MATH 219: MULTIVARIABLE CALCULUS (SECTION 4)
SPRING 2024

Midterm Exam #1 (Total Points: 100)

Name: Solutions

(1) (a) (15 pts) Let L be the line in 3-space that's parametrized by $\vec{x}(t) = (-7t-2, -3-t, 5t)$. Find an equation of the plane that contains L and the point $(1,1,5)$.

(b) (5 pts) What is a parametrization of this plane?

1a Need a pt on the plane & a vector $\perp$ to the plane.

Take pt to be $(1,1,5)$.

We can find a normal vector by taking 2 vectors on the plane & computing their cross product.

2 pts on line L:

- $t=0 \Rightarrow (-2,-3,0) \leftarrow p_0$
- $t=1 \Rightarrow (-9,-4,5) \leftarrow p_1$

$\Rightarrow$

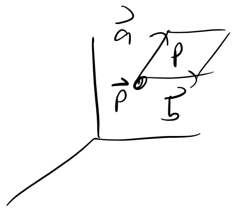
2 vectors on plane:

- $\vec{p_0} - \overrightarrow{(1,1,5)} = \overrightarrow{(-3,-4,-5)}$
- $\vec{p_1} - \overrightarrow{(1,1,5)} = \overrightarrow{(-10,-5,0)}$

$$\overrightarrow{(-3,-4,-5)} \times \overrightarrow{(-10,-5,0)} = \overrightarrow{(-25,50,-25)}.$$

$\Rightarrow$ Equation of plane: $-25(x-1)+50(y-1)-25(z-5)=0.$

1b



Plane P has parametrization:

$\vec{x}(s,t) = \vec{p} + s\vec{a} + t\vec{b}.$

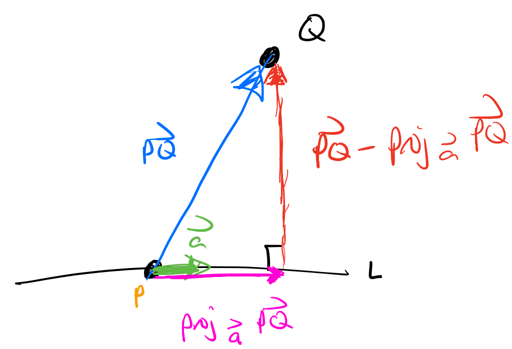
$\Rightarrow$ Our plane can be parametrized by:

$$\vec{x}(s,t) = \overrightarrow{(1,1,5)} + s\overrightarrow{(-3,-4,-5)} + t\overrightarrow{(-10,-5,0)}$$

(2) (15 pts) Let L be the line in 3-space that's parametrized by $\vec{x}(t) = (2t - 5, 3 - t, 4)$. What is the distance between L and the point $(1, -2, 3)$?

call this pt Q

Schematic picture:



$$\vec{x}(t) = \underbrace{(-5, 3, 4)}_{\text{pt } P} + t\underbrace{(2, -1, 0)}_{\text{the direction vector for line } L \text{ (call it } \vec{a})}.$$

The distance we're looking for is $\| \vec{PQ} - \mathrm{proj}_{\vec{a}} \vec{PQ} \|$.

$$\vec{PQ} = (1, -2, 3) - (-5, 3, 4) = (6, -5, -1).$$

$$\mathrm{proj}_{\vec{a}} \vec{PQ} = \left(\frac{\vec{a} \cdot \vec{PQ}}{\vec{a} \cdot \vec{a}} \right) \vec{a}$$

$$= \left(\frac{17}{5} \right) (2, -1, 0) = \left(\frac{34}{5}, \frac{-17}{5}, 0 \right).$$

$$\Rightarrow \| \vec{PQ} - \mathrm{proj}_{\vec{a}} \vec{PQ} \| = \left\| \left(\frac{-4}{5}, \frac{-8}{5}, -1 \right) \right\|$$

$$= \sqrt{\frac{16}{25} + \frac{64}{25} + \frac{25}{25}}$$

$$= \sqrt{\frac{105}{25}}.$$

(3) In this problem, $\overrightarrow{a} = \overrightarrow{(1, 2, 3)}$, $\overrightarrow{b} = \overrightarrow{(-1, 1, 2)}$, and $\overrightarrow{c} = \overrightarrow{(2, 1, 4)}$.

(a) (5 pts) Compute the area of the parallelogram whose edge vectors are $\overrightarrow{a}$ and $\overrightarrow{b}$.

(b) (5 pts) Find the volume of the parallelepiped whose edge vectors are $\overrightarrow{a}$, $\overrightarrow{b}$, and $\overrightarrow{c}$.

(c) (5 pts) Does the ordering:

- 1. $\overrightarrow{a}$
- 2. $\overrightarrow{b}$
- 3. $\overrightarrow{c}$

satisfy the right-hand rule? Why or why not?

