SIAM J. DISCRETE MATH.
Vol. 12, No. 4, pp. 524-534

ON THE SIZE OF MINIMUM SUPER ARROVIAN DOMAINS*

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Abstract. Arrow's celebrated impossibility theorem states that a sufficiently diverse domain of voter preference profiles cannot be mapped into social orders of the alternatives without violating at least one of three appealing conditions. Following Fishburn and Kelly, we define a set of strict preference profiles to be super Arrovian if Arrow's impossibility theorem holds for this set and each of its strict preference profile supersets. We write $\sigma(m,n)$ for the size of the smallest super Arrovian set for m alternatives and n voters. We show that $\sigma(m,2) = \lceil \frac{2m}{m-2} \rceil$ and $\sigma(3,3) = 19$. We also show that $\sigma(m,n)$ is bounded by a constant for fixed n and bounded on both sides by a constant times 2^n for fixed m. In particular, we find that $\lim_{n\to\infty} \sigma(3,n)/2^n = 3$. Finally, we answer two questions posed by Fishburn and Kelly on the structure of minimum and minimal super Arrovian sets.

Key words. Arrow's impossibility theorem, voter preference profiles, minimum profile sets

AMS subject classifications. 05A05, 90A08

PII. S0895480198332521

1. Introduction.

Arrow's impossibility theorem [1] states that a sufficiently diverse domain of voter preference profiles cannot be mapped into social orders of the alternatives without violating at least one of three appealing conditions. Fishburn and Kelly [2] consider the smallest domains of profiles of strict rankings for voters that induce an Arrovian dictator and have the property that every superset domain also induces an Arrovian dictator. In this paper, we continue their analysis and answer some of the questions left open in their work.

We consider a finite set X of $m \geq 3$ alternatives and a set of $n \geq 2$ voters, labeled $i = 1, \ldots, n$. Let $\mathbf{R}$ denote the set of all weak orders (transitive and complete binary relations that need not be asymmetric—that is, ties are allowed) on X, and let $\mathbf{S}$ be the set of all linear orders (strict rankings) on X. A *profile* of voter preferences is an n-tuple of strict rankings $d = (S_1, \ldots, S_n) \in \mathbf{S}^n$. Here the linear order S_i represents the preferences of voter i in the profile. A *domain* is a set of profiles: $D \subseteq \mathbf{S}^n$. A *social choice rule* on D is a mapping $f : D \to \mathbf{R}$ that assigns a weak order $\succsim_d$ on X to every $d \in D$. For $S \in \mathbf{S}$, the notation xSy means that x is preferred to y in S, and $S = x_1 x_2 \ldots x_m$ means that x_j is preferred to x_k whenever $j < k$. The strict part of a weak order $\succsim$ in $\mathbf{R}$ is denoted by $\succ$, that is, $x \succ y$ if $x \succsim y$ and $y \not\succsim x$. Given a subset of the alternatives $Y \subseteq X$, any weak order $\succsim$ in $\mathbf{R}$ induces a weak order $\succsim|_Y$ on Y. Similarly, any $S \in \mathbf{S}$ induces a linear order $S|_Y$ on Y. Given a profile $d = (S_1, \ldots, S_n)$, we write $d|_Y = (S_1|_Y, \ldots S_n|_Y)$ and say that d restricts to $d|_Y$.

A domain D is called *Arrovian* if there is no social choice rule f satisfying the following three conditions of Arrow [1], for all $x, y \in X$ and $d, e \in D$:

(P) *Pareto condition.* If $d = (S_1, \ldots, S_n)$ and xS_iy for $i = 1, \ldots, n$, then $x \succ_d y$.

(IIA) *Independence of irrelevant alternatives.* For $Y \subseteq X$, if $d|_Y = e|_Y$ then $\succsim_d|_Y$ is equal to $\succsim_e|_Y$.

*Received by the editors January 12, 1998; accepted for publication January 8, 1999; published electronically October 19, 1999. This research was supported by the National Science Foundation (grant DMS 9531373-001) and the National Security Agency (grant MDA-904-96-1-0044).
http://www.siam.org/journals/sidma/12-4/33252.html

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(ND) *Nondictatorship.* There is no $i \in \{1, \ldots, n\}$ such that $\succ_d$ equals S_i for all $d = (S_1, \ldots, S_n) \in D$.

If $d = (S_1, \ldots, S_n)$ and the hypothesis of the Pareto condition xS_iy for $i = 1, \ldots, n$ holds, then we write $x \gg y$ in d. We say that x and y are a *Pareto pair* and are involved in a *Pareto relationship.* A mapping f that satisfies the first two conditions is called a (P) + (IIA) function. If the function f is such that $\succ_d$ equals S_i for each $d = (S_1, \ldots, S_n)$, then voter i is called a *dictator* for f. An Arrovian domain is one for which every (P) + (IIA) function has a dictator. A domain D is called *super Arrovian* if it is Arrovian and every superset domain $D' \supset D$ is also Arrovian. A super Arrovian domain on m alternatives and n voters that has the smallest number of profiles is called *minimum*, and this number of profiles is denoted $\sigma(m, n)$. A super Arrovian domain is called *minimal* if no proper subset is super Arrovian.

