and the above equations, our result boils down to

$$\mathcal{E}(\kappa, \sigma) = \prod_{\ell | N} (1 - \chi_F \psi \kappa \sigma^{-1}(\ell)/\ell) = \frac{P(f, f, \psi, \sigma)}{P(\mathrm{Sym}^2 f, \psi, \sigma)}. \tag{68}$$

Now the Euler factors of the *imprimitive* L-functions $L(f, f, \psi, s)$ and $L(\mathrm{Sym}^2 f, \psi, s)$ agree at primes $\ell \mid N$, namely, they both equal $(1 - a_\ell(f)^2\psi(\ell)\ell^{-s})$. Furthermore, in view of the decomposition $\rho_f \otimes \rho_f \otimes \psi \cong (\mathrm{Sym}^2 f \otimes \psi) \oplus (\chi\psi\epsilon^{k-1})$, the Euler factors of the *primitive* L-functions $L(f \otimes f \otimes \psi, s)$ and $L(\mathrm{Sym}^2 f \otimes \psi, s)$ disagree by a factor of $(1 - \chi\psi\ell^{-s+k-1})$. Therefore, the ratio of the polynomials $P_\ell(f, f, \psi, x)$ and $P_\ell(\mathrm{Sym}^2, \psi, x)$ is exactly $(1 - (\chi\psi)(\ell)\ell^{k-1}x)$. Plugging in $\sigma^{-1}(\ell)$ for x and substituting $\kappa = \nu_{k,\chi_p}$ in (68), the desired result follows. □

9. Functional equations

For the remainder of the paper, we assume that p does not divide the level N of the newform f. In order to deduce Theorem 1 on the half of weight space satisfying $\sigma(-1) = \psi(-1)$ from the other half where we have already proven the result, we will prove functional equations for $L_p(f \otimes f \otimes \psi, \sigma)$ and $L_p(\mathrm{Sym}^2 f, \sigma)$ that switch the two halves.

9.1 Symmetric Square L-series

Theorem 9.1 (Jacquet-Gelbart). *Let f be a newform of weight k and nebentype character χ. Let ψ be a Dirichlet character. Define*

$$\Lambda(\mathrm{Sym}^2 f \otimes \psi, s) = \Gamma_{\mathbf{R}}(s - k + 2 - a_{\chi\psi})\Gamma_{\mathbf{C}}(s) L(\mathrm{Sym}^2 f \otimes \psi, s).$$

There is an analytic function $\epsilon(\mathrm{Sym}^2 f, \psi, s) = A \cdot B^s$ with $A \in \overline{\mathbf{Q}}^*$, $B = \mathrm{cond}(\mathrm{Sym}^2 f \otimes \psi) \in \mathbf{Z}^{>0}$, such that

$$\Lambda(\mathrm{Sym}^2 f \otimes \psi, 2k - 1 - s) = \epsilon(\mathrm{Sym}^2 f, \psi, s) \Lambda(\mathrm{Sym}^2 f \otimes \psi^{-1}\chi^{-2}, s).$$

Mirroring the notation from §3.2, we let $\widetilde{\epsilon}(\mathrm{Sym}^2 f, \psi, \sigma) = A\sigma(B)$ be the analytic function on $\mathcal{W}$ that agrees with $\epsilon(\mathrm{Sym}^2 f, \psi, s)$ for $s \in \mathbf{Z}$.

Theorem 9.2 (Schmidt). *Let f be a newform of weight k and nebentype character χ_f. Let ψ be a Dirichlet character of conductor N_ψ. Suppose that $p \nmid N_f N_\psi$. We have*

$$L_p(\mathrm{Sym}^2 f \otimes \psi, \nu_{2k-1}/\sigma) = \widetilde{\epsilon}(\mathrm{Sym}^2 f, \psi, \sigma) L_p(\mathrm{Sym}^2 f \otimes \psi^{-1}\chi_f^{-2}, \sigma)$$

Proof. Theorem 5.5(b) of [Sc] states that

$$\mathfrak{C}_\lambda(\sigma) = C(\Sigma, \lambda)\sigma^{-1}(M_\lambda)\mathfrak{C}_{\lambda^{-1}}(\nu_1/\sigma),$$

where $C(\Sigma, \lambda)$ and M_λ are defined on page 607 of *loc. cit.* Substituting (33) and (34) into this equation with $\lambda = \psi^{-1}\chi^{-1}$ yields the desired result through a tedious but straightforward calculation. ☐

9.2 Rankin L-series

In vast generality, L-functions of motives are expected to satisfy certain functional equations. For the L-function $L(f \otimes g \otimes \psi, s)$ associated to the tensor product of two modular forms, this functional equation is known due to the Rankin–Selberg formula and the functional equation satisfied by non-holomorphic Eisenstein series. We will not require an exact formula for the root number, only that it is an algebraic number (see [LLZ, Prop 4.1.5] for the statement below).