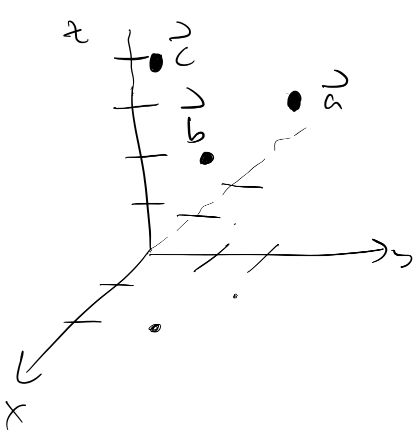
3a Area $= \|\vec{a} \times \vec{b}\|$

$\vec{a} \times \vec{b} = \overrightarrow{(1, -5, 3)} \Rightarrow \|\vec{a} \times \vec{b}\| = \sqrt{35}$.

3b Volume $= |(\vec{a} \times \vec{b}) \cdot \vec{c}| =$

$= |\overrightarrow{(1, -5, 3)} \cdot \overrightarrow{(2, 1, 4)}| = 9$.

3c



Yes b/c if you point your right fingers (minus thumb) in the direction of $\vec{a}$ & curl your fingers towards $\vec{b}$, your thumb pts towards $\vec{c}$.

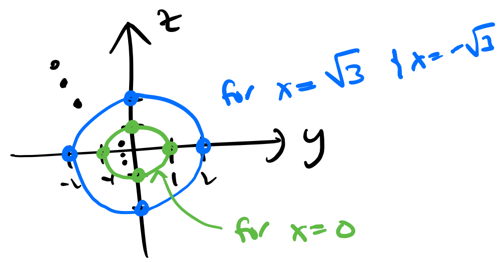
Another way to see this: Form the matrix $\begin{bmatrix} | & | & | \\ \vec{a} & \vec{b} & \vec{c} \\ | & | & | \end{bmatrix} = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 1 \\ 3 & 2 & 4 \end{bmatrix}$.

It has positive determinant (it's 9), so $\vec{a}$, followed by $\vec{b}$, then $\vec{c}$, is in right-hand order.

(4) (a) (10 pts) Sketch a graph of the surface $y^2 + z^2 - x^2 = 1$ using the level curves and contour curves approach. Label one of the level curves and one of the contour curves.

(b) (5 pts) Suppose we stretch this surface by a factor of 3 in the y-direction. We get a new surface; find an equation for this new surface.

(a) level curves:

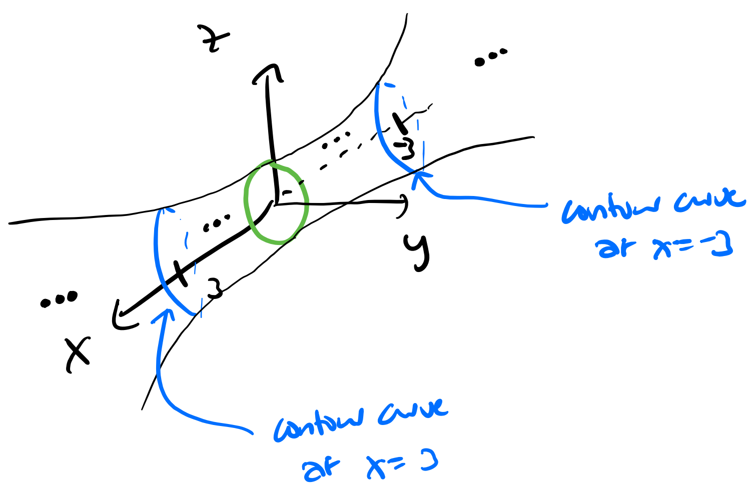


$y^2 + z^2 - (\sqrt{3})^2 = 1 \Rightarrow y^2 + z^2 = 4$

$y^2 + z^2 - (-\sqrt{3})^2 = 1 \Rightarrow y^2 + z^2 = 4$

$y^2 + z^2 - (0)^2 = 1 \Rightarrow y^2 + z^2 = 1$

graph:



(b) $\left(\frac{1}{3}y\right)^2 + z^2 - x^2 = 1$ or $\frac{y^2}{9} + z^2 - x^2 = 1$

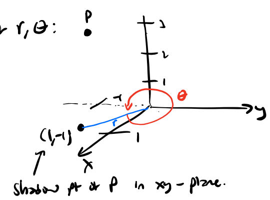
Note: We're using a coefficient of $\frac{1}{3}$ instead of 3, b/c then the pt $(0,1,0)$ is on the graph & $(0,3,0)$ is what you get when you stretch the pt $(0,1,0)$ out by a factor of 3.