Fishburn and Kelly [2] proved that $\sigma(3, 2) = 6$ and showed $\sigma(m, n) = o(4^n)$ for fixed m and $\sigma(m, n) = o((\log m)^{2+\epsilon})$ for fixed n and $\epsilon > 0$. They also proved $\sigma(m, 2) \leq \min\{6 \cdot 2^{m-3}, (7 \log m)^2\}$ and $\sigma(m, n) > 2^n - 2$. In this paper, we find improved lower and upper bounds and calculate the value of $\sigma(m, 2)$ exactly.

THEOREM 1. *For $m \geq 3$, we have*

$$\sigma(m, 2) = \left\lceil \frac{2m}{m-2} \right\rceil = \begin{cases} 6 & \text{if } m = 3, \\ 4 & \text{if } m = 4 \text{ or } 5, \\ 3 & \text{if } m \geq 6. \end{cases}$$

THEOREM 2. *For $m, n \geq 3$, we have*

$$\frac{m}{m-2}(2^n - 2) < \sigma(m, n) \leq \left\lceil \frac{2m}{m-2} \right\rceil (2^{n-1} - 1) + \left\lceil \frac{2^n - 2n - 2}{\lfloor m/2 \rfloor} \right\rceil + \left\lceil \frac{n(n-1)}{2(m-2)} \right\rceil + 1.$$

For $n \geq 4$, the upper bound can be decreased by 1.

In Section 2, we establish the lower bound for $\sigma(m, n)$ given in Theorem 2. In Section 3, we prove an upper bound for $\sigma(m, n)$ in terms of $\sigma(m, 2)$. In Section 4, we find a general class of super Arrovian domains which yields a tighter upper bound in the case $m = 3$. In Section 5 we evaluate $\sigma(m, 2)$ explicitly, completing the proofs of Theorems 1 and 2. In Section 6, we show that $\sigma(3, 3) = 19$ and answer two questions posed in [2] by giving examples of minimal super Arrovian domains that are not minimum, and by constructing distinct minimum super Arrovian domains for the same m and n that are not related by a permutation of voters and alternatives.

2. Lower bounds on $\sigma(m, n)$. A *doubles profile* is an n-tuple of linear orders on a two element subset of X. Given a profile $d \in \mathbf{S}^n$ and a two-element subset of the alternatives $Y \subset X$, the restriction $d|_Y$ is a doubles profile. This doubles profile represents the preferences of the voters on the two alternatives of Y in the profile d. A doubles profile is said to be *nonunanimous* if the n voters do not all agree on the ordering of the two alternatives. Let P denote the set of all nonunanimous doubles profiles. A domain $D \subseteq \mathbf{S}^n$ satisfies the *near-free doubles condition* if every nonunanimous doubles profile appears as the restriction of some profile $d \in D$. Fishburn and Kelly found the following necessary and sufficient conditions for a domain to be super Arrovian using the near-free doubles condition.

THEOREM 3 ([2],Theorem 1). *A domain D is super Arrovian if and only if it is Arrovian and satisfies the near-free doubles condition.* □

We begin with a lemma generalizing the "only if" part of this result, motivated by [2, Proof of Part 2 of Lemma 2]. If $S \in \mathbf{S}$ and xSy, then we say that x and y are *nonadjacent* if there is a $w \in X$ such that xSw and wSy. Otherwise, x and y are said to be *adjacent.*

LEMMA 4. *Given a super Arrovian domain D, a nonunanimous doubles profile p, and a voter v, there is a profile $d \in D$ restricting to p such that the two alternatives of p are nonadjacent in d for voter v.*

Proof. For concreteness, let v be voter 1, let p be a doubles profile on $\{x, y\} \subset X$, and suppose that x is preferred to y in p for voter 1. Let $C \subset D$ be the set of profiles that restrict to p, and assume that each profile of C has x and y adjacent for voter 1. Then we can define a (P) + (IIA) function $f : D \to \mathbf{R}$ by letting voter 1 be a dictator on $D - C$ and letting voter 1 be a dictator on C *except* for reversing the order of x and y. In other words, f chooses the preferences of voter 1 on every doubles profile other than p. Because D is super Arrovian, f must have a dictator. Since C is nonempty by Theorem 3, voter 1 is not a dictator. Without loss of generality, let voter 2 be a dictator. Then voters 1 and 2 agree on all elements of $D - C$, so that no doubles profile of the form $(yx, xy, \dots)$ appears as the restriction of a profile in D. This contradicts Theorem 3 and proves the lemma. □

COROLLARY 5. *For $m \geq 3$ and $n \geq 2$, we have $\sigma(m, n) \geq \frac{m}{m-2}(2^n - 2)$.*

Proof. Let D be super Arrovian, and let H be the set of all ordered pairs $(p, d) \in P \times D$ such that d restricts to p, and the two alternatives of p are nonadjacent in d for voter 1. By the lemma, $|H| \geq |P| = \binom{m}{2}(2^n - 2)$. Yet by counting the greatest possible contribution of each $d \in D$, we see that $|H| \leq \binom{m-1}{2}|D|$. Combining these inequalities gives the desired result. □

LEMMA 6. *For $m, n \geq 3$, we have $\sigma(m, n) > \frac{m}{m-2}(2^n - 2)$.*

Proof. We need only eliminate the possibility that equality holds in the corollary above. Assume that there exists a super Arrovian domain D of this size. The proof of the corollary implies that for any profile $d \in D$, the alternatives in every Pareto relationship must be adjacent for every voter (otherwise, the contribution of d to $|H|$ is less than $\binom{m-1}{2}$). We claim that this restriction makes it possible to define a (P) + (IIA) function $f : D \to \mathbf{S} \subset \mathbf{R}$ by choosing, for every profile, the reverse of voter 1's preferences on every pair of alternatives except those involved in a Pareto relationship. That is, for $d = (S_1, \dots, S_n)$ we let $f(d)$ be the linear order $\succ_d$ defined by

$$x \succ_d y \text{ if } (x \gg y \text{ in } d) \text{ or } (yS_1x \text{ but not } y \gg x \text{ in } d).$$