Theorem 9.3. *Let f and g be newforms of weights $k \geqslant \ell$ and nebentype characters χ_f, χ_g, respectively. Let ψ be a Dirichlet character. Define*

$$\Lambda(f \otimes g \otimes \psi, s) = \Gamma_{\mathbf{C}}(s)\Gamma_{\mathbf{C}}(s - \ell + 1)L(f \otimes g \otimes \psi, s).$$

There is an analytic function $\epsilon(f, g, \psi, s) = A \cdot B^s$ with $A \in \overline{\mathbf{Q}}^*$, $B = \mathrm{cond}(f \otimes g \otimes \psi) \in \mathbf{Z}^{>0}$, such that

$$\Lambda(f \otimes g \otimes \psi, k + \ell - 1 - s) = \epsilon(f, g, \psi, s)\Lambda(f \otimes g \otimes \psi^{-1}\chi_f^{-1}\chi_g^{-1}, s).$$

Theorem 9.4. *Let f and g be newforms of weights $k \geqslant \ell$ and nebentype characters χ_f, χ_g, respectively. Let ψ be a Dirichlet character of conductor N_ψ. Suppose that $p \nmid N_f N_g N_\psi$. We have*

$$L_p(f \otimes g \otimes \psi, \nu_{k+\ell-1}/\sigma) = \widetilde{\epsilon}(f, g, \psi, \sigma)L_p(f \otimes g \otimes \psi^{-1}\chi_f^{-1}\chi_g^{-1}, \sigma).$$

Proof. Consider the functional equations for classical and p-adic Rankin L-series given in [LLZ, Thm 4.2.3 and Prop 5.4.4]. For any N divisible by N_f, N_g and indivisible by p, we have

$$D(f, g, 1/N, k + \ell - 1 - s) = N^{1-s} \sum_{y \in \frac{1}{N}\mathbf{Z}/\mathbf{Z}} e^{2\pi i y} D(f|_{W_N}, g|_{W_N}, y, s), \tag{69}$$

$$D_p(f, g, 1/N, z^{k+\ell-1}/\sigma) = N\sigma^{-1}(N) \sum_{y \in \frac{1}{N}\mathbf{Z}/\mathbf{Z}} e^{2\pi i y} D_p(f|_{W_N}, g|_{W_N}, y, \sigma). \tag{70}$$

The functions D and D_p are defined in [LLZ, §4.2 and §5.4]. We apply this with g replaced by g_ψ and $N = \mathrm{lcm}(N_f, N_g \cdot N_\psi^2)$. Now $f|_{W_N}$ is a constant multiple of $(f_{\chi_f^{-1}})(nz)$ for $n = N/N_f$.

Meanwhile $(g_\psi)|_{W_N}$ shares the same T_ℓ-eigenvalues as $g_{\chi_g^{-1}\psi^{-1}}$ for $\ell \nmid N$. As a result, for each $y \in \frac{1}{N}\mathbf{Z}/\mathbf{Z}$, the function $D(f|_{W_N}, (g_\psi)|_{W_N}, y, s)$ is a multiple of $D(f_{\chi_f^{-1}}, g_{\chi_g^{-1}\psi^{-1}}, 1/N, s)$, where the multiple is rational function with algebraic coefficients in terms of the form d^s with $d \mid N$. In summary, we have

$$D(f, g_\psi, 1/N, k+\ell-1-s) = R(s) D(f_{\chi_f^{-1}}, g_{\chi_g^{-1}\psi^{-1}}, 1/N, s), \tag{71}$$

$$D_p(f, g_\psi, 1/N, \nu_{k+\ell-1}/\sigma) = \widetilde{R}(\sigma) D_p(f_{\chi_f^{-1}}, g_{\chi_g^{-1}\psi^{-1}}, 1/N, \sigma), \tag{72}$$

where $R(s)$ is a rational function in terms d^s as above and $\widetilde{R}$ is the unique meromorphic function on $\mathcal{W}$ such that $\widetilde{R}(\nu_s) = R(s)$ for $s \in \mathbf{Z}$. Now by [LLZ, Theorem 4.2.3], we have