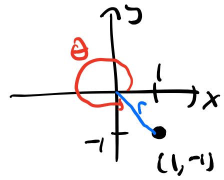
(5) (a) (15 pts) Let P be the point in 3-space with rectangular coordinates $(1, -1, \sqrt{6})$. Write down cylindrical and spherical coordinates for P. You might find it helpful to recall that $\frac{\sqrt{6}}{\sqrt{8}} = \frac{\sqrt{3}}{2}$.

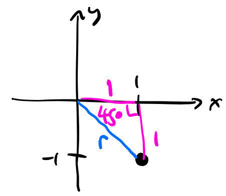
(b) (10 pts) Let $f(x, y) = \frac{\cos(3x)}{e^{2y}}$ and $a = (\frac{\pi}{3}, -1)$. Find the gradient vector $\nabla f(a)$.

5a cylindrical coordinates : (r, θ, z), $z = \sqrt{6}$

To get r, θ:







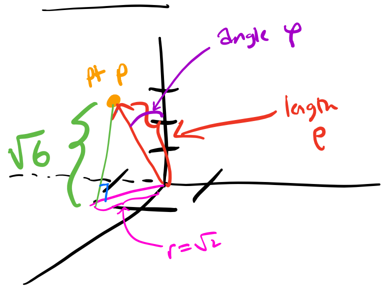
$\Rightarrow r = \sqrt{2}$ &

$\theta = 360° - 45°$
$= 315°$

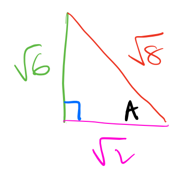
Pt P in cylindrical coordinate is $(\sqrt{2}, 315°, \sqrt{6})$.

5b Spherical coordinates (ρ, φ, θ) : $\theta = 315°$

To get ρ & φ:



$\Rightarrow \rho^2 = (\sqrt{6})^2 + (\sqrt{2})^2$
$= 8$
$\Rightarrow \rho = \sqrt{8}$.



$\sin A = \frac{\sqrt{6}}{\sqrt{8}} = \frac{\sqrt{3}}{2}$

$\Rightarrow A = 60°$

$\Rightarrow \varphi = 90° - A = 30°$

Pt P in spherical coordinates is $(\sqrt{8}, 30°, 315°)$

5b $\nabla f = \overrightarrow{\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)} = \overrightarrow{\left(\frac{-3\sin(3x)}{e^{2y}}, \cos(3x)\cdot\left(-2e^{-2y}\right)\right)}$

$\Rightarrow \nabla f\left(\frac{\pi}{3}, -1\right) = \overrightarrow{\left(\frac{-3\sin(3\cdot\frac{\pi}{3})}{e^{2(-1)}}, \cos\left(3\cdot\frac{\pi}{3}\right)\cdot\left(-2e^{-2(-1)}\right)\right)}$

$= \overrightarrow{(0, 2e^2)}$.

(6) (10 pts) Below are some level curves for a function $z = f(x, y)$.

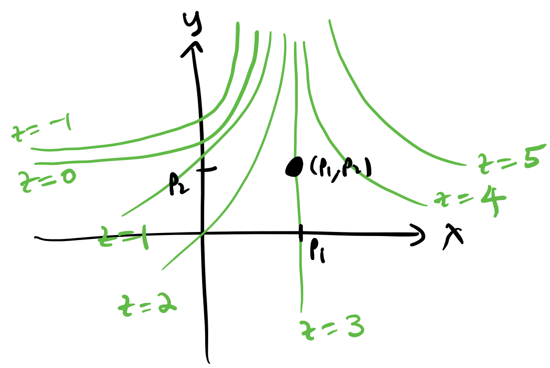


FIGURE 1.

Assume that the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist. Also assume that the level curves not drawn occur at values of z that are strictly between those that are drawn and that segments that look vertical are actually vertical. Now consider the point (p_1, p_2) above.

- Is $\frac{\partial f}{\partial x}(p_1, p_2)$ positive, negative, or 0? Why?
- Is $\frac{\partial f}{\partial y}(p_1, p_2)$ positive, negative, or 0? Why?

Let a be a small > 0.

$\frac{\partial f}{\partial x}(p_1, p_2)$ is approximately $\dfrac{f(p_1+a, p_2) - f(p_1, p_2)}{(p_1+a) - p_1}$

$= \dfrac{f(p_1+a, p_2) - f(p_1, p_2)}{a}$ >0 >0

>0

$\frac{\partial f}{\partial y}(p_1, p_2)$ is approximately $\dfrac{f(p_1, p_2+a) - f(p_1, p_2)}{(p_2+a) - p_2}$

$= \dfrac{\overbrace{f(p_1, p_2+a)}^{3} - \overbrace{f(p_1, p_2)}^{3}}{a}$

$= \frac{0}{a}$ $= 0$.