Suppose that there is some profile $d = (S_1, \dots, S_n) \in D$ for which this definition of $f(d)$ results in a relation that is not transitive. Then there are three alternatives $x, y, z \in X$ such that the rule defining f yields $x \succ_d y \succ_d z \succ_d x$. First suppose that none of xy, yz, or zx is a Pareto pair. Then we must have yS_1x, zS_1y, and xS_1z, contradicting the transitivity of S_1. Hence we may assume without loss of generality that zx is a Pareto pair and that the others are not. But then we have zS_1y and yS_1x, and the alternatives of the Pareto relationship are nonadjacent. Since this contradicts the assumption about D, we conclude that $f(d)$ is indeed transitive for each profile $d \in D$.

The map f is clearly a (P) + (IIA) function, and voter 1 is not a dictator for f. If there is some other dictator i, then we can choose any other voter j and note that D does not contain any doubles profiles for which 1 and i agree but j differs. This contradicts the near-free doubles condition for D and concludes the proof of the lemma. □

3. An upper bound on $\sigma(m, n)$. We begin by generalizing some notation used by Fishburn and Kelly [2]. Given a subset A of the set $\{1, \dots, n\}$ of all voters and distinct $x, y \in X$, we write xyA for the doubles profile in which the preference ranking of each voter in A is xy and the preference ranking for each voter not in A is yx. Write $\overline{A}$ for $\{1, \ldots, n\} - A$, and note that $xyA = yx\overline{A}$. For a (P) + (IIA) function f on a domain D, we write xAy if $x \succ_d y$ for each $d \in D$ restricting to xyA. If xAy for all ordered pairs of distinct alternatives $x, y \in X$, we write that $A+$ holds. For example, the f constructed in the proof of Lemma 6 has $A+$ if and only if $1 \notin A$. A domain D is super Arrovian if and only if the only (P) + (IIA) functions f on any domain containing D are those defined by: "$A+$ holds whenever $i \in A$" for some fixed i.

LEMMA 7. *For $m \geq 3$ and $n \geq 4$, we have*

$$\sigma(m, n) \leq \sigma(m, 2)(2^{n-1} - 1) + 2^n + \frac{n^2 - 5n - 4}{2}.$$

Proof. We shall construct a super Arrovian domain D of the required size.

Let $\{A, \overline{A}\}$ be a partition of $\{1, \ldots, n\}$, where neither A nor $\overline{A}$ is empty. If $d' = (S'_1, S'_2)$ is a profile on m alternatives and 2 voters, let $g(d')$ be the profile on m alternatives and n voters $(S_1, \ldots, S_n)$ where $S_i = S'_1$ if $i \in A$ and $S_i = S'_2$ if $i \in \overline{A}$. Consider the $\sigma(m, 2)$ profiles of the form $g(d')$ as d' ranges through a minimum super Arrovian domain on m alternatives and 2 voters. Any (P) + (IIA) function f on a domain containing these profiles must have either $A+$ or $\overline{A}+$. Let Z_1 be the set of all such profiles, for the $2^{n-1} - 1$ possibilities for $\{A, \overline{A}\}$. Note that Z_1 satisfies the near-free doubles condition.

Choose three distinct alternatives $x, y, z \in X$. For each pair $i, j \in \{1, \ldots, n\}$ with $i < j$, consider any profile d in which voter i has preference ranking zxy, voter j has preference ranking yzx, and all the other voters have preference ranking zyx. Let f be a (P) + (IIA) function on a domain containing this profile. By the Pareto condition, $z \succ_d x$. Now $\{i\}+$ would imply $x \succ_d y$, and $\{j\}+$ would imply $y \succ_d z$. Since these three orderings are incompatible, this profile shows that $\{i\}+$ and $\{j\}+$ cannot both hold for f. Let Z_2 be any domain containing such a profile d for every pair i, j. It is clear that we can choose Z_2 such that $|Z_2| \leq \binom{n}{2}$.

Finally, for each subset $A \subset \{1, \ldots, n\}$ with $2 \leq |A| \leq n - 2$, choose an element $i \in A$, and write $B = A - \{i\}$. Consider any profile d in which voter i has preference ranking xzy, the voters in B have preference ranking yxz, and all the other voters have preference ranking zyx. If f is any (P) + (IIA) function on a domain containing this profile, then $\overline{\{i\}}+, \overline{B}+$, and $A+$ cannot all hold for f, since otherwise $y \succ_d x, z \succ_d y$, and $x \succ_d z$. Let Z_3 be any domain containing such a d for each set A. We can choose Z_3 so that $|Z_3| \leq 2^n - (2n + 2)$.

Now let $D = Z_1 \cup Z_2 \cup Z_3$. The domain D has the required number of profiles and satisfies the near-free doubles condition; it remains to show that it is Arrovian. We will refer to the profiles of each Z_i as the "stage i profiles." Let f be a (P) + (IIA) function on D. By the stage one profiles, we have either $\{i\}+$ or $\overline{\{i\}}+$ for each i. The stage two profiles show that for no two distinct i and j can we have both $\{i\}+$ and $\{j\}+$.