$$D(f, g_\psi, 1/N, s) = C(s)\Lambda(f \otimes g \otimes \psi, s) P(f, g, \psi, s) \tag{73}$$

where $C(s) = 2^{1-k} i^{k-\ell} N^{2s+2-k-\ell}$ and $P(f, g, \psi, s)$ is as in (25). We then rewrite (71) and (72) as

$$\frac{D(f, g_\psi, 1/N, k+\ell-1-s)}{C(k+\ell-1-s)P(f, g, \psi, k+\ell-1-s)} = R_1(s) \frac{D(f_{\chi_f^{-1}}, g_{\chi_g^{-1}\psi^{-1}}, 1/N, s)}{C(s)P(f, g, \psi, s)}, \tag{74}$$

$$\frac{D_p(f, g_\psi, 1/N, \nu_{k+\ell-1}/\sigma)}{C(\nu_{k+\ell-1}/\sigma)P(f, g, \psi, \nu_{k+\ell-1}/\sigma)} = \widetilde{R}_1(\sigma) \frac{D_p(f_{\chi_f^{-1}}, g_{\chi_g^{-1}\psi^{-1}}, 1/N, \sigma)}{C(\sigma)P(f, g, \chi_f^{-1}\chi_g^{-1}\psi, \sigma)}, \tag{75}$$

where

$$R_1(s) = R(s) \frac{C(s)P(f, g, \psi, s)}{C(k+\ell-1-s)P(f, g, \psi, k+\ell-1-s)}.$$

Comparing Theorem 9.3, (73), and (74), we obtain $R_1(s) = \epsilon(f, g, \psi, s)$ and hence $\widetilde{R}_1(\sigma) = \widetilde{\epsilon}(f, g, \psi, \sigma)$. The desired result now follows from (75), since the quotients on the left and right sides of this equation are (up to the same constant) by definition $L_p(f \otimes g \otimes \psi, \nu_{k+\ell-1}/\sigma)$ and $L_p(f \otimes g \otimes \psi^{-1}\chi_f^{-1}\chi_g^{-1})$, respectively. $\square$

9.3 Conclusion of the the proof of Theorem 1

One readily verifies that

$$\Lambda(f \otimes f \otimes \psi, s) = \Lambda(\mathrm{Sym}^2 f \otimes \psi, s)\Lambda(\chi\psi, s-k+1)$$

using (1) and the duplication formula for the Gamma function. Comparing the functional equations in Theorems 3.2, 9.1, and 9.3, it follows that

$$\epsilon(f, f, \psi, s) = \epsilon(\mathrm{Sym}^2 f, \psi, s)\epsilon(\chi\psi, s-k+1),$$

and hence

$$\widetilde{\epsilon}(f, f, \psi, \sigma) = \widetilde{\epsilon}(\mathrm{Sym}^2 f, \psi, \sigma)\widetilde{\epsilon}(\chi\psi, \sigma/\nu_{k-1}).$$

Theorem 1 for $\sigma(-1) = \psi(-1)$ now follows from the result for $\sigma(-1) = -\psi(-1)$ by applying the functional equations for the three p-adic L-functions involved, since the map $\sigma \mapsto \nu_{2k-1}/\sigma$ switches these two halves of weight space.

10. Greenberg's conjecture

We conclude by filling in details for the proof of Theorem 4 sketched in the introduction.

10.1 Conjecture at $s = 1$

We must prove equation (14), which we recall:

$$L_p^{[k]}(f \otimes f \otimes \chi^{-1}, k) = \mathscr{L}_{\mathrm{GS}}(\operatorname{ad} f) \cdot S(f) \left(1 - \frac{1}{p}\right) L^{\mathrm{alg}}(\operatorname{ad} f, 1). \tag{76}$$

By definition (see (3) and (37)), we have

$$L_p^{[k]}(f \otimes f \otimes \chi^{-1}, k) = L_p(f \otimes f \otimes \chi^{-1}, \nu_k) = S(f) \cdot \frac{L_p(F, F_{\chi^{-1}}, \nu_k, \nu_k, \nu_k)}{P(f, f, \chi^{-1}, \nu_k)}. \tag{77}$$

Theorem 5.1d' of [Hi3] together with the observation that the derivative of $1 - a_p(F_\kappa)/a_p(f)$ at $\kappa = \nu_k$ is $\frac{1}{2}\mathscr{L}_{\mathrm{GS}}(\operatorname{ad} f)$ implies that

$$L_p(F, F_{\chi^{-1}}, \nu_k, \nu_k, \nu_k) = \frac{1}{2}\mathscr{L}_{\mathrm{GS}}(\operatorname{ad} f) \prod_{\ell | Np} (1 - \ell^{-1}) \cdot \frac{1}{N\varphi(N)}. \tag{78}$$