Suppose first that $\overline{\{i\}}+$ holds for all i. We show by induction that $\overline{A}+$ holds for all $A \subset \{1, \ldots, n\}$ with $1 \leq |A| \leq n - 2$. For any A with $2 \leq |A| \leq n - 2$, choose the $i \in A$ and $B \subset A$ from the third stage of the construction. Note that $\overline{\{i\}}+$ and $\overline{B}+$ hold by induction, so that $A+$ is impossible by the stage three profiles. Then by the stage one profiles, $\overline{A}+$ must hold, completing the induction. However, this is a contradiction for any $|A|$ with $n/2 \leq |A| \leq n-2$ since $A+$ and $\overline{A}+$ are not compatible.

Hence we must have $\{i\}+$ for exactly one voter i. We must show that this voter is a dictator, that is, $A+$ holds for all subsets A containing i. The assumption about voter i implies that this is true for $|A| = 1$ or $|A| = n-1$. The inductive argument in the preceding paragraph shows that $\overline{A}+$ holds for each subset A not containing i with $2 \leq |A| \leq n-2$. This is the desired result. □

Note that the proof of Lemma 7 above works for $n = 3$ except for the paragraph that precludes the possibility that $\overline{\{i\}}+$ holds for all i, since now $n/2 > n-2$. Here one extra profile is needed, namely any profile restricting to (xzy, yxz, zyx), to complete the argument. Henceforth, we will refer to this as a stage three profile in the case $n = 3$. We obtain $\sigma(m,3) \leq 3\sigma(m,2) + 4$, which is the result of [2, Lemma 7].

The profiles of stages two and three are wasteful in the sense that we only draw information from three of the alternatives in each profile. For example, given a stage two profile, we can choose some other pair i', j' and alternative w and alter the position of w so that the profile restricts to zxw for voter i', wzx for voter j', and zwx for the other voters. This saves one profile from Z_2. Continuing in this way, we can choose Z_2 such that $|Z_2| \leq \lceil \binom{n}{2}/(m-2) \rceil$. Similarly, for any stage three profile, we can choose two unused alternatives u and v, another triple i', B', A', and alter the profile so that it restricts to uzv for voter i', vuz for the voters in B', and zvu for the other voters. Using up all the alternatives efficiently in this manner allows us to choose Z_3 such that $|Z_3| \leq \lceil (2^n - 2n - 2)/\lfloor m/2 \rfloor \rceil$. We have proven:

Corollary 8. *For $m, n \geq 3$,*

$$\sigma(m,n) \leq \sigma(m,2)(2^{n-1}-1) + \left\lceil \frac{n(n-1)}{2(m-2)} \right\rceil + \left\lceil \frac{2^n - 2n - 2}{\lfloor m/2 \rfloor} \right\rceil + 1.$$

For $n \geq 4$, the bound can be decreased by 1. □

4. An improved bound for three alternatives. We now present a separate but similar construction of super Arrovian domains which will allow us to find a stronger upper bound in the case $m = 3$. A domain D is a said to be *basic* if any (P) + (IIA) function f on any domain containing D satisfies $\{i\}+$ or $\overline{\{i\}}+$ for every i, and such that $\{i\}+$ and $\{j\}+$ cannot both hold for distinct i and j.

A domain D is said to be *recursive* for a subset $A \subset \{1, \ldots, n\}$ with $1 \leq |A| \leq n-2$ if for every ordered pair of alternatives $x, z \in X$, there exists an alternative $y \in X$ and a profile $d \in D$ such that d restricts to xyz for the voters in A, yzx for some non-empty proper subset of the other voters, and zxy for the rest. If D is recursive for every A with $1 \leq |A| \leq n-2$, we say that D is recursive. A domain D which is both basic and recursive is called *inductive*. Note that any inductive domain satisfies the near-free doubles condition.

Lemma 9. *Any inductive domain D is super Arrovian.*

Proof. Let D be inductive. Since D is basic, any (P) + (IIA) function f on D must have $\{i\}+$ or $\overline{\{i\}}+$ for each i. Also, there can be no two distinct i and j such that both $\{i\}+$ and $\{j\}+$ hold. First suppose that $\overline{\{i\}}+$ holds for all i. We shall show by induction on $|A|$ that $\overline{A}+$ holds for each A with $1 \leq |A| \leq n-1$. The base case, when $|A| = 1$, follows from our assumption. For larger $|A|$, let $x, z \in X$; we need to show that $x\overline{A}z$ holds. Consider the profile $d \in D$ which restricts to xyz for the voters in $\overline{A}$, yzx for the voters in B, and zxy for the voters in C, where $B \cup C = A$ and B and C are nonempty. By the induction, we have $x\overline{B}y$ and $y\overline{C}z$, from which it follows that $x \succ_d y \succ_d z$. Hence we conclude that $x\overline{A}z$ holds, and the induction is complete. Now, the assertions $A+$ and $\overline{A}+$ are contradictory for any A, so we must have $\{i\}+$

for exactly one i. Repeating the induction above shows that $\overline{A}+$ holds for each A not containing i, so that i is a dictator. Hence D is super Arrovian. ∎

THEOREM 10. *For $n \geq 2$, we have $\sigma(3,n) \leq 3(2^n-2)+6\binom{n}{\lfloor (n+1)/2 \rfloor}+\binom{n}{2}$.*

Proof. We construct an inductive domain D of the required size. We begin by letting Y_1 consist of the stage two profiles from the proof of Lemma 7 and the $n\cdot\sigma(3,2)$ stage one profiles corresponding to $\{A,\overline{A}\}=\{\{i\},\overline{\{i\}}\}$, for $i=1,\dots,n$. Y_1 is basic, with $|Y_1|=6n+\binom{n}{2}$.