The extra factor of $(N\varphi(N))^{-1}$ comes from a difference in conventions for the interpolation property of $L_p(F, F_{\chi^{-1}})$. With $k(P) = k(Q) = k$ in the notation of [Hi3], there is a factor of N in [Hi3, Theorem 5.1d] whereas there is none in our Theorem 3.7; also Hida's period (see [Hi3, 4.13]) involves $\langle f, f \rangle_{\Gamma_0(N)} = \langle f, f \rangle_{\Gamma_1(N)}/\varphi(N)$. Combining (77) and (78), we see that (76) is equivalent to

$$L^{\mathrm{alg}}(\operatorname{ad} f, 1) = \frac{1}{2} P(f, f, \chi^{-1}, \nu_k)^{-1} \prod_{\ell | N} (1 - \ell^{-1}) \cdot \frac{1}{N\varphi(N)}. \tag{79}$$

Recall that $L^{\mathrm{alg}}(\operatorname{ad} f, 1)$ is by definition the algebraic part of $L(\operatorname{ad} f, 1)$ using the same period as in the interpolation formula for the definition of $L_p(\operatorname{ad} f, s)$. (Note that the choice of period is therefore inessential, as scaling the period by a factor scales both $L^{\mathrm{alg}}(\operatorname{ad} f, 1)$ and $L_p'(\operatorname{ad} f, 1)$ by the same factor, leaving $\mathscr{L}_{\mathrm{an}}(\operatorname{ad} f, 1)$ independent of choice of period.) From (32), we see that

$$L^{\mathrm{alg}}(\operatorname{ad} f, 1) = \frac{L(\operatorname{ad} f, 1)\Gamma(k)}{2 \cdot 4^k \pi^{k+1} \langle f, f \rangle}. \tag{80}$$

Now the following formula for the imprimitive $L(\operatorname{ad} f, 1)$ is well-known from Rankin's method (for example, combine the last displayed equation on [Hi3, pg. 5] with the factorization $D_N(s, f, f^*) = L^{\mathrm{imp}}(\operatorname{ad} f, s - k + 1)\zeta_N(s - k + 1)$):

$$L^{\mathrm{imp}}(\operatorname{ad} f, 1) = \frac{4^k \pi^{k+1} \langle f, f \rangle}{\Gamma(k) N \varphi(N)}. \tag{81}$$

Now

$$L(\operatorname{ad} f, 1) = L^{\mathrm{imp}}(\operatorname{ad} f, 1) P(\operatorname{Sym}^2 f, \chi^{-1}, \nu_k)^{-1} \tag{82}$$

and

$$P(f, f, \chi^{-1}, \nu_k) = P(\operatorname{Sym}^2 f, \chi^{-1}, \nu_k) \prod_{\ell | N} (1 - \ell^{-1}) \tag{83}$$

as discussed at the end of §8. Combining (80)–(83) yields (79) and therefore completes the proof.

10.2 Conjecture at $s = 0$

The functional equation (Theorem 9.2) yields

$$L_p^{[0]\prime}(\mathrm{ad}\, f, 0) \cdot \widetilde{\epsilon}(\mathrm{Sym}^2 f, \chi^{-1}, \nu_{k-1}) = -L_p^{[1]\prime}(\mathrm{ad}\, f, 1).$$

Meanwhile we have

$$L^{\mathrm{alg}}(\mathrm{ad}\, f, 0) = \frac{L(\mathrm{ad}\, f, 0)\Gamma(k-1)}{4^k \pi^{k-1}\langle f, f\rangle} = L^{\mathrm{alg}}(\mathrm{ad}\, f, 1)/\epsilon(\mathrm{Sym}^2 f, \chi^{-1}, k-1),$$

where the first inequality follows from the interpolation formula (31) and the second by combining (80) and the classical functional equation (Theorem 9.1). Since

$$\widetilde{\epsilon}(\mathrm{Sym}^2 f, \chi^{-1}, \nu_{k-1}) = \epsilon(\mathrm{Sym}^2 f, \chi^{-1}, k-1),$$

we obtain

$$\mathscr{L}_{\mathrm{an}}(\mathrm{ad}\, f, 0) = \mathscr{L}_{\mathrm{an}}(\mathrm{ad}\, f, 1)$$

as desired, in view of the sign in (11). This concludes the proof.

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