For any subset $B \subset \{1,\dots,n\}$ with $2 \leq |B| \leq n-1$ and any $i \in B$, consider the 6 profiles which have the following form, as a, b, and c range over the permutations of the alternatives x, y, and z: the voters in $B-\{i\}$ have preference ranking abc, voter i has preference ranking bca, and the rest of the voters have preference ranking cab. Any domain containing these profiles will be recursive for $B-\{i\}$ and $\overline{B}$. Hence if there is a way to choose, for each subset B with $2 \leq |B| \leq \lfloor \frac{n+1}{2} \rfloor$, an element $i \in B$ such that the sets $B-\{i\}$ range over all possible subsets of size between 1 and $\lfloor \frac{n-1}{2} \rfloor$, then we can create a recursive domain Y_2 containing the corresponding

$$6\sum_{i=2}^{\lfloor (n+1)/2 \rfloor}\binom{n}{i}=\begin{cases}3(2^n+\binom{n}{n/2}-2n-2) & \text{if } n \text{ is even}\\ 3(2^n+2\binom{n}{(n+1)/2}-2n-2) & \text{if } n \text{ is odd}\end{cases}$$

profiles. Then $D=Y_1 \cup Y_2$ will be inductive of the required size. We now demonstrate the existence of such a choice.

Let $j<n/2$. Consider the bipartite graph whose vertices consist of the subsets of $\{1,\dots,n\}$ of size j (partite class X) and $j+1$ (partite class Y), where $A \in X$ is connected to $B \in Y$ if $A \subset B$. We want to show the existence of a matching of X into Y, and we will do so using the Hall Matching Condition. Given any subset of the first partite class $Z \subset X$, write $N(Z)$ for the set of neighbors of vertices in Z. We need to show that $|N(Z)| \geq |Z|$ for every $Z \subset X$. Note that each vertex in Z is connected to $n-j$ vertices of Y. This gives $(n-j)|Z|$ edges between vertices in Z and vertices in Y. Since each vertex in Y has degree exactly $j+1$, we have $|N(Z)| \geq (n-j)|Z|/(j+1) \geq |Z|$. Hence the Hall Matching Condition is satisfied, and a matching exists. ∎

COROLLARY 11. *We have $\lim_{n\to\infty}\frac{\sigma(3,n)}{2^n}=3$.*

5. Two voters. In this section we show that the lower bound of Corollary 5 is an equality for $n=2$. Fishburn and Kelly [2, Lemma 2] do this for $(m,n)=(3,2)$. For notational purposes, we will use $X=\{1,\dots,m\}$, and we will label the two voters I and J, with the preferences of voter I written first in each profile. We simply write xIy for $x\{I\}y$. Also, xIY for $Y \subset X$ means xIy for all $y \in Y$. Finally, as an abuse of notation, we write xIX for $xI(X-\{x\})$. We begin by recalling the construction for $m=3$ in [2].

LEMMA 12 ([2], Lemma 2). *We have $\sigma(3,2)=6$.*

Proof. Let D_3 be composed of the six profiles in the table below.

	Profile	Pareto	Conclusion
p_1	$(321,213)$	$2 \gg 1$	$3I1$ or $2J3$
p_2	$(231,123)$	$2 \gg 3$	$2I1$ or $1J3$
p_3	$(213,132)$	$1 \gg 3$	$2I3$ or $1J2$
p_4	$(123,312)$	$1 \gg 2$	$1I3$ or $3J2$
p_5	$(132,321)$	$3 \gg 2$	$1I2$ or $3J1$
p_6	$(312,231)$	$3 \gg 1$	$3I2$ or $2J1$

Note that the near-free doubles condition is satisfied. Let f be a (P) + (IIA) function on D_3. The condition $2 \gg 1$ in the first profile implies $3I1$ or $2J3$. Suppose that the first of these holds for f. The condition $2 \gg 3$ in the second profile implies that $2I1$ or $1J3$. But $3I1$ contradicts $1J3$, so $2I1$ must hold for f. Continuing in this way for each profile, we find that xIy for all distinct $x, y \in X$, so that voter I is a dictator on f. Had we assumed that $2J3$ held in the first profile, we would have similarly found that voter J was a dictator. Hence D_3 is super Arrovian. □

LEMMA 13. *We have* $\sigma(4,2) = \sigma(5,2) = 4$.

Proof. For $m = 4$, let D_4 be the domain below.

	Profile	Pareto	Conclusion
p_1	$(1234, 3412)$	$3 \gg 4$	$1I4$ or $3J1$
p_2	$(2413, 1324)$	$2 \gg 4$	$2I1$ or $1J4$
p_3	$(3142, 4231)$	$4 \gg 2$	$1I2$ or $4J1$
p_4	$(4321, 2143)$	$4 \gg 3$	$4I1$ or $1J3$

Note that the near-free doubles condition is satisfied.

Restricting to the three alternatives $\{1, 2, 3\}$, we obtain four of the six profiles of the minimal super Arrovian domain D_3 above. The only information used to show that D_3 is super Arrovian supplied by the two missing profiles is that if f is any (P) + (IIA) function, then $2I1$ or $1J3$ holds and $1I2$ or $3J1$ holds for f. Yet we can conclude this from our domain D_4 using alternative 4. For example, the Pareto relationship $3 \gg 4$ in p_1 implies that $1I4$ or $3J1$ holds. Similarly, from the other profiles we can conclude that $2I1$ or $1J4$ holds, that $1I2$ or $4J1$ holds, and that $4I1$ or $1J3$ holds. These four conclusions combine to supply the "missing" information above. Hence we can conclude that any (P) + (IIA) function f is dictatorial on the three alternatives $\{1, 2, 3\}$.

Suppose that voter I is dictatorial on $\{1, 2, 3\}$. We can conclude from $4 \gg 3$ in p_4 that $4I2$ and $4I1$ hold. Then in p_2 we have $4 \succ_{p_2} 1 \succ_{p_2} 3$ so that $4I3$ holds. Similarly, $3 \gg 4$ in p_1 implies that $1I4$ and $2I4$ hold. Then from p_3, we have $3I4$. Therefore, voter I is a dictator on all four alternatives. Now suppose that voter J is dictatorial on $\{1, 2, 3\}$. We can conclude from $4 \gg 2$ in p_3 that $4J1$ and $4J3$ hold. Hence from p_1, we see $4J2$ holds. Similarly, $2 \gg 4$ in p_2 implies that $1J4$ and $3J4$ hold. Then from p_4, we have $2J4$. Therefore, voter J is a dictator on all four alternatives. Since either voter I or J is a dictator and the near-free doubles condition is satisfied, D_4 is super Arrovian.

For $m = 5$, let D_5 be the domain below.

q_1	$(15234, 35412)$
q_2	$(24135, 51324)$
q_3	$(53142, 42315)$
q_4	$(43251, 21453)$

Note that the restriction of D_5 to $\{1, 2, 3, 4\}$ is the super Arrovian domain D_4 above. Hence there is a dictator on these four alternatives. Suppose that this dictator is voter I. From $5 \gg 2$ in q_1, we can conclude $5I3$. Then from q_3, we have $5I\{1, 2, 4\}$. Similarly, the condition $2 \gg 5$ in q_4 shows that $3I5$, which implies that $\{1, 2, 4\}I5$ from q_2. Hence voter I is dictatorial on all five alternatives. Now suppose that voter J is dictatorial on $\{1, 2, 3, 4\}$. From $5 \gg 4$ in q_1, we can conclude $5J1$. Then from q_2, we have $5J\{2, 3, 4\}$. Similarly, the condition $4 \gg 5$ in q_4 shows that $1J5$, which implies that $\{2, 3, 4\}J5$ from q_3. Hence voter J is dictatorial on all five alternatives. Therefore, D_5 is super Arrovian. □

Note that since $\sigma(m,n) = \frac{m}{m-2}(2^n-2)$ for $(m,n)=(4,2)$, the proof of Corollary 5 uniquely determines the preferences of each voter, up to permutations of the alternatives, in a minimal super Arrovian domain. The same is true for $m=6$, and one finds that the super Arrovian domains for $m=4$ and $m=6$ are unique up to permutations of the alternatives.

LEMMA 14. *We have* $\sigma(6,2)=\sigma(7,2)=3$.

Proof. For $m=6$, let D_6 be the profile below.

p_1	$(123456, 563412)$	(id, τ)
p_2	$(536142, 426153)$	$(\pi, \pi\tau)$
p_3	$(462513, 132546)$	$(\pi^2, \pi^2\tau)$

If we define permutations $\pi=(154)(236)$ and $\tau=(15)(26)$, then the profiles have the form given in the table above, with $\pi^3=\tau^2=\mathrm{id}$. The permutation τ switches the voters, and the permutation π rotates the profiles. This symmetry will be important in proving that D_6 is Arrovian.

Let f be a (P) + (IIA) map on D_6. The profile p_1 has the three Pareto pairs $1 \gg 2, 3 \gg 4$ and $5 \gg 6$. Hence, the first alternative in $\succsim_{p_1}$ is 1, 3, or 5, or some combination of these in a tie. Suppose that 3 is in the first position, with or without a tie. Then $3I6$ holds, implying that the order $\succ_{p_2}$ restricts to 5361 on those alternatives. Hence $5I1$ holds, so that the order $\succ_{p_3}$ restricts to 2513 on those alternatives. Thus $2I3$ holds, contradicting the fact that 3 appears first in $\succsim_{p_1}$.

Now suppose that 1 appears first in $\succsim_{p_1}$, possibly tied with 5. Then $1I4$ holds. By the symmetry of the profiles, 5 or 4 appears first in $\succsim_{p_2}$, but 4 cannot appear first because we have $1I4$. Hence 5 appears alone in first in $\succsim_{p_2}$, so we conclude that $5I\{1,2,4,6\}$. Then we have $5 \succ_{p_3} 1 \succ_{p_3} 3$, so $5I3$ as well, and hence $5IX$.

In summary, by assuming that 1 appears first in $\succsim_{p_1}$, we are able to show that 5 appears alone in first in $\succsim_{p_2}$ and that $5IX$ holds. Then by symmetry, we also have that 4 appears alone in first in $\succsim_{p_3}$ and that $4IX$ holds, and that 1 appears alone in first in $\succsim_{p_1}$ and that $1IX$ holds. Now $4I5$ implies $3I\{5,6\}$ from p_1, so that $3IX$ holds from p_2. Similarly, we conclude that $2IX$ and $6IX$ hold, and voter I is a dictator for f.

It follows by symmetry that if we had assumed that 5 appeared first in p_1, then we would have found that voter J was a dictator. Hence D is Arrovian, and since it satisfies the near-free doubles condition, it is super Arrovian.

This domain can be extended to a super Arrovian domain D_7 for $m=7$ by inserting alternative 7 as the middle preference of each voter in all three profiles (e.g. between the 3 and 4 for both voters in p_1). The domain D_7 is clearly Arrovian since there is a dictator on $\{1,\ldots,6\}$ and the place of alternative 7 is uniquely determined by the Pareto relationships with the alternatives directly before and after it. Since the near-free doubles condition is satisfied, D_7 is super Arrovian. ∎

This method of inserting alternatives into super Arrovian domains is the fundamental technique used in showing $\sigma(m,2)=3$ for $m \geq 8$. We state without proof a few more base cases whose verifications are nearly identical to the "insertion" arguments used already.

LEMMA 15. *The domain D below, and its restrictions to $\{1,\ldots,7\} \cup K$ for $K \subseteq \{8,9,\alpha,\beta,\delta,\gamma\}$, are all super Arrovian:*

$$\begin{aligned} D = \{&(\beta 1\delta 2387\alpha 45\gamma 69,\ 95\alpha 63\gamma 7\delta 4182\beta),\\ &(\alpha 5936\delta 7\gamma 14\beta 28,\ 84\gamma 26\beta 7915\delta 3\alpha),\\ &(\gamma 486297\beta 51\alpha 3\delta,\ \delta 1\beta 32\alpha 785496\gamma)\}. \end{aligned}$$ ∎

A super Arrovian domain D of size 3 for $n = 2$ is called *extremely expandable* if an alternative can be inserted to yield a super Arrovian domain such that for some profile, the new alternative is in first place for voter I and in last place for voter J. Let T be the set of m for which there exists an extremely expandable D for m alternatives. The lemma above implies that $\{7, 8, 9, 10, 11, 12\} \subseteq T$. If $U = \{m : \sigma(m, 2) = 3\}$, then $T \cup (T + 1) \subseteq U$.

LEMMA 16. *If $m \in T$, then $\{m + 8, m + 9, m + 10\} \subset T$.*

Proof. Let $D = \{(l_1, l_2), (l_3, l_4), (l_5, l_6)\}$ be super Arrovian for m alternatives, where an alternative A can be inserted to yield the super Arrovian domain $D' = \{(n_1, n_2), (n_3, n_4), (n_5, n_6)\}$ with $n_5 = Al_5$ and $n_6 = l_6A$. Then consider the domain with $m + 8$ alternatives:

$$\begin{aligned} D'' = \{&(1n_1237456,\ 5637n_2412),\\ &(536n_37142,\ 426715n_43),\\ &(462751A3l_5,\ l_61A327546)\}. \end{aligned}$$

We first show that D'' is Arrovian. Let f be a (P) + (IIA) function on D'', and let B be any alternative of D. The previous lemma shows that there is a dictator for f on $\{1, \ldots, 7, B\}$, say voter I. Restricting to the alternatives of D', there is also a dictator since D' is super Arrovian. But in the third profile, $A \gg 3$ and $3 \succ B$, so $A \succ B$, and this second dictator must also be voter I. Also, the conditions $1 \gg A \gg 3$ in the third profile uniquely determine the position of A for this profile. Since this argument holds for every B, we see that voter I is in fact a dictator on all the alternatives. Similarly, if we had assumed that voter J was a dictator on $\{1, \ldots, 7, B\}$, then we would have found that voter J was a dictator on all the alternatives.

Except for possibly the doubles $(3A, A3)$ and $(1A, A1)$, the near-free doubles condition holds for D'' because it holds on D' and the restriction of D'' to $\{1, \ldots, 7, B\}$ for every B. Since these two doubles do appear in the second profile, D'' is super-Arrovian. D'' is extremely expandable because an alternative can be inserted in the position of γ in the previous lemma to yield a super Arrovian domain. Hence $m + 8 \in T$. To show that $m + 9, m + 10 \in T$, one inserts alternatives 8 and 9 into D'' as indicated by the previous lemma. □

COROLLARY 17. *For $m \geq 8$, we have $\sigma(m, 2) = 3$.*

Proof. The previous lemma and the base cases $\{7, \ldots, 12\} \subset T$ imply that $\{m \geq 7 : m \neq 13, 14\} \subseteq T$. Hence $\{m \geq 6 : m \neq 14\} \subseteq U$. For $m = 14$, the standard insertion argument shows that

$$\begin{aligned} D = \{&(\beta\omega1\delta2387\alpha45\gamma69,\ 95\alpha63\gamma7\delta418\omega2\beta),\\ &(\alpha5936\delta7\omega\gamma14\beta28,\ 84\delta26\beta791\omega5\delta3\alpha),\\ &(\gamma486297\beta51\alpha3\delta\omega,\ \omega\delta1\beta32\alpha785496\gamma)\} \end{aligned}$$

is super Arrovian, so $U = \{m \geq 6\}$. □

Corollary 8, Lemmas 6, 12, 13, and 14, and Corollary 17 combine to prove Theorems 1 and 2.

6. Miscellaneous results and conclusion. In this section we answer some questions asked by Fishburn and Kelly [2].

PROPOSITION 18. $\sigma(3, 3) = 19$.

Proof. Theorem 2 shows that $\sigma(3, 3) \geq 19$. We create a super Arrovian set of size 19 by starting with the construction of Corollary 8 and eliminating three unnecessary stage one profiles. Let

$$\begin{aligned} D = \{&(zyx, yxz, yxz), && (yxz, zyx, yxz), && (yxz, yxz, zyx), && (yzx, zyx, zxy), \\ &(yzx, xyz, xyz), && (xzy, yxz, xzy), && (xyz, xyz, yzx), && (yxz, yzx, xyz), \\ &(yxz, xzy, xzy), && (zxy, xzy, zyx), && (xzy, xzy, yxz), && (xyz, zxy, xzy), \\ &(xzy, zyx, zyx), && (zyx, xzy, zyx), && (zxy, zxy, xyz), && (xyz, yzx, zxy), \\ &(zxy, yzx, yzx), && (yzx, zxy, yzx), && (zyx, zyx, xzy)\}. \end{aligned}$$

The first three columns are stage one profiles for the three partitions $\{\{1\},\{2,3\}\}$, $\{\{2\},\{1,3\}\}$, and $\{\{3\},\{1,2\}\}$, respectively. There are three stage one profiles missing from D: (xyz, zxy, zxy), (xyz, yzx, xyz) and (zyx, zyx, xzy). The fourth column contains the profiles from stages two and three, with some permutation of the alternatives applied for each profile. Note that permuting the alternatives in the profiles for stages two and three does not affect the proof of Corollary 8.

The information provided by (xyz, zxy, zxy) is that any (P)+(IIA) function on a domain containing this profile has $x1z$ or $z\{2,3\}y$. However, we can conclude this directly from the profiles in D. Suppose that $z\{2,3\}y$ does not hold. Then in the first profile of the fourth column, we have $y \succsim z \succ x$, so that $y\{1,2\}x$ holds. Hence $x3y$ does not hold, and using the argument of Lemma 12 on the profiles in the third column, we find $y\{1,2\}z$. Similarly, the third profile of the fourth column and the profiles of the second column show $x\{1,3\}y$. Then in the last profile of the fourth column, we have $x \succ y \succ z$, so that $x1z$ holds. Therefore, the information provided by (xyz, zxy, zxy) is already provided by the profiles in D, and this profile is unnecessary. A similar argument works for the other two missing profiles. Since the near-free doubles condition is satisfied, D is super Arrovian. □

PROPOSITION 19. *There exists an infinite family of minimal super Arrovian domains that are not minimum.*

Proof. Fix $n = 2$, and let $m \geq 4$. Let x and y be the greatest even and odd integers less than or equal to m, respectively. Define E to be the string of alternatives $567 \cdots m$, let $F = 68 \cdots x$, and let $G = 579 \cdots y$. For $L = E, F$, or G, we let $\overline{L}$ be the string of alternatives of L in reverse order. Finally, define

$$\begin{aligned} D = \{&(\overline{E}1234, 3412E), \\ &(3142E, \overline{E}4231), \\ &(2\overline{F}41G3, 1\overline{G}32F3), \\ &(4\overline{G}32F1, 2\overline{F}14G3)\}. \end{aligned}$$

For $m = 4$, we have $D = D_4$ from Lemma 13, so D is super Arrovian. The standard insertion argument shows that D is super Arrovian for larger m as well.

For $m \geq 6$, removing the first profile causes D not to be super Arrovian because Lemma 4 is not satisfied with $p = (m(m-2), (m-2)m)$ on either voter. Similarly, removing the second profile violates Lemma 4 with $p = ((m-2)m, m(m-2))$. Removing the third profile violates the lemma with $p = (m(m-1), (m-1)m)$ for m even and with $p = ((m-1)m, m(m-1))$ for m odd. The same is true with the cases reversed for the fourth profile. Hence D is minimal. By Theorem 1, D is not minimum for $m \geq 6$, and the proof is complete. □

PROPOSITION 20. *Minimum super Arrovian domains need not be unique up to permutations of the voters and alternatives.*

Proof. Let D' be the restriction of the domain D in Lemma 15 to the alternatives $\{1, \ldots, 9\}$, and let D'' be the restriction of D to the alternatives $\{1, \ldots, 8, \alpha\}$. No matter which permutation of alternatives or voters is applied to D', no profile will have the first choice of each voter in the last place for the other voter. Since D'' does contain such a profile, the two minimum super Arrovian domains are not related by permutations. □

Theorems 1 and 2 along with these propositions essentially answer all of the open questions posed in [2]. The exact value of $\sigma(m, 2)$ is calculated for all m, as well as bounds which show that $\sigma(m, n)$ is $\Theta(1)$ for fixed n and $\Theta(2^n)$ for fixed m. Open questions suggested by Theorem 1 are whether $\sigma(m, n)$ is always decreasing in m for fixed n and whether $\sigma(m, n)$ always reaches $2^n - 1$ for m large enough. Another topic for further study is the behavior of $\sigma(m, n)/2^n$ for fixed m as n grows large. We have also left open the question of precisely which pairs (m, n) have the property that there is a unique minimum super Arrovian set on m alternatives and n voters, up to permutations of the alternatives and voters.

Acknowledgments. This work was done under the supervision of Joseph Gallian at the University of Minnesota, Duluth. The author wishes to thank Joseph Gallian, Daniel Biss, and David Witte for their suggestions and encouragement.